Fredholm and elliptic boundary conditions for general-order elliptic differential operators on compact manifolds

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The setting

$\mathcal{M}$ smooth manifold with smooth measure μ.

$(\mathcal{E}, \mathrm{h}^{\mathcal{E}}) \to \mathcal{M}$ and $(\mathcal{F}, \mathrm{h}^{\mathcal{F}}) \to \mathcal{M}$ Hermitian bundles.

$\mathrm{D} : \mathrm{C}^{\infty}(\mathcal{M}; \mathcal{E}) \to \mathrm{C}^{\infty}(\mathcal{M}; \mathcal{F})$ order $m \geq 1$ differential operator.

D *elliptic* $\iff$ $\sigma_{\mathrm{D}}(x, \xi) : \mathcal{E}_x \to \mathcal{F}_x$ invertible for $0 \neq \xi \in \mathrm{T}^*_x\mathcal{M}$.

Formal adjoint $\mathrm{D}^{\dagger} : \mathrm{C}^{\infty}(\mathcal{M}; \mathcal{F}) \to \mathrm{C}^{\infty}(\mathcal{M}; \mathcal{E})$, i.e.,

$$\langle \mathrm{D}u, v \rangle_{\mathrm{L}^2(\mathcal{F}; \mathrm{h}^{\mathcal{F}}, \mu)} = \langle u, \mathrm{D}^{\dagger} v \rangle_{\mathrm{L}^2(\mathcal{E}; \mathrm{h}^{\mathcal{E}}, \mu)}$$

$\forall u \in \mathrm{C}^{\infty}_{\mathrm{c}}(\mathring{\mathcal{M}}; \mathcal{E}),\ v \in \mathrm{C}^{\infty}_{\mathrm{c}}(\mathring{\mathcal{M}}; \mathcal{F})$.

Define:

$$\mathrm{D}_{\max} := \left((\mathrm{D}^\dagger)|_{\mathrm{C}_c^\infty(\mathring{\mathcal{M}};\mathcal{F})}\right)^* \qquad \text{and} \qquad \mathrm{D}_{\min} := \overline{\mathrm{D}|_{\mathrm{C}_c^\infty(\mathring{\mathcal{M}};\mathcal{E})}}.$$

I.e.

$$\mathrm{dom}(\mathrm{D}_{\max}) := \Big\{u \in \mathrm{L}^2(\mathcal{E};\mathrm{h}^{\mathcal{E}},\mu) : \\ \exists C_u \quad |\langle u, \mathrm{D}^\dagger v\rangle| \leq C_u \|v\|_{\mathrm{L}^2(\mathcal{F};\mathrm{h}^{\mathcal{F}},\mu)} \quad \forall v \in \mathrm{C}_c^\infty(\mathring{\mathcal{M}};\mathcal{E})\Big\}.$$

Goal: Understand *all* (not necessarily closed) extensions $\mathrm{D}_{\mathrm{ext}}$

$$\mathrm{D}_{\min} \subset \mathrm{D}_{\mathrm{ext}} \subset \mathrm{D}_{\max}.$$

Equivalently, understand *all* subspaces of

$$\mathrm{dom}(\mathrm{D}_{\max}) \big/ \mathrm{dom}(\mathrm{D}_{\min}).$$

More precisely:

(i) a Banach space $\check{\mathrm{H}}(\mathrm{D})$;

(ii) map $\gamma : \mathrm{dom}(\mathrm{D}_{\mathrm{max}}) \to \check{\mathrm{H}}(\mathrm{D})$ *bounded surjection* satisfying

$$\ker \gamma = \mathrm{dom}(\mathrm{D}_{\mathrm{min}}).$$

Open mapping theorem:

$$\gamma : {\mathrm{dom}(\mathrm{D}_{\mathrm{max}})}/{\mathrm{dom}(\mathrm{D}_{\mathrm{min}})} \to \check{\mathrm{H}}(\mathrm{D})$$

Banach space isomorphism.

