Geometric singularities and a flow tangent to the Ricci flow

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The flow for smooth metrics

Let $\mathcal{M}$ be a smooth compact manifold, and g a smooth metric. Let $\rho_t^{\mathrm{g}} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ be the heat kernel of the Laplacian Δ_{g}.

Fix $t > 0$, $x \in \mathcal{M}$ and $v \in \mathrm{T}_x\mathcal{M}$. Let $\varphi_{t,x,v}$ be a solution to:

$$\begin{aligned} &-\mathrm{div}_{\mathrm{g}}(\rho_t^{\mathrm{g}}(x,y)\nabla\varphi_{t,x,v})(y) = (\mathrm{d}_x\rho_t^{\mathrm{g}}(x,y))(v) \\ &\int_{\mathcal{M}} \varphi_{t,x,v}(y)\ d\mu_{\mathrm{g}}(y) = 0. \end{aligned} \tag{CE}$$

Define g_t, a metric evolving in time by:

$$\begin{aligned} \mathrm{g}_t(u,v)(x) &= \int_{\mathcal{M}} \mathrm{g}(\nabla\varphi_{t,x,u}(y), \nabla\varphi_{t,x,v}(y))\ \rho_t^{\mathrm{g}}(x,y)\ d\mu_{\mathrm{g}}(y) \\ &= \langle \rho_t^{\mathrm{g}}(x,\cdot\,)\nabla\varphi_{t,x,u}, \nabla\varphi_{t,x,v}\rangle_{\mathrm{L}^2(\mathcal{M},\mathrm{g})} \end{aligned} \tag{GM}$$

Connection to the Ricci flow

Let $\gamma : [0,1] \to \mathcal{M}$ be an g-geodesic. Then,

$$\partial_t \mathrm{g}_t(\dot\gamma(s), \dot\gamma(s))|_{t=0} = -2\mathrm{Ric}_{\mathrm{g}}(\dot\gamma(s), \dot\gamma(s)),$$

That is, the metrics $t \mapsto \mathrm{g}_t$ is *tangential* to the Ricci flow almost-everywhere along g-geodesics.

Note: this is not saying it is a linearisation of the Ricci flow.

Main redeeming feature: this can be defined as a flow of distance metrics d_t for metric spaces $(\mathcal{X}, \mathrm{d}, \mu)$ that satisfy the *Riemannian Curvature Dimension* (RCD) condition.

Wasserstein space

Let $(\mathcal{X}, \mathrm{d}, \mu)$ be a compact measure metric geodesic space. Denote set of probability measures by $\mathscr{P}(\mathcal{X})$.

For $\nu, \sigma \in \mathscr{P}(\mathcal{X})$, a *transport plan* between ν and σ is measure π on $\mathcal{X} \times \mathcal{X}$ such that

$$\pi(A \times \mathcal{X}) = \nu(A) \quad \text{and} \quad \pi(\mathcal{X} \times B) = \sigma(B).$$

Define:

$$W_2(\nu, \sigma)^2 = \inf \left\{ \int_{\mathcal{X} \times \mathcal{X}} \mathrm{d}(x, y)^2 \, d\pi : \pi \text{ transport map from } \nu \text{ to } \sigma \right\},$$

which is the *Wasserstein metric*.

The space $(\mathscr{P}(\mathcal{X}), W_2)$ is the *Wasserstein space* and it is a geodesic space.

Relative entropy and synthetic Ricci curvature

Let $\nu \in \mathscr{P}(\mathcal{X})$ as before. The relative entropy of ν with respect to μ is then given by

$$\operatorname{Ent}_\mu(\nu) = \begin{cases} \int_{\mathcal{X}} \rho \log \rho \, d\mu, & \nu \ll \mu, \quad d\nu = \rho \, d\mu, \\ +\infty, & \text{otherwise.} \end{cases}$$

Suppose that $\nu_0,\ \nu_1 \in \mathscr{P}(\mathcal{X})$ and let ν_t be the geodesic between ν_0 and ν_1.

