Fredholm and elliptic boundary conditions for general-order elliptic differential operators on compact manifolds

Lashi Bandara
(with Magnus Goffeng - Lund, Hemanth Saratchandran - Adelaide)

Department of Mathematics
Brunel University London

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$\forall u \in \mathrm{C}^\infty_{\mathrm{c}}(\mathring{\mathcal{M}}; \mathcal{E}),\ v \in \mathrm{C}^\infty_{\mathrm{c}}(\mathring{\mathcal{M}}; \mathcal{F})$.

Define:

$$\mathrm{D}_{\max} := ((\mathrm{D}^{\dagger})|_{\mathrm{C}_c^\infty(\mathring{\mathcal{M}};\mathcal{F})})^* \qquad \text{and} \qquad \mathrm{D}_{\min} := \overline{\mathrm{D}|_{\mathrm{C}_c^\infty(\mathring{\mathcal{M}};\mathcal{E})}}.$$

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Goal: describe topology of $\check{\mathrm{H}}(\mathrm{D})$ in terms of data on $\partial\mathcal{M}$.

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Note: This does not imply P acts bounded only $\check{\mathrm{H}}(\mathrm{D})$.

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$$\mathrm{dom}(\Delta_{\mathrm{Dir}}) = \mathrm{H}^2(\mathcal{M};\mathcal{E}) \cap \mathrm{H}^1_0(\mathcal{M};\mathcal{E}).$$

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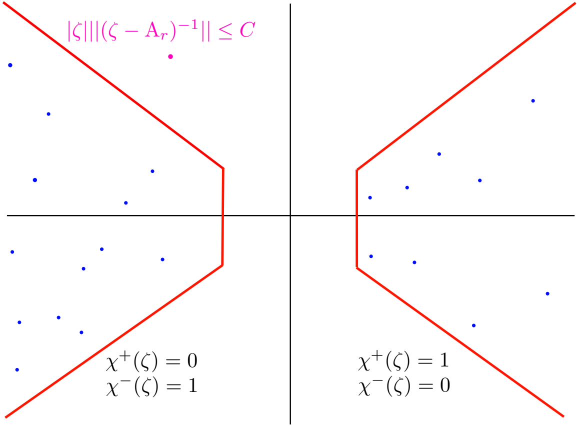

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☣ **Warning: This does not imply $\mathcal{P}_\mathcal{C} - \chi^+(\mathrm{A})$ is compact!** ☣

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Choose $\alpha \in \mathrm{C}_{\mathrm{c}}^{\infty}(0,1]$ such that $\alpha'(1) \neq 0$

$$u \in \chi^+(\mathrm{A})\mathrm{H}^{-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E}) \cap \mathcal{C}_{\mathrm{D}} \quad \Longleftrightarrow \quad \begin{cases} \chi^+(\mathrm{A})u = u \\ \mathcal{P}_{\mathcal{C}_{\mathrm{D}}} u = u \end{cases}$$

$$\Longleftrightarrow \quad \begin{cases} \chi^+(\mathrm{A})u = u \\ (\chi^+(\mathrm{A}) - \mathcal{P}_{\mathcal{C}_{\mathrm{D}}})u = 0 \end{cases}$$

$$\Longleftrightarrow \quad \begin{cases} \chi^+(\mathrm{A})u = u \\ \mathcal{L}u = 0 \end{cases}$$

$$\mathcal{L} := \chi^+(\mathrm{A}) - \mathcal{P}_{\mathcal{C}_{\mathrm{D}}} - \sigma\frac{\alpha'(1)}{4}(1+\Delta)^{-\frac{1}{2}}\chi^-(\mathrm{A}) \in \Psi\mathrm{DO}(-1).$$

Symbol:

$$\sigma_{-1}(\mathcal{L},\xi) = \frac{\alpha'(1)}{4}\begin{pmatrix} 0 & \frac{1}{|\xi|} \\ -\frac{1}{\xi} & 0 \end{pmatrix}.$$

Choose $\alpha \in \mathrm{C}^\infty_{\mathrm{c}}(0,1]$ such that $\alpha'(1) \neq 0 \implies \sigma_{-1}(\mathcal{L},\xi)$ invertible for $\xi \neq 0$

$$u \in \chi^+(\mathrm{A})\mathrm{H}^{-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E}) \cap \mathcal{C}_\mathrm{D} \iff \begin{cases} \chi^+(\mathrm{A})u = u \\ \mathcal{P}_{\mathcal{C}_\mathrm{D}} u = u \end{cases}$$

$$\iff \begin{cases} \chi^+(\mathrm{A})u = u \\ (\chi^+(\mathrm{A}) - \mathcal{P}_{\mathcal{C}_\mathrm{D}})u = 0 \end{cases}$$

$$\iff \begin{cases} \chi^+(\mathrm{A})u = u \\ \mathcal{L}u = 0 \end{cases}$$

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Choose $\alpha \in \mathrm{C}^\infty_\mathrm{c}(0,1]$ such that $\alpha'(1) \neq 0 \implies \sigma_{-1}(\mathcal{L},\xi)$ invertible for $\xi \neq 0 \implies$ $\ker\mathcal{L} < \infty$

$$u \in \chi^+(\mathrm{A})\mathrm{H}^{-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E}) \cap \mathcal{C}_{\mathrm{D}} \quad \iff \quad \begin{cases} \chi^+(\mathrm{A})u = u \\ \mathcal{P}_{\mathcal{C}_{\mathrm{D}}} u = u \end{cases}$$

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Choose $\alpha \in \mathrm{C}^\infty_{\mathrm{c}}(0,1]$ such that $\alpha'(1) \neq 0 \implies \sigma_{-1}(\mathcal{L},\xi)$ invertible for $\xi \neq 0 \implies$ $\ker \mathcal{L} < \infty \iff \dim\left(\chi^+(\mathrm{A})\mathrm{H}^{-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E}) \cap \mathcal{C}_{\mathrm{D}}\right) < \infty.$