Examples

(i) $(\mathcal{M}, \mathrm{g})$ complete Riemannian, $\mathcal{E} = \mathcal{F}$, $\mathrm{D} = \mathrm{D}^\dagger$ first-order (symmetric). Assume: $\exists C < \infty \quad |\sigma_{\mathrm{D}}(x, \xi)|_{\mathrm{op}} \leq C|\xi|$. Then, for all $k \in \mathbb{N}_+$, $\mathrm{dom}((\mathrm{D}^k)_{\max}) = \mathrm{dom}((\mathrm{D}^k)_{\min})$. I.e.,

$$\mathrm{dom}((\mathrm{D}^k)_{\max}) \big/ \mathrm{dom}((\mathrm{D}^k)_{\min}) = 0.$$

(ii) $(\mathcal{N}, \mathrm{g})$ "manifold" with conic singularity at $x \in \mathcal{N}$. I.e., in "polar coordinates" near x, we have $\mathrm{g} = dr^2 + r^2 \mathrm{g}_{\mathcal{P}}$, for $(\mathcal{P}, \mathrm{g}_{\mathcal{P}})$ $(n-1)$-dim Riemannian manifold. Set $\mathcal{M} = \mathcal{N} \setminus \{x\}$, $\mathcal{E} = \mathcal{F} \to \mathcal{M}$ Clifford bundle, D Dirac operator on $\mathcal{E}$. Then,

$$\dim \left(\mathrm{dom}(\mathrm{D}_{\max}) \big/ \mathrm{dom}(\mathrm{D}_{\min}) \right) < \infty.$$

The situation of boundary

Suppose $\mathcal{M}$ has a smooth boundary $\partial\mathcal{M}$.

Let $\vec{T}$ inward pointing vectorfield, and τ associated inward pointing co-vectorfield.

Consider $\gamma : \mathrm{C}^\infty_\mathrm{c}(\mathcal{M};\mathcal{E}) \to \bigoplus_{j=0}^{m-1} \mathrm{C}^\infty_\mathrm{c}(\partial\mathcal{M};\mathcal{E})$

$$\gamma(u) = \Big(\, u|_{\partial\mathcal{M}}, \; (\partial_{\vec{T}} u)|_{\partial\mathcal{M}}, \; \dots \; , \; (\partial_{\vec{T}}^{m-1} u)|_{\partial\mathcal{M}} \Big) .$$

Want:

- extend γ to act on all of $\mathrm{dom}(\mathrm{D}_{\max})$, $\ker\gamma = \mathrm{dom}(\mathrm{D}_{\min})$,
- $\check{\mathrm{H}}(\mathrm{D}) := \gamma\,\mathrm{dom}(\mathrm{D}_{\max})$.

Now suppose $\mathcal{M}$ is compact.

Classic result (Seeley '66, Lions-Magenes '63 (Eng '72)):
$\gamma : \mathrm{C}^\infty(\mathcal{M};\mathcal{E}) \to \bigoplus_{j=0}^{m-1} \mathrm{C}^\infty(\partial\mathcal{M};\mathcal{E})$ extends to a bounded mapping

$$\gamma : \mathrm{dom}(\mathrm{D}_{\max}) \to \bigoplus_{j=0}^{m-1} \mathrm{H}^{-\frac{1}{2}-\mathrm{j}}(\partial\mathcal{M};\mathcal{E})$$

- $\check{\mathrm{H}}(\mathrm{D}) := \mathrm{ran}\,\gamma$ dense in $\bigoplus_{j=0}^{m-1} \mathrm{H}^{-\frac{1}{2}-\mathrm{j}}(\partial\mathcal{M};\mathcal{E})$,
- $\ker\gamma = \mathrm{H}^{\mathrm{m}}_0(\mathcal{M};\mathcal{E}) = \mathrm{dom}(\mathrm{D}_{\min})$.

Topologise $\check{\mathrm{H}}(\mathrm{D})$ such that $\gamma : {}^{\mathrm{dom}(\mathrm{D}_{\max})}\big/_{\mathrm{dom}(\mathrm{D}_{\min})} \rightarrowtail\!\!\!\!\twoheadrightarrow \check{\mathrm{H}}(\mathrm{D})$.

Goal: describe topology of $\check{\mathrm{H}}(\mathrm{D})$ in terms of data on $\partial\mathcal{M}$.

Boundary conditions

- *Generalised boundary condition*: $B \subset \check{\mathrm{H}}(\mathrm{D})$ subspace.
  D_B extension satisfying $\mathrm{D}_{\min} \subset \mathrm{D}_B \subset \mathrm{D}_{\max}$ with
  $$\mathrm{dom}(\mathrm{D}_B) = \{u \in \mathrm{dom}(\mathrm{D}_{\max}) : \gamma u \in B)\}.$$

- *Boundary condition*: $B \subset \check{\mathrm{H}}(\mathrm{D})$ *closed* subspace.
  $\rightsquigarrow \quad \mathrm{D}_B$ *closed* operator.

