Quadratic estimates and perturbations of Dirac type operators on doubling measure metric spaces

Lashi Bandara

maths.anu.edu.au/~bandara

Mathematical Sciences Institute
Australian National University

February 10, 2011

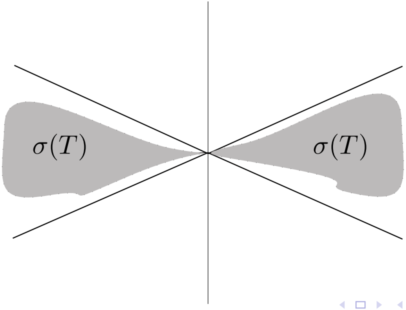
AMSI International Conference on Harmonic Analysis and Applications

Macquarie University

Bisectorial operators

Let $\mathscr{H}$ be a Hilbert space. We say that an operator $T : \mathcal{D}(T) \subset \mathscr{H} \to \mathscr{H}$ is *bi-sectorial* if

(i) T is closed and densely defined,
(ii) $\sigma(T) \subset S_\omega$ for some $\omega < \frac{\pi}{2}$ where $S_\omega = \{\zeta \in \mathbb{C} : \arg(\pm\zeta) \leq \omega\}$, and
(iii) For all $\mu > \omega$, there exists $C_\mu > 0$ such that $\left\|(1 + \xi T)^{-1}\right\| \leq C_\mu$ for all $\xi \in \mathbb{C} \setminus \{0\}$ with $|\arg \xi| \geq \mu$.



The Ψ-functional calculus

Let $S_{\mu}^{o} = \{\zeta \in \mathbb{C} : \zeta \neq 0,\ \arg(\pm\zeta) < \omega\}$ for $\omega < \mu < \frac{\pi}{2}$ and $\Psi(S_{\mu}^{o})$ to be the collection of holomorphic functions $\psi : S_{\mu}^{o} \to \mathbb{C}$ such that

$$|\psi(\zeta)| \leq C \frac{|\zeta|^{s}}{1 + |\zeta|^{2s}}$$

for some $s > 0$.

As in [McIntosh], we can define a bounded operator $\psi(T)$ by

$$\psi(T) = \frac{1}{2\pi\imath} \oint_{\gamma} \psi(\zeta)(\zeta - T)^{-1}\, d\zeta$$

where $\gamma = \left\{\pm e^{\pm\theta} : \omega < \theta < \mu\right\}$ parametrised counterclockwise around S_{ω}.

H^∞ bounded functional calculi

We say that T has a bounded S^o_μ functional calculus if

$$\exists C > 0 \quad \text{such that} \quad \|\psi(T)\| \leq C \|\psi\|_\infty$$

for all $\psi \in \Psi(S^o_\mu)$.

In this case, we can define $f(T)$ for every bounded $f : S^o_\mu \cup \{0\} \to \mathbb{C}$ holomorphic on S^o_μ by

$$f(T) = f(0)P_0 u + \lim_{n\to\infty} \psi_n(T)u$$

where $u \in \mathscr{H}$, $P_0 : \mathscr{H} \to \mathcal{N}(T)$ is the bounded projection corresponding to the decomposition $\mathscr{H} = \mathcal{N}(T) \oplus \overline{\mathcal{R}(T)}$, and $\psi_n \to f$ uniformly on compact subsets of S^o_μ.

See [McIntosh].

Connection to Harmonic analysis

Theorem

A bisectorial operator T has a bounded H^∞ functional calculus if and only if

$$\int_0^\infty \left\| tT(1 + t^2T^2)^{-1}u \right\|^2 \frac{dt}{t} \simeq \|u\|^2$$

for each $u \in \overline{\mathcal{R}(T)}$.

A version of this Theorem can be found as Theorem F in [ADM].

Motivation

In [AKM], Axelsson, Keith and McIntosh consider $\mathscr{H} = \mathrm{L}^2(\mathbb{R}^n, \wedge)$ such that $B \in \mathrm{L}^\infty(\mathbb{R}^n, \mathcal{L}(\wedge))$ with B invertible and *strictly accretive*.

They consider the perturbation of the Hodge-Dirac operator

$$D_B = d + B^{-1} d^* B.$$

They show $\mathcal{D}(\sqrt{D_B^2}) = \mathcal{D}(D_B)$ and

$$\left\| \sqrt{D_B^2} u \right\| \simeq \|D_B u\| \simeq \|du\| + \|dBu\|$$

for all $u \in \mathcal{D}(D_B)$.

This is an extension of the Kato Square Root problem for Differential forms on $\mathbb{R}^n$.

A more general problem

To prove this, [AKM] consider a more general setup.

(H1) The operator $\Gamma : \mathcal{D}(\Gamma) \to \mathscr{H}$ is closed, densely defined and *nilpotent*.