Now, suppose that there exists $\kappa \in \mathbb{R}$ such that

$$\operatorname{Ent}_\mu(\nu_t) \leq (1-t)\operatorname{Ent}_\mu(\nu_0) + t\operatorname{Ent}_\mu(\nu_1) - \frac{\kappa}{2}(1-t)tW_2^2(\nu_0, \nu_1).$$

Then, we say that $(\mathcal{X}, \mathrm{d}, \mu)$ has Ricci curvature bounded below by κ, or is said to be CD(κ, ∞).

Cheeger Energy

For a Lipschitz function $\xi \in \mathrm{Lip}(\mathcal{X}, \mathrm{d})$, recall the *pointwise Lipschitz constant*:

$$\mathbf{Lip}\, \xi(x) = \limsup_{y \to x} \frac{|\xi(x) - \xi(y)|}{\mathrm{d}(x, y)},$$

for non-isolated points $x \in \mathcal{X}$.

For $f \in \mathrm{L}^2(\mathcal{X}, \mu)$, if $f_n \to f$ with $f_n \in \mathrm{Lip}(\mathcal{X}, \mathrm{d})$, define the *Cheeger energy*:

$$\mathrm{Ch}(f) = \inf_{\mathrm{Lip}(\mathcal{X}, \mathrm{d}) \ni f_n \to f} \lim_{n \to \infty} \frac{1}{2} \int_{\mathcal{X}} |\,\mathbf{Lip}\, f_n|^2 \, d\mu.$$

If no such such sequence exists, $\mathrm{Ch}(f) = +\infty$.

Infinitesimally Hilbertian

The first-order Sobolev space is defined as:

$$\mathrm{W}^{1,2}(\mathcal{X}) = \left\{ f \in \mathrm{L}^2(\mathcal{X}, \mu) : \mathrm{Ch}(f) < \infty \right\}.$$

It is a Banach space with respect to the norm

$$\|f\|_{\mathrm{W}^{1,2}}^2 = \|f\|_2^2 + 2\mathrm{Ch}(f).$$

If this norm polarises, i.e., $(\mathrm{W}^{1,2}(\mathcal{X}), \|\cdot\|_{\mathrm{W}^{1,2}})$ is a Hilbert space, then we say that $(\mathcal{X}, \mathrm{d}, \mu)$ is *infinitesimally Hilbertian*.

The space $(\mathcal{X}, \mathrm{d}, \mu)$ is RCD if it is $\mathrm{CD}(\kappa, \infty)$ and it is infinitesimally Hilbertian.

This is equivalent to the Laplacian associated to the energy Ch being linear.

Heat kernels and action for RCD spaces

For an RCD space $(\mathcal{X}, \mathrm{d}, \mu)$, the heat kernel ρ_t exists and it is *Lipschitz*.

There is an induced heat action on $(\mathscr{P}(\mathcal{X}), W_2)$, which is a map $\mathrm{H}_t : \mathscr{P}(\mathrm{spt}\ \mu) \to \mathscr{P}(\mathrm{spt}\ \mu)$ such that for all $\nu, \sigma \in \mathscr{P}(\mathcal{X})$ with $\mathrm{spt}\ \nu,\ \mathrm{spt}\ \sigma \subset \mathrm{spt}\ \mu$,

$$W_2(\mathrm{H}_t(\nu), \mathrm{H}_t(\sigma)) \leq \mathrm{e}^{-\kappa t}\ W_2(\nu, \sigma).$$

For $(\mathcal{X}, \mathrm{d}, \mu) = (\mathcal{M}, \mathrm{d}_{\mathrm{g}}, \mu_{\mathrm{g}}/\mu_{\mathrm{g}}(\mathcal{M}))$, if $s \mapsto \gamma_s$ is an absolutely continuous curve, then

$$\mathrm{H}_t(\delta_{\gamma_s}) = \rho_t^{\mathrm{g}}(\gamma_s, \cdot\,)\ d\mu_{\mathrm{g}}.$$

The flow for RCD spaces

Define: $\tilde{\mathrm{d}}_t(x,y) = W_2(\mathrm{H}_t(\delta_x), \mathrm{H}_t(\delta_y))$. The spaces $(\mathcal{X}, \tilde{\mathrm{d}}_t)$ are pseudo-metric spaces for each $t > 0$.