- $\mathrm{D}_{\min} \subset \mathrm{D}_{\mathrm{ext}} \subset \mathrm{D}_{\max}$ (non-closed) closed extension
  $\iff \quad B_{\mathrm{ext}} := \{\gamma u : u \in \mathrm{dom}(\mathrm{D}_{\mathrm{ext}})\}$ (generalised) boundary condition with
  $\mathrm{D}_{B_{\mathrm{ext}}} = \mathrm{D}_{\mathrm{ext}}$.

- *Adjoint condition*: $\mathrm{D}_B^* = \mathrm{D}^\dagger_{B^*}$ where
  $$B^* := \left\{ v \in \check{\mathrm{H}}(\mathrm{D}^\dagger) : \quad \langle u, v \rangle_{\check{\mathrm{H}}(\mathrm{D}) \times \check{\mathrm{H}}(\mathrm{D}^\dagger)} = 0 \quad \forall u \in B \right\}.$$

  where $\langle u, v \rangle_{\check{\mathrm{H}}(\mathrm{D}) \times \check{\mathrm{H}}(\mathrm{D}^\dagger)} = \langle \mathrm{D}_{\max} u, v \rangle - \langle u, \mathrm{D}^\dagger_{\max} v \rangle$ induced pairing.

- *Fredholm boundary condition*: B boundary condition such that D_B is a Fredholm operator.
- *Semi-elliptically regular boundary condition*: $\mathrm{dom}(\mathrm{D}_B) \subset \mathrm{H}^{\mathrm{m}}(\mathcal{M};\mathcal{E}) \iff$

$$B \subset \mathbb{H}^{\mathrm{m},\mathrm{m}-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E}) := \bigoplus_{j=0}^{m-1} \mathrm{H}^{\mathrm{m}-\frac{1}{2}-\mathrm{j}}(\partial\mathcal{M};\mathcal{E}).$$

- *Elliptically regular boundary condition*: D_B and D_B^* semi-elliptically regular I.e.

$$\mathrm{dom}(\mathrm{D}_B) \subset \mathrm{H}^{\mathrm{m}}(\mathcal{M};\mathcal{E}) \quad \text{and} \quad \mathrm{dom}(\mathrm{D}_B^*) \subset \mathrm{H}^{\mathrm{m}}(\mathcal{M};\mathcal{F}).$$

Note: B Elliptically regular $\implies$ B Fredholm.

Seeley and Calderón projectors

Cauchy data space: $\mathcal{C}_{\mathrm{D}} := \gamma \ker(\mathrm{D}_{\max})$. Define:

$$\mathbb{H}^{\mathrm{m,s}}(\partial\mathcal{M};\mathcal{E}) := \bigoplus_{j=0}^{m-1} \mathrm{H}^{\mathrm{s-j}}(\partial\mathcal{M};\mathcal{E}).$$

There exists a classical pseudo-differential projector $\mathcal{P}_{\mathcal{C}\mathrm{D}}$ of order zero such that

$$\mathcal{C}_{\mathrm{D}} = \mathcal{P}_{\mathcal{C}\mathrm{D}}\mathbb{H}^{\mathrm{m},-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E}),$$

and

$$\check{\mathrm{H}}(\mathrm{D}) = (1-\mathcal{P}_{\mathcal{C}\mathrm{D}})\mathbb{H}^{\mathrm{m,m}-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E}) \bigoplus \mathcal{P}_{\mathcal{C}\mathrm{D}}\mathbb{H}^{\mathrm{m},-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E}).$$

First-order: $\check{\mathrm{H}}(\mathrm{D}) = (1-\mathcal{P}_{\mathcal{C}\mathrm{D}})\mathrm{H}^{\frac{1}{2}}(\partial\mathcal{M};\mathcal{E}) \bigoplus \mathcal{P}_{\mathcal{C}\mathrm{D}}\mathrm{H}^{-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})$.

Induced pairing $\langle u, v\rangle_{\check{\mathrm{H}}(\mathrm{D})\times\check{\mathrm{H}}(\mathrm{D}^\dagger)}$ described in terms of this description.