(H2) The operators $B_1, B_2 \in \mathcal{L}(\mathscr{H})$ satisfy

$$\begin{aligned} \mathrm{Re}\ \langle B_1 u, u\rangle &\geq \kappa_1 \|u\| && \text{whenever } u \in \mathcal{R}(\Gamma^*) \\ \mathrm{Re}\ \langle B_2 u, u\rangle &\geq \kappa_2 \|u\| && \text{whenever } u \in \mathcal{R}(\Gamma) \end{aligned}$$

where $\kappa_1, \kappa_2 > 0$ are constants.

(H3) The operators B_1, B_2 satisfy $B_1 B_2(\mathcal{R}(\Gamma)) \subset \mathcal{N}(\Gamma)$ and $B_2 B_1(\mathcal{R}(\Gamma^*)) \subset \mathcal{N}(\Gamma^*)$.

Let

$$\Gamma_B^* = B_1 \Gamma^* B_2, \qquad \Pi_B = \Gamma + \Gamma_B^* \qquad \text{and} \qquad \Pi = \Gamma + \Gamma^*.$$

Quadratic estimates and Kato

Given quadratic estimates and hence a bounded H^∞ functional calculus, they conclude

$$\mathcal{D}(\sqrt{\Pi_B^2}) = \mathcal{D}(\Pi_B) = \mathcal{D}(\Gamma) \cap \mathcal{D}(\Gamma_B^*)$$

with

$$\left\| \sqrt{\Pi_B^2} u \right\| \simeq \|\Pi_B u\| \simeq \|\Gamma u\| + \|\Gamma_B^* u\| .$$

[AKM] give additional necessary conditions in order to prove quadratic estimates for Π_B on $\mathscr{H} = \mathrm{L}^2(\mathbb{R}^n, \mathbb{C}^N)$.

Our setting

We let $(\mathcal{X}, d)$ be a *complete, connected* metric space.

Let μ be a *Borel* measure on $\mathcal{X}$ satisfying the doubling condition:

$$\exists C > 1 \quad \text{such that} \quad \mu(B(x, 2r)) \leq C\mu(B(x, r))$$

for all $x \in \mathcal{X}, r > 0$.

We further assume that $0 < \mu(B(x, r)) < \infty$ for all $x \in \mathcal{X}$ and $r > 0$.

Such a space exhibits "polynomial" growth in the sense

$$\mu(B(x, tr)) \leq Ct^p \mu(B(x, r))$$

for all $t > 1$ and where $p = \log_2 C$.

Additional hypothesis

To get the quadratic estimates, additional hypothesis are needed. The following three hypothesis easily translate to our context.

(H4) Then set $\mathscr{H} = \mathrm{L}^2(\mathcal{X}, \mathbb{C}^N; d\mu)$.

(H5) $B_i \in \mathrm{L}^\infty(\mathcal{X}, \mathcal{L}(\mathbb{C}^N))$ for $i = 1, 2$.

(H7) For B an open ball,

$$\int_B \Gamma u \, d\mu = 0 \quad \text{and} \quad \int_B \Gamma^* v \, d\mu = 0$$

for all $u \in \mathcal{D}(\Gamma)$ with $\mathrm{spt}\ u \subset B$ and for all $v \in \mathcal{D}(\Gamma^*)$ with $\mathrm{spt}\ v \subset B$.

Upper gradients

One of the key components in the proof of [AKM] is to use cutoff functions with good gradient bounds.

In our situation the lack of a differential structure means that we do not have the luxury of a gradient and smooth functions.

Motivated by Cheeger in [Cheeger], we define:

Definition (Lipschitz pointwise constant)

For $\xi : \mathcal{X} \to \mathbb{C}^N$ Lipschitz, define $\operatorname{Lip} \xi : \mathcal{X} \to \mathbb{R}$ by

$$\operatorname{Lip} \xi(x) = \limsup_{y \to x} \frac{|\xi(x) - \xi(y)|}{d(x, y)}.$$

We let $\mathbf{Lip}\, \xi$ denote the *Lipschitz constant* ξ.

The Leibniz hypothesis

(H6) For every bounded Lipschitz function $\xi : \mathcal{X} \to \mathbb{C}$, multiplication by ξ preserves $\mathcal{D}(\Gamma)$ and $\mathbb{M}_\xi = [\Gamma, \xi I]$ is a multiplication operator. Furthermore, there exists a constant $m > 0$ such that $|\mathbb{M}_\xi(x)| \leq m\, |\mathrm{Lip}\, \xi(x)|$ for almost all $x \in \mathcal{X}$.

In [AKM], $\nabla\xi$ was used in place of $\mathrm{Lip}\, \xi$.

The Poincaré hypothesis

Since we do not have a gradient, we replace the Coercivity hypothesis of [AKM] as a Poincaré hypothesis in terms of the operator Π.