Noting that $s \to \gamma_s$ is d-Lipschitz implies that it is also $\tilde{\mathrm{d}}_t$ Lipschitz, define

$$\mathrm{d}_t(x,y) = \inf_{\gamma\ \mathrm{d}-\text{Lipschitz}} \int |\dot{\gamma}_s|_{\tilde{\mathrm{d}}_t}\ ds,$$

where

$$|\dot{\gamma}_s|_{\tilde{\mathrm{d}}_t} = \lim_{h \to 0} \frac{\tilde{\mathrm{d}}_t(\gamma_{s+h}, \gamma_s)}{h}.$$

The family of spaces $(\mathcal{X}, \mathrm{d}_t)$ are metric spaces for all $t > 0$, $\lim_{t \to 0} \mathrm{d}_t = \mathrm{d}$.

Theorem (Gigli-Mantegazza, [GM])

When $(\mathcal{X}, \mathrm{d}, \mu) = (\mathcal{M}, \mathrm{d}_{\mathrm{g}}, \mu_{\mathrm{g}}/\mu_{\mathrm{g}}(\mathcal{M}))$, *we have that* $\mathrm{d}_t = \mathrm{d}_{\mathrm{g}_t}$.

Main Theorem

Theorem (Theorem 1.1, [BLM])

Let $\mathcal{M}$ be a smooth, compact manifold with rough metric g that induces a distance metric d_{g}. Moreover, suppose there exists $K \in \mathbb{R}$ and $N > 0$ such that $(\mathcal{M}, \mathrm{d}_{\mathrm{g}}, \mu_{\mathrm{g}}) \in \mathrm{RCD}(K, N)$. If $\mathcal{S} \neq \mathcal{M}$ is a closed set and $\mathrm{g} \in \mathrm{C}^k(\mathcal{M} \setminus \mathcal{S})$, there exists a family of metrics $\mathrm{g}_t \in \mathrm{C}^{k-1,1}$ on $\mathcal{M} \setminus \mathcal{S}$ evolving according to (GM) on $\mathcal{M} \setminus \mathcal{S}$. For two points $x, y \in \mathcal{M}$ that are g_t-admissible, the distance $\mathrm{d}_t(x, y)$ given by the $\mathrm{RCD}(K, N)$ Gigli-Mantegazza flow is induced by g_t.

Note: $x, y \in \mathcal{M} \setminus \mathcal{S}$ are g_t-admissible if for any abs. cts. $\gamma : I \to \mathcal{M}$ connecting these points, there is another abs. cts. $\gamma' : I \to \mathcal{M}$ with d_t-length less than γ and for which $\gamma'(s) \in \mathcal{M} \setminus \mathcal{S}$

Rough metrics

Let $\mathrm{g} \in \mathbf{\Gamma}(\mathcal{T}^{(2,0)}\mathcal{M})$ be symmetric, with measurable coefficients. Suppose for each $x \in \mathcal{M}$, there exists some chart (ψ, U) containing x and a constant $C \geq 1$ (dependent on U), such that, for y-a.e. in U,

$$C^{-1}|u|_{\psi^*\delta(y)} \leq |u|_{\mathrm{g}(y)} \leq C|u|_{\psi^*\delta(y)},$$

where $u \in \mathrm{T}_y\mathcal{M}$, $|u|^2_{\mathrm{g}(y)} = \mathrm{g}(u,u)$ and $\psi^*\delta$ is the pullback of the Euclidean metric inside $\psi(U) \subset \mathbb{R}^n$. Then, g is called a *rough metric*.