Towards Fredholmness - closed range

Theorem. Suppose B generalised boundary condition for D elliptic differential operator of order $m \geq 1$. Then, the following hold:

(i) $\ker(\mathrm{D}_B)$ is finite-dimensional $\iff$ $B \cap \mathcal{C}_{\mathrm{D}}$ is finite-dimensional.

(ii) $\mathrm{ran}(\mathrm{D}_B) = \mathrm{ran}(D_{B+\mathcal{C}_D})$ and it is closed $\iff$ $B + \mathcal{C}_{\mathrm{D}}$ is a boundary condition. I.e. $B + \mathcal{C}_{\mathrm{D}}$ is closed in $\check{\mathrm{H}}(D)$.

(iii) $\mathrm{ran}(\mathrm{D}_B)$ has finite algebraic codimension $\iff$ $B + \mathcal{C}_{\mathrm{D}}$ has finite algebraic codimension in $\check{\mathrm{H}}(\mathrm{D})$ $\iff$ $\mathrm{ran}(\mathrm{D}_B)$ is closed and $\mathrm{ran}(\mathrm{D}_B)^\perp$ is finite-dimensional.

Examples

(i) $B := \mathbb{H}^{\mathrm{m},\mathrm{m}-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E}) = \bigoplus_{j=0}^{m-1} \mathrm{H}^{\mathrm{m}-\frac{1}{2}-\mathrm{j}}(\partial\mathcal{M};\mathcal{E})$.
Easy to check: $\mathrm{dom}(\mathrm{D}_B) = \mathrm{H}^{\mathrm{m}}(\mathcal{M};\mathcal{E})$.
B dense subspace of $\check{\mathrm{H}}(\mathrm{D}) \implies \mathrm{D}_B$ is *not* closed.

$$\begin{aligned} &B + \mathcal{C}_{\mathrm{D}} \\ &\quad = \mathbb{H}^{\mathrm{m},\mathrm{m}-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E}) + (1 - \mathcal{P}_{\mathcal{C}_{\mathrm{D}}})\mathbb{H}^{\mathrm{m},\mathrm{m}-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E}) \bigoplus \mathcal{P}_{\mathcal{C}_{\mathrm{D}}}\mathbb{H}^{\mathrm{m},-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E}) \\ &\quad = \check{\mathrm{H}}(\mathrm{D}) \end{aligned}$$

$\implies \mathrm{ran}(\mathrm{D}_B) = \mathrm{ran}(\mathrm{D}_{\max})$ closed.

(ii) B semi-elliptically regular BC $\iff B \subset \bigoplus_{j=0}^{m-1} \mathrm{H}^{\mathrm{m}-\frac{1}{2}-\mathrm{j}}(\partial\mathcal{M};\mathcal{E})$.
Then $\mathrm{ran}(\mathrm{D}_B)$ is closed.

$$\mathrm{ran}(\mathrm{D}_B) = \mathrm{ran}(\mathrm{D}_{B+\mathcal{C}_{\mathrm{D}}})$$

$\implies B + \mathcal{C}_{\mathrm{D}}$ boundary condition.

Characterising Fredholmness

X Banach space, A, B closed subspaces of X.

(A, B) is a *Fredholm pair in* X if:

- $A + B$ is closed;
- ${}^{X}\!/_{(A+B)}$ is finite dimensional.

$$\mathrm{ind}(A, B) := \dim(A \cap B) - \dim\left({}^{X}\!/_{(A+B)}\right).$$

Theorem. D_B is a Fredholm operator $\iff$ $(B, \mathcal{C}_{\mathrm{D}})$ is a Fredholm pair in $\check{\mathrm{H}}(\mathrm{D})$.

$$B^* \cap \check{\mathrm{H}}(\mathrm{D}^\dagger) \cong {}^{\check{\mathrm{H}}(\mathrm{D})}\!/_{(B + \mathcal{C}_{\mathrm{D}})}$$

$$\mathrm{ind}(\mathrm{D}_B) = \mathrm{ind}(B, \mathcal{C}_{\mathrm{D}}) + \dim\ker(\mathrm{D}_{\min}) - \dim\ker(\mathrm{D}^\dagger_{\min}).$$

Elliptic regularity

Theorem. $P : \mathbb{H}^{\mathrm{m},\mathrm{m}-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E}) \to \mathbb{H}^{\mathrm{m},\mathrm{m}-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})$ bounded projection satisfying:

(i) $\mathcal{P}_{\mathcal{C}\mathrm{D}} - (1-P)$ Fredholm on $\mathbb{H}^{\mathrm{m},\mathrm{m}-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})$;

(ii) $\mathcal{P}_{\mathcal{C}\mathrm{D}} - (1-P)$ extends by continuity to $\mathbb{H}^{\mathrm{m},-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})$ and this extension is Fredholm on $\mathbb{H}^{\mathrm{m},-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})$.