(H8) There exist a $C > 0$ and $c > 0$ such that for all balls $B \subset \mathcal{X}$,

$$\int_B |u(x) - \mathrm{m}_B u|^2 \ d\mu(x) \leq C \,\mathrm{rad}(B)^2 \int_{cB} |\Pi u(x)|^2 \ d\mu(x)$$

for all $u \in \mathcal{D}(\Pi) \cap \mathcal{R}(\Pi)$.

The main result

Theorem

If (Γ, B_1, B_2) *satisfies the hypothesis (H1)-(H8), then*

$$\int_0^\infty \left\| t\Pi_B(1 + t^2\Pi_B^2)^{-1}u \right\|^2 \ \frac{dt}{t} \simeq \|u\|^2$$

for all $u \in \overline{\mathcal{R}(\Pi_B)} \subset \mathrm{L}^2(\mathcal{X}, \mathbb{C}^N)$.

Our work closely follows that of [Morris], where this Theorem is proved as a local estimate on a complete Riemannian Manifold with at most exponential volume growth.

Christ's dyadic cubes

Theorem (Michael Christ's dyadic cubes [Christ])

There exists a collection of open subsets $\left\{Q_\alpha^k \subset \mathcal{X} : k \in \mathbb{Z}, \alpha \in I_k\right\}$ *with each* $z_\alpha^k \in Q_\alpha^k$, *where* I_k *are index sets (possibly finite), and constants* $\delta \in (0,1)$, $a_0 > 0$, $\eta > 0$ *and* $C_1, C_2 < \infty$ *satisfying:*

(i) *For all* $k \in \mathbb{Z}$, $\mu(\mathcal{X} \setminus \cup_\alpha Q_\alpha^k) = 0$,

(ii) *If* $l \geq k$, *either* $Q_\beta^l \subset Q_\alpha^k$ *or* $Q_\beta^l \cap Q_\alpha^k = \varnothing$,

(iii) *For each* (k,α) *and each* $l < k$ *there exists a unique* β *such that* $Q_\alpha^k \subset Q_\beta^l$,

(iv) $\operatorname{diam} Q_\alpha^k \leq C_1 \delta^k$,

(v) $B(z_\alpha^k, a_0\delta^k) \subset Q_\alpha^k$,

(vi) *For all* k, α *and for all* $t > 0$,
$\mu\left\{x \in Q_\alpha^k : d(x, \mathcal{X} \setminus Q_\alpha^k) \leq t\delta^k\right\} \leq C_2 t^\eta \mu(Q_\alpha^k)$.

Important associated bounded operators

As in [AKM], for $t \neq 0$ we define the following bounded operators

$$\begin{aligned} R_t^B &= (1 + \imath t \Pi_B)^{-1} \\ P_t^B &= (1 + t^2 \Pi_B^2)^{-1} \\ Q_t^B &= t\Pi_B (1 + t^2 \Pi_B^2)^{-1} \\ \Theta_t^B &= t\Gamma_B^* (1 + t^2 \Pi_B^2)^{-1}. \end{aligned}$$

The operators R_t, P_t, Q_t are defined by setting $B_1, B_2 = I$.

Dyadic averaging and the principal part

Let $\mathscr{Q}_t = \mathscr{Q}^j$ when $\delta^{j+1} < t \leq \delta^j$, where $\mathscr{Q}^j = \left\{Q^j_\alpha\right\}$.

We also need the dyadic averaging operator. Let $Q \in \mathscr{Q}_t$ be the unique $Q \ni x$ and define

$$\mathcal{A}_t(x) = \frac{1}{\mu(Q)} \int_Q u \, d\mu.$$

Furthermore, we define the *principal part* of Θ^B_t for $w \in \mathbb{C}^N$ considered as the function $\tilde{w}(x) = w$ by

$$\gamma_t(x) w = (\Theta^B_t \tilde{w})(x).$$

The estimate break up

It is enough to prove that there exists a $C > 0$ such that

$$\int_0^\infty \left\| \Theta_t^B P_t u \right\|^2 \frac{dt}{t} \leq C \left\| u \right\|.$$

As in [AKM], we break up the integral in the following way

$$\begin{aligned} \int_0^\infty \left\| \Theta_t^B P_t u \right\|^2 \frac{dt}{t} \lesssim & \int_0^\infty \left\| (\Theta_t^B - \gamma_t \mathcal{A}_t) P_t u \right\|^2 \ \frac{dt}{t} \\ & + \int_0^\infty \left\| \gamma_t \mathcal{A}_t (P_t - I) u \right\|^2 \ \frac{dt}{t} \\ & + \int_0^\infty \int_{\mathcal{X}} \left| \mathcal{A}_t(x) \right|^2 \left| \gamma_t(x) \right|^2 \ d\mu(x) \frac{dt}{t}. \end{aligned}$$