- By the usual expression $d\mu_{\mathrm{g}} = \sqrt{\det \mathrm{g}_{ij}}\, d\mathscr{L}$ inside local comparable charts, obtain a Borel measure μ_{g}, finite on compact sets.
- A priori, there may not be an induced length structure.

- The L^p spaces exist, and differentiation on functions $\nabla = \mathrm{d}$ is densely-defined and closable.
- Sobolev space $\mathrm{W}^{1,2}(\mathcal{M}) = \mathcal{D}(\overline{\nabla})$ and the Laplacian is a self-adjoint operator $\Delta_{\mathrm{g}} = -\operatorname{div} \nabla$, where $\operatorname{div} = \nabla^*$.
- Two rough metrics g and $\tilde{\mathrm{g}}$ are C-close for some $C \geq 1$ if

$$C^{-1}|u|_{\tilde{\mathrm{g}}(y)} \leq |u|_{\mathrm{g}(y)} \leq C|u|_{\tilde{\mathrm{g}}(y)},$$

  for y-a.e. in $\mathcal{M}$.
- In this situation, $\Delta_{\mathrm{g}} = -\uptheta^{-1} \operatorname{div}_{\tilde{\mathrm{g}}} \uptheta \mathrm{B} \nabla$, where

$$\mathrm{g}(u,v) = \tilde{\mathrm{g}}(\mathrm{B}u, v) \quad \text{and} \quad \uptheta = \sqrt{\det \mathrm{B}}.$$

Main fact: for $\mathcal{M}$ compact, for every rough metric g, there exists a smooth metric $\tilde{\mathrm{g}}$ that is C-close to g.

We have that $\varphi_{t,x,v} \in \mathrm{W}^{1,2}(\mathcal{M})$ solves:

$$-\operatorname{div}_{\mathrm{g}}(\rho^{\mathrm{g}}_t(x,y)\nabla\varphi_{t,x,v})(y) = (\mathrm{d}_x\rho^{\mathrm{g}}_t(x,y))(v)$$
$$\int_{\mathcal{M}} \varphi_{t,x,v}(y)\ d\mu_{\mathrm{g}}(y) = 0.$$

if and only if

$$-\operatorname{div}_{\tilde{\mathrm{g}}}(\mathrm{B}(y)\theta(y)\rho^{\mathrm{g}}_t(x,y)\nabla\varphi_{t,x,v})(y) = \theta(y)(\mathrm{d}_x\rho^{\mathrm{g}}_t(x,y))(v)$$
$$\int_{\mathcal{M}} \varphi_{t,x,v}(y)\ d\mu_{\mathrm{g}}(y) = 0.$$

So, it suffices to study divergence form operators with L^∞ coefficients for smooth metrics $\tilde{\mathrm{g}}$.

L^∞-coefficient divergence form operators

Fix $\mathcal{M}$ smooth compact manifold and $\tilde{\mathrm{g}}$ a smooth Riemannian metric. Let $A \in \mathbf{\Gamma}(\mathrm{L}^\infty(\mathcal{T}^{(1,1)}\mathcal{M}))$ real-symmetric and elliptic:

(i) there exist $\kappa > 0$ such that for x a.e. $\tilde{\mathrm{g}}_x(A(x)u, u) \geq \kappa |u|_x^2$

(ii) there exists a $\Lambda < \infty$ such that $\mathrm{esssup}_{x \in \mathcal{M}} |A(x)| < \Lambda$.

- Associated energy: $J_A[u, v] = \langle A \nabla u, \nabla v \rangle$ for $\mathcal{D}(J_A) = \mathrm{W}^{1,2}(\mathcal{M})$.
- Ellipticity gives: $\kappa \|\nabla u\|^2 \leq J_A[u, u] \leq \Lambda \|\nabla u\|^2$.
- Lax-Milgram theorem yields $\mathrm{L}_A = -\operatorname{div} A \nabla$ with domain

$$\mathcal{D}(\mathrm{L}_A) = \left\{ u \in \mathrm{W}^{1,2}(\mathcal{M}) : v \mapsto J_A[u, v] \text{continuous} \right\}$$

as a non-negative self-adjoint operator. Moreover, $\mathcal{D}(\sqrt{\mathrm{L}_A}) = \mathrm{W}^{1,2}(\mathcal{M})$.