Then,

$$B_P = (1-P)\mathbb{H}^{\mathrm{m},\mathrm{m}-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})$$

defines an elliptically regular boundary condition.

In particular, $(1-\mathcal{P}_{\mathcal{C}})\mathbb{H}^{\mathrm{m},\mathrm{m}-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})$ elliptically regular boundary condition.

Note: This does not imply P acts bounded only $\check{\mathrm{H}}(\mathrm{D})$.

Example - Dirichlet Laplacian

$\nabla : \mathrm{C}^\infty(\mathcal{E}) \to \mathrm{C}^\infty(\mathrm{T}^*\mathcal{M} \otimes \mathcal{E})$, $\Delta := \nabla^\dagger \nabla : \mathrm{C}^\infty(\mathcal{E}) \to \mathrm{C}^\infty(\mathcal{E})$, and $m = 2$:

$$\mathbb{H}^{\mathrm{m},\mathrm{m}-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E}) = \mathbb{H}^{2,2-1/2}(\partial\mathcal{M};\mathcal{E}) = \oplus_{j=0}^{1} \mathrm{H}^{\frac{3}{2}-\mathrm{j}}(\partial\mathcal{M};\mathcal{E}) = \mathrm{H}^{\frac{3}{2}}(\partial\mathcal{M};\mathcal{E}) \oplus \mathrm{H}^{\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})$$
$$\mathbb{H}^{\mathrm{m},-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E}) = \mathbb{H}^{2,-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E}) = \mathrm{H}^{-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E}) \oplus \mathrm{H}^{-\frac{3}{2}}(\partial\mathcal{M};\mathcal{E}).$$

Boundary trace: $\gamma(u) = (u|_{\partial\mathcal{M}}, \partial_{\vec{T}} u|_{\partial\mathcal{M}})$.

Dirichlet Laplacian: $\mathrm{dom}(\Delta_{\mathrm{Dir}}) := \left\{ u \in \mathrm{dom}(\Delta_{\max}) : u|_{\partial\mathcal{M}} = 0 \right\}$.

Dirichlet BC:

$$B_{\mathrm{Dir}} := \left\{ u|_{\partial\mathcal{M}} : u|_{\partial\mathcal{M}} = 0 \right\}.$$

Elliptic regularity of boundary condition is not obvious.

Projector defining BC (i.e., $B_{\mathrm{Dir}} = \mathrm{ran}(1 - P_{\mathrm{Dir}})$):

$$P_{\mathrm{Dir}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Principal symbol of $\mathcal{P}_{\mathcal{C}}$:

$$\sigma_0(\mathcal{P}_{\mathcal{C}})(x,\xi) = \begin{pmatrix} \mathrm{Id}_{\mathcal{E}} & |\xi|^{-1}\mathrm{Id}_{\mathcal{E}} \\ |\xi|^{-1}\mathrm{Id}_{\mathcal{E}} & \mathrm{Id}_{\mathcal{E}} \end{pmatrix}.$$

Then,

$$\sigma_0(\mathcal{P}_{\mathcal{C}} - (1 - P_{\mathrm{Dir}})) = \begin{pmatrix} \mathrm{Id}_{\mathcal{E}} & |\xi|^{-1}\mathrm{Id}_{\mathcal{E}} \\ |\xi|^{-1}\mathrm{Id}_{\mathcal{E}} & -\mathrm{Id}_{\mathcal{E}} \end{pmatrix},$$

bounded on both $\mathbb{H}^{2,\frac{3}{2}}(\partial\mathcal{M};\mathcal{E})$ and $\mathbb{H}^{2,-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})$. Theorem gives Δ_{Dir} is elliptically regular. In fact,

$$\mathrm{dom}(\Delta_{\mathrm{Dir}}) = \mathrm{H}^2(\mathcal{M};\mathcal{E}) \cap \mathrm{H}^1_0(\mathcal{M};\mathcal{E}).$$

