First-order elliptic boundary value problems beyond self-adjoint adapted boundary operators

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- ⋆ Fredholmness and index theorems?

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STOP Rarita-Schwinger $\not{D}_{\mathrm{RS}}$ does not give rise to a symmetric A_{RS}.

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$T > 0$ determined by (A1)-(A6).

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Admissible cut $r \in \mathbb{R}$: the line $l_r := \{\zeta \in \mathbb{C} : \mathrm{Re}\ \zeta = r\}$ is *not* in the spectrum of A.

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Norm: $\quad \|u\|^2_{\check{H}(A)} := \|\chi^-(A_r)u\|^2_{\mathrm{H}^{\frac{1}{2}}} + \|\chi^+(A_r)u\|^2_{\mathrm{H}^{-\frac{1}{2}}}$.

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(v) Higher regularity:

$$\mathcal{D}(D_{\max}) \cap \mathrm{H}^{\mathrm{k}+1}_{\mathrm{loc}}(E)$$
$$= \{ u \in \mathcal{D}(D_{\max}) : Du \in \mathrm{H}^{\mathrm{k}}_{\mathrm{loc}}(F) \text{ and } \chi^+(A_r)(u|_\Sigma) \in \mathrm{H}^{\mathrm{k}+\frac{1}{2}}(E_\Sigma) \}.$$

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- Boundary condition $B \subset \mathrm{H}^{\frac{1}{2}}(E_\Sigma)$ if and only if $D_B = D_{B,\max}$.
- Adjoint boundary condition B^{ad} so that $D_B^{\mathrm{ad}} = D_{B^{\mathrm{ad}}}$:

$$B^{\mathrm{ad}} := \left\{v \in \check{H}(-\tilde{A}) : \langle \sigma_0 u, v\rangle_{\mathrm{L}^2(F_\Sigma)} = 0 \quad \forall u \in B\right\}.$$

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(iii) $g : V_- \to V_+$ bounded linear map with $g(V_-^{\frac{1}{2}}) \subset V_+^{\frac{1}{2}}$ and $g^*((V_+^*)^{\frac{1}{2}}) \subset (V_-^*)^{\frac{1}{2}}$

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(i) $W_\pm$, $V_\pm$ are mutually complementary subspaces such that

$$V_\pm \oplus W_\pm = \chi^\pm(A_r)\mathrm{L}^2(E_\Sigma),$$

(ii) $W_\pm$ are finite dimensional with $W_\pm, W_\pm^* \subset \mathrm{H}^{\frac{1}{2}}(E_\Sigma)$, and

(iii) $g : V_- \to V_+$ bounded linear map with $g(V_-^{\frac{1}{2}}) \subset V_+^{\frac{1}{2}}$ and $g^*((V_+^*)^{\frac{1}{2}}) \subset (V_-^*)^{\frac{1}{2}}$ such that

$$B = W_+ \oplus \left\{ v + gv : v \in V_-^{\frac{1}{2}} \right\}.$$

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$$\sigma_0^*(B^{\mathrm{ad}}) = W_-^* \oplus \left\{ u - g^* u : u \in (V_+^*)^{\frac{1}{2}} \right\}.$$

Pseudo-local and local boundary conditions

- Classical pseudo-differential projector P of order zero (not necessarily orthogonal), the space

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- Boundary condition $B \subset \mathrm{H}^{\frac{1}{2}}(E_\Sigma)$ a *local boundary condition* if there exists a sub-bundle $E' \subset E_\Sigma$ such that

$$B = \mathrm{H}^{\frac{1}{2}}(E').$$

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In particular, if $D_B u$ is smooth, then u is smooth up to the boundary.