- $\mathrm{L}^2(\mathcal{M}) = \mathcal{N}(\mathrm{L}_A) \oplus^{\perp} \overline{\mathcal{R}(\mathrm{L}_A)}$,
- $\mathcal{N}(\mathrm{L}_A) = \mathcal{N}(\nabla)$ and crucially,

$$\overline{\mathcal{R}(\mathrm{L}_A)} = \mathcal{R} := \left\{ u \in \mathrm{L}^2(\mathcal{M}) : \int u = 0 \right\},$$

- The operator $\mathrm{L}_A^{\mathcal{R}} = \mathrm{L}_A$ with $\mathcal{D}(\mathrm{L}_A^{\mathcal{R}}) = \mathcal{D}(\mathrm{L}_A) \cap \mathcal{R}$ is an unbounded operator $\mathrm{L}_A^{\mathcal{R}} : \mathcal{R} \to \mathcal{R}$.
- $\sigma(\mathrm{L}_A) = \{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots\}$, and
- $\sigma(\mathrm{L}_A^{\mathcal{R}}) = \{0 < \lambda_1 \leq \lambda_2 \leq \dots\}$.

Proposition

Let $f \in \mathrm{L}^2(\mathcal{M})$ with $\int f \, d\mu_{\tilde{g}} = 0$. Then, there exists a unique $u \in \mathrm{W}^{1,2}(\mathcal{M})$ with $\int u \, d\mu_{\tilde{g}} = 0$ such that $\mathrm{L}_A u = f$. Explicitly, $u = (\mathrm{L}_A^{\mathcal{R}})^{-1} f$.

Back to continuity equations on rough metrics

Let g be a rough metric, $\mathrm{g}(u,v) = \tilde{\mathrm{g}}(\mathrm{B}u, v)$, and let $(x,y) \mapsto \omega_x(y) \in \mathrm{C}^{0,1}(\mathcal{M}^2)$, $\omega_x > 0$. Suppose there exists $\varnothing \neq \mathcal{N} \subset \mathcal{M}$ open set on which $x \mapsto \omega_x(\cdot\,) \in \mathrm{C}^k(\mathcal{N})$, for $k \geq 1$. Let

$$\mathrm{D}_x = -\operatorname{div}_{\mathrm{g}} \omega_x \nabla = -\theta^{-1} \operatorname{div}_{\tilde{\mathrm{g}}} \mathrm{B}\theta\omega_x \nabla.$$

The continuity equation is then

$$\mathrm{D}_x \varphi_x = \eta_x. \tag{F}$$

By previous proposition,

Proposition

Let $\eta_x \in \mathrm{L}^2(\mathcal{M})$ with $\int \eta_x \, d\mu_{\mathrm{g}} = 0$. Then there exists a unique $\varphi_x \in \mathrm{W}^{1,2}(\mathcal{M})$ with $\int \varphi_x \, d\mu_{\mathrm{g}} = 0$ solving (F).

Regularity

To understand regularity, we need to understand the behaviour of the operators $x \mapsto \mathrm{D}_x$. Two crucial facts:

- $\mathcal{D}(\mathrm{D}_x) = \mathcal{D}(\Delta_{\mathrm{g}})$ and $\mathrm{D}_x u = \omega_x \Delta_{\mathrm{g}} u - \mathrm{g}(\nabla u, \nabla \omega_x)$,
- $\mathcal{M} \ni x \mapsto \mathrm{D}_x : (\mathcal{D}(\Delta_{\mathrm{g}}), \|\cdot\|_{\Delta_{\mathrm{g}}}) \to \mathrm{L}^2(\mathcal{M})$ is a uniformly bounded family of operators and $\|u\|_{\mathrm{D}_x} \simeq \|u\|_{\Delta_{\mathrm{g}}}$ holds with the implicit constant independent of $x \in \mathcal{M}$.