Back to the topology of $\check{\mathrm{H}}(\mathrm{D})$

$P : \mathbb{H}^{\mathrm{m},-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E}) \to \mathbb{H}^{\mathrm{m},-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})$ projection, restricts to a projection on $\mathbb{H}^{\mathrm{m},\mathrm{m}-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})$ is *boundary decomposing* if:

$$\|u\|_{\check{\mathrm{H}}(\mathrm{D})} \simeq \|(1-P)u\|_{\mathbb{H}^{\mathrm{m},\mathrm{m}-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})} + \|Pu\|_{\mathbb{H}^{\mathrm{m},-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})}.$$

Theorem. $P : \mathbb{H}^{\mathrm{m},\mathrm{m}-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E}) \to \mathbb{H}^{\mathrm{m},\mathrm{m}-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})$ bounded projection satisfying:

(i) $\mathcal{P}_{\mathcal{C}_\mathrm{D}} - (1-P)$ Fredholm on $\mathbb{H}^{\mathrm{m},\mathrm{m}-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})$;

(i) $\mathcal{P}_{\mathcal{C}_\mathrm{D}} - (1-P)$ extends by continuity to $\check{\mathrm{H}}(\mathrm{D})$ and $\mathbb{H}^{\mathrm{m},-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})$ and this extension is Fredholm on $\check{\mathrm{H}}(\mathrm{D})$.

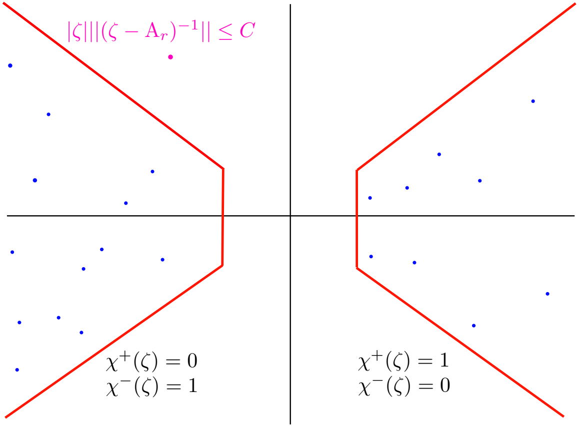
Then, P is boundary decomposing.

The first-order case

Adapted boundary operator A on $\partial\mathcal{M}$:

$$\sigma_{\mathrm{A}}(x,\xi) = \sigma_{\mathrm{D}}(x,\tau(x))^{-1} \circ \sigma_{\mathrm{D}}(x,\xi).$$

Elliptic differential operator of order 1, can be chosen ω-bisectorial $\exists\omega < \pi/2$.



The spaces, $m = 1$

$$\mathbb{H}^{1,\mathrm{s}}(\partial\mathcal{M};\mathcal{E}) = \bigoplus_{j=0}^{m-1} \mathrm{H}^{\mathrm{s}-\mathrm{j}}(\partial\mathcal{M};\mathcal{E}) = \mathrm{H}^{\mathrm{s}}(\partial\mathcal{M};\mathcal{E}).$$

We have $\chi^{+}(\mathrm{A})$ is boundary decomposing, i.e.,

$$\|u\|_{\check{\mathrm{H}}(\mathrm{D})} \simeq \|\chi^{-}(\mathrm{A})u\|_{\mathrm{H}^{\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})} + \|\chi^{+}(\mathrm{A})u\|_{\mathrm{H}^{\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})}.$$

$B := \chi^{-}(\mathrm{A})\mathrm{H}^{\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})$ - Atiyah-Patodi-Singer boundary condition for A.

I.e.,

$$\mathrm{D}_{\chi^{-}(\mathrm{A})\mathrm{H}^{\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})}$$

elliptically regular and hence Fredholm.

In particular $\dim\ker \mathrm{D}_{\chi^{-}(\mathrm{A})\mathrm{H}^{\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})} < \infty$.

What about “anti-APS” $B' := \chi^+(\mathrm{A})\mathrm{H}^{-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})$?

Have $(1-\mathcal{P}_{\mathcal{C}_\mathrm{D}})\mathrm{H}^{\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})$ and $\chi^-(\mathrm{A})\mathrm{H}^{\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})$ are elliptic boundary conditions.