Separation via Lipschitz functions

Lemma (Lipschitz separation lemma)

Let (X, d) *be a metric space and suppose* $E, F \subset X$ *satisfying* $d(E, F) > 0$*. Then, there exists a Lipschitz function* $\eta : X \to [0, 1]$*, a* $\tilde{E} \supset E$ *with* $d(\tilde{E}, F) > 0$ *such that*

$$\eta|_E = 1, \quad \eta|_{X \setminus \tilde{E}} = 0 \quad \text{and} \quad \mathbf{Lip}\, \eta \leq 4/d(E, F).$$

This is a crucial tool to obtain cutoff functions with good upper-gradient bounds.

Off diagonal estimates

Proposition (Off diagonal estimates)

Let U_t be either R_t^B for $t \in \mathbb{R}$ or P_t^B, Q_t^B, Θ_t^B for $t > 0$. Then, for each $M \in \mathbb{N}$, there exists a constant $C_M > 0$ (that depends only on M and H1-H6) such that

$$\|U_t u\|_{\mathrm{L}^2(E)} \leq C_M \left\langle \frac{\mathrm{dist}(E, F)}{t} \right\rangle^{-M} \|u\|_{\mathscr{H}}$$

where $E, F \subset \mathcal{X}$ are Borel sets and $u \in \mathscr{H}$ with $\mathrm{spt}\ u \subset F$.

Principle and second part approximation

We prove

$$\int_0^\infty \left\|(\Theta_t^B - \gamma_t \mathcal{A}_t) P_t u\right\|^2 \ \frac{dt}{t} \lesssim \|u\|$$

by application of the Off diagonal estimates and a Poincaré hypothesis (H8).

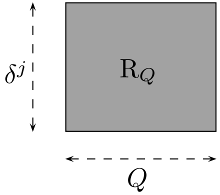
The proof of

$$\int_0^\infty \|\gamma_t \mathcal{A}_t (P_t - I) u\|^2 \ \frac{dt}{t} \lesssim \|u\|$$

relies on the off diagonal estimates and the cancellation property (H7).

Carleson estimate

We define the *Carleson box* over a cube $Q \in \mathscr{Q}^j$ as $\mathrm{R}_Q = \overline{Q} \times (0, \delta^j]$.



We say that a Borel measure ν on $\mathcal{X} \times \mathbb{R}^+$ is Carleson if there exists a $C > 0$ such that

$$\int_{\mathrm{R}_Q} |d\nu| \leq C\mu(Q)$$

and define

$$\|\nu\|_{\mathcal{C}} = \sup_Q \frac{1}{\nu(Q)} \int_{\mathrm{R}_Q} |d\nu|$$

Carleson estimate (cont.)

Proposition

For all $u \in \mathscr{H}$, we have

$$\iint_{\mathcal{X} \times \mathbb{R}^+} |\mathcal{A}_t u(x)|^2 \; d\nu(x,t) \lesssim \|\nu\|_{\mathcal{C}} \|u\|^2$$

for every Carleson measure ν.

The proof of this relies on obtaining a Carelson Theorem in the doubling setting.

Carleson estimate (cont.)

Proposition (Carleson Measure)

For all $Q \in \mathscr{Q}$, we have

$$\iint_{\mathrm{R}_Q} |\gamma_t(x)|^2 \ d\mu(x) \frac{dt}{t} \lesssim \mu(Q).$$

The tools we have constructed for the doubling setting allows us to repeat the argument found in §5.3 in [AKM] with minimal alteration to prove this Proposition.

References I

D. Albrecht, X. Duong, and A. McIntosh.
Operator theory and harmonic analysis.
In *Instructional Workshop on Analysis and Geometry, Part III (Canberra, 1995)*, volume 34 of *Proc. Centre Math. Appl. Austral. Nat. Univ.*, pages 77–136. Austral. Nat. Univ., Canberra, 1996.

A. Axelsson, S. Keith, A. McIntosh.
Quadratic estimates and functional calculi of perturbed Dirac operators.
Invent. math., 163:455–497, 2006.

J. Cheeger.
Differentiability of Lipschitz functions on metric measure spaces.
GAFA, Geom. funct. anal., 9:428–517, 1999.

References II

M. Christ.
A T(b) theorem with remarks on analytic capacity and the Cauchy integral.
Colloq. Math, 60:601–628, 1990.

A. McIntosh.
Operator theory - spectra and functional calculi.
http://maths.anu.edu.au/~alan/lectures/optheory.pdf, 2009.

A. Morris.
Local Hardy Spaces and Quadratic Estimates for Dirac Type Operators on Riemannian Manifolds.
PhD thesis, Australian National University, 2010.