Let $v \in \mathrm{T}_x\mathcal{M}$ and $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ such that $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. Let $f : \mathcal{N} \to \mathcal{V}$, where $\mathcal{V}$ where $\mathcal{V}$ is some normed vector space.

- Difference quotient: $Q_s^v f(x) = \frac{f(x) - f(\gamma(s))}{s}$.
- *Directional derivative* of f (when it exists and it is independent of the generating curve γ): $(\mathrm{d}_x f(x))(v) = \lim_{s \to 0} Q_s^v f(x)$.

For us, $\mathcal{V} = \mathrm{L}^2(\mathcal{M})$ with the weak topology for the choice $f(x) = \mathrm{D}_x$. More precisely, if there exists $\tilde{\mathrm{D}}_x : \mathcal{D}(\Delta_{\mathrm{g}}) \to \mathrm{L}^2(\mathcal{M})$ satisfying $\lim_{s\to 0}\langle Q_s^v \mathrm{D}_x u, w\rangle = \langle \tilde{\mathrm{D}}_x u, w\rangle$, for every $w \in \mathrm{W}^{1,2}(\mathcal{M})$, say that D_x has a *(weak) derivative* at x and write $(\mathrm{d}_x \mathrm{D}_x) = \tilde{\mathrm{D}}_x$.

Proposition

Let $x \mapsto u_x : \mathcal{N} \to \mathcal{D}(\Delta_{\mathrm{g}})$, $v \in \mathrm{T}_x\mathcal{M}$ *and suppose that* $(\mathrm{d}_x u_x)(v)$ *exists weakly. Then* $(\mathrm{d}_x \mathrm{D}_x u_x)(v)$ *exists weakly if and only if* $\mathrm{D}_x((\mathrm{d}_x u_x)(v))$ *exists weakly and*

$$(\mathrm{d}_x \mathrm{D}_x u_x)(v) = (\mathrm{d}_x \mathrm{D}_x)(v) u_x + \mathrm{D}_x((\mathrm{d}_x u_x)(v)).$$

Regularity of solutions

Theorem

Suppose that $k \geq 1$ *and* $(x, y) \mapsto \omega_x(y) \in \mathrm{C}^{0,1}(\mathcal{M}^2)$ *and* $x \mapsto \omega_x \in \mathrm{C}^k(\mathcal{N})$*. Moreover, suppose that* $(x, y) \mapsto \eta_x(y) \in \mathrm{C}^0(\mathcal{N} \times \mathcal{M})$ *and* $x \mapsto \eta_x(y) \in \mathrm{C}^l(\mathcal{N})$ *where* $l \geq 1$*. If at* $x \in \mathcal{N}$*,* φ_x *solves* (F) *with* $\int_{\mathcal{M}} \varphi_x \, d\mu_{\mathrm{g}} = \int_{\mathcal{M}} \eta_x \, d\mu_{\mathrm{g}} = 0$*, the map* $x \mapsto \langle \eta_x, \varphi_x \rangle \in \mathrm{C}^{\min\{k,l\}-1,1}(\mathcal{N})$*.*

Back to the flow

Set $\omega_x(y) = \rho_t^{\mathrm{g}}(x,y)$, $\eta_x = \mathrm{d}_x(\rho_t^{\mathrm{g}}(x,y))(v)$ for $x \in \mathcal{N}$.

(i) The heat kernel ρ_t^{g} is Lipschitz for each $t > 0$ because we assume that $(\mathcal{M}, \mathrm{g}, \mu_{\mathrm{g}})$ induces an RCD space.

(ii) Backward uniqueness of the heat flow (via semigroup argument to avoid maximum principles) gives us that $\mathrm{d}_x(\rho_t^{\mathrm{g}}(x,y))(v) \neq 0$ if $v \neq 0$.