By construction: $\dim\ker\left(\mathrm{D}_{\mathcal{P}_{\mathcal{C}_\mathrm{D}}\mathrm{H}^{-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})}\right) = \dim\ker(\mathrm{D}_{\max}) = \infty$.

Is $\dim\ker\left(\mathrm{D}_{\chi^+(\mathrm{A})\mathrm{H}^{-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})}\right) = \infty$?

- Principal symbol of $\mathcal{P}_\mathcal{C}$ is the same as principal symbol of $\chi^+(\mathrm{A})$.
- $\mathcal{P}_\mathcal{C} - \chi^+(\mathrm{A})$ is an operator of order -1.
- $\mathcal{P}_\mathcal{C} - (1-\chi^+(\mathrm{A})) = \mathcal{P}_\mathcal{C} - \chi^-(\mathrm{A})$ elliptic.

☣ **Warning: This does not imply $\mathcal{P}_\mathcal{C} - \chi^+(\mathrm{A})$ is compact!** ☣

Concrete counterexample

$\mathcal{M} = \mathbb{D} = \left\{x \in \mathbb{R}^2 : |x|_{\mathbb{R}^2} \leq 1\right\}$ unit disc. Boundary $\partial\mathcal{M} = \partial\mathbb{D} = \mathrm{S}^1$.

$\mathcal{E} = \mathcal{F} = \mathcal{M} \times \mathbb{C}^2$.

In polar coordinates (r, θ):

$$\mathrm{D}_0 := \begin{pmatrix} 0 & \partial_r + \frac{\imath}{r}\partial_\theta \\ -\partial_r + \frac{\imath}{r}\partial_\theta & 0 \end{pmatrix} = \sigma(\partial_r + \mathrm{A} + R_{00}),$$

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{A} = \begin{pmatrix} -\imath\partial_\theta & 0 \\ 0 & \imath\partial_\theta \end{pmatrix}.$$

For $\alpha \in \mathrm{C}^\infty_{\mathrm{c}}(0, 1]$, $\alpha(1) = 0$,

$$\mathrm{D}_\alpha := \sigma(\partial_r + \mathrm{A} + (R_{00} - \imath\alpha(r)\sigma\partial_\theta\mathrm{Id})) = \begin{pmatrix} \imath\alpha(r)\partial_\theta & \partial_r + \frac{\imath}{r}\partial_\theta \\ -\partial_r + \frac{\imath}{r}\partial_\theta & \imath\alpha(r)\partial_\theta \end{pmatrix}.$$

$$u \in \chi^+(\mathrm{A})\mathrm{H}^{-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E}) \cap \mathcal{C}_{\mathrm{D}} \iff \begin{cases} \chi^+(\mathrm{A})u = u \\ \mathcal{P}_{\mathcal{C}_{\mathrm{D}}}u = u \end{cases}$$

$$\iff \begin{cases} \chi^+(\mathrm{A})u = u \\ (\chi^+(\mathrm{A}) - \mathcal{P}_{\mathcal{C}_{\mathrm{D}}})u = 0 \end{cases}$$

$$\iff \begin{cases} \chi^+(\mathrm{A})u = u \\ \mathcal{L}u = 0 \end{cases}$$

$$\mathcal{L} := \chi^+(\mathrm{A}) - \mathcal{P}_{\mathcal{C}_{\mathrm{D}}} - \sigma\frac{\alpha'(1)}{4}(1+\Delta)^{-\frac{1}{2}}\chi^-(\mathrm{A}) \in \Psi\mathrm{DO}(-1).$$

Symbol:

$$\sigma_{-1}(\mathcal{L},\xi) = \frac{\alpha'(1)}{4}\begin{pmatrix} 0 & \frac{1}{|\xi|} \\ -\frac{1}{\xi} & 0 \end{pmatrix}.$$

Choose $\alpha \in \mathrm{C}^\infty_\mathrm{c}(0,1]$ such that $\alpha'(1) \neq 0 \implies \sigma_{-1}(\mathcal{L},\xi)$ invertible for $\xi \neq 0 \implies$ $\ker\mathcal{L} < \infty \iff \dim\left(\chi^+(\mathrm{A})\mathrm{H}^{-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E}) \cap \mathcal{C}_{\mathrm{D}}\right) < \infty.$