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Lemma (Lemma 4.1 in [BB12])

On the cylinder $Z_{[0,T)}$,

$$D = \sigma_t(\partial_t + B + R_t),$$
$$D^* = -\sigma_t^*(\partial_t + \tilde{B} + \tilde{R}_t),$$

for any pair of adapted boundary operators B *and* $\tilde{B}$ *to* D *and* D^**. Remainder terms* R_t *and* $\tilde{R}_t$ *are* Ψ*DO's of order at most one, their coefficients depend smoothly on* t*, and*

$$\|R_t u\|_{L^2(\Sigma)} \lesssim t\|Bu\|_{L^2(\Sigma)} + \|u\|_{L^2(\Sigma)}, \text{ and}$$
$$\|\tilde{R}_t v\|_{L^2(\Sigma)} \lesssim t\|\tilde{B}v\|_{L^2(\Sigma)} + \|v\|_{L^2(\Sigma)}.$$

for $u \in C^\infty(E_\Sigma)$ *and* $v \in C^\infty(F_\Sigma)$.

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$$\|u\|^2_{\mathrm{H}^1_{\mathrm{D}}} := \|\eta u\|^2_{\mathrm{H}^1} + \|Du\|^2 + \|u\|,$$

where η is a compactly supported cutoff near the boundary.

Lemma

For $\theta \in (\omega_r, \pi/2)$ *fixed, there exists an inner product* $\langle \cdot, \cdot \rangle_{N,\theta}$ *such that* $|A_r|$ *is m-*θ*-accretive and for which the estimate*

$$\|(\partial_t + \mathrm{A})u\|^2_{\mathrm{L}^2(Z_{[0,\infty)})} \simeq \|u'\|^2_{\mathrm{L}^2(Z_{[0,\infty)})} + \|Au\|^2_{\mathrm{L}^2(Z_{[0,\infty)})} - \mathrm{Re}\,\langle |A_r|\,\mathrm{sgn}(A_r)\ u_0, u_0\rangle_{N,\theta} - r\|u_0\|^2_{N,\theta},$$

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Higher regularity

Banach-valued Cauchy problem: $f \in \mathrm{L}^2(Z_{[0,\rho]}, E)$,

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Define:

$$S_{0,r} u(t) = \int_0^t e^{-(t-s)|A_r|} \sigma_0^{-1} \chi^+(A_r) u(s)\, ds - \int_t^\rho e^{-(s-t)|A_r|} \sigma_0^{-1} \chi^-(A_r) u(s)\, ds$$

Let $(C_\rho u)(s) = u(\rho - s)$,

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(v) $D_{0,r} : \mathrm{H}^{\mathrm{k}+1}(Z_{[0,\rho]}, E; B_0) \to \mathrm{H}^{\mathrm{k}}(Z_{[0,\rho]}, E)$ isomorphism, where $B_0 = \chi^-(A_r) \mathrm{H}^{\frac{1}{2}}(E_\Sigma) \oplus \chi^+(A_r) \mathrm{H}^{\frac{1}{2}}(E_\Sigma)$

(vi) $(I - S_{0,r} D_{0,r}) u = e^{-t|A_r|}(\chi^+(A_r) u(0))$ whenever $\chi^-(A_r)(u(\rho)) = 0$,

Let $(C_\rho u)(s) = u(\rho - s)$,

(i) $S_{0,r} u(t) = W(t; \sigma_0^{-1} \chi^+(A_r) u) - W(\rho - t; \sigma_0^{-1} \chi^-(A_r) C_\rho u)$

(ii) $\chi^+(A_r)(S_{0,r} u)(0) = \chi^-(A_r)(S_{0,r} u)(\rho) = 0$.

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Key estimate:

$$\int_0^\rho \|\partial_t W(t; f)\|^2_{\mathrm{L}^2(E_\Sigma)} dt + \int_0^\rho \||A_r| W(t; f)\|^2_{\mathrm{L}^2(E_\Sigma)} \lesssim \int_0^\rho \|f(t)\|^2_{\mathrm{L}^2(E_\Sigma)}.$$

Elliptic boundary conditions

Prove: $B \subset \mathrm{H