(iii) For each $t > 0$, there exists $\kappa_t > 0$ and $\Lambda_t < \infty$ such that $\kappa_t \leq \rho_t^{\mathrm{g}}(x,y) \leq \Lambda_t$.

This gives that: g_t is non-degenerate, symmetric, linear.

Regularity $x \mapsto \mathrm{g}_t(x)$: previous theorem.

General rough metric spaces

In the situation that g does *not* necessarily induce an RCD structure, $(\mathcal{M}, \mathrm{g})$ is only guaranteed to be a measure space.

- Considering $\Delta_{\mathrm{g}} = -\theta^{-1} \operatorname{div}_{\tilde{\mathrm{g}}} \mathrm{B}\theta\nabla$ on a smooth $\tilde{\mathrm{g}}$,
- Parabolic Harnack estimates exist for such operators as proved in [SC],
- Beurling-Deny condition for Δ_{g}: $f \in \mathcal{D}(\sqrt{\Delta_{\mathrm{g}}})$ $(= \mathrm{W}^{1,2}(\mathcal{M}))$ implies $|f| \in \mathcal{D}(\sqrt{\Delta_{\mathrm{g}}})$ and $\|\sqrt{\Delta_{\mathrm{g}}}|f|\| \lesssim \|\sqrt{\Delta_{\mathrm{g}}}f\|$, so that $\mathrm{e}^{-t\Delta_{\mathrm{g}}}$ is positive-preserving.
- Obtain a heat kernel $(x, y) \mapsto \rho_t^{\mathrm{g}}(x, y) \in \mathrm{C}^{\alpha}(\mathcal{M}^2)$ for some $\alpha > 0$.

This means we can still make sense of the equation (CE) and define (GM) on a non-singular region.

Best expected regularity is only continuity.

Theorem for non-RCD rough spaces

Theorem (Theorem 3.4 [BCon])

Let $\mathcal{M}$ be a smooth, compact manifold, and $\varnothing \neq \mathcal{N} \subset \mathcal{M}$, an open set. Suppose that $\tilde{\mathrm{g}}$ is a rough metric and that $\rho_t^{\tilde{\mathrm{g}}} \in \mathrm{C}^1(\mathcal{N}^2)$. Then, g_t as defined by (GM) *exists on $\mathcal{N}$ and it is continuous.*

Suffices to know that $\|\sqrt{\mathrm{D}_x}u - \sqrt{\mathrm{D}_y}u\|$ is small whenever the coefficients ω_x are close in L^∞.

This amounts to proving a homogeneous Kato square root estimate.

Homogeneous Kato square root problem

Let $B \in \mathbf{\Gamma}(\mathrm{L}^\infty(\mathcal{T}^{(1,0)}\mathcal{M}))$, possibly non-symmetric, and complex, and $b \in \mathrm{L}^\infty(\mathcal{M})$. Let

$$\Pi_B = \begin{pmatrix} 0 & -b \operatorname{div} B \\ \nabla & 0 \end{pmatrix}.$$

Theorem (Theorem 4.3 [BCon])

The operator Π_B admits a bounded functional calculus. In particular, $\mathcal{D}(\sqrt{-b \operatorname{div} B \nabla}) = \mathrm{W}^{1,2}(\mathcal{M})$ and $\|\sqrt{-b \operatorname{div} B \nabla} u\| \simeq \|\nabla u\|$. Moreover, whenever $\|\tilde{b}\|_\infty < \eta_1$ and $\|\tilde{B}\|_\infty < \eta_2$, where $\eta_i < \kappa_i$, we have the following Lipschitz estimate

$$\|\sqrt{-b \operatorname{div} B \nabla} u - \sqrt{-(b + \tilde{b}) \operatorname{div}(B + \tilde{B}) \nabla} u\| \lesssim (\|\tilde{b}\|_\infty + \|\tilde{B}\|_\infty) \|\nabla u\|$$

whenever $u \in \mathrm{W}^{1,2}(\mathcal{M})$. The implicit constant depends on b, B and η_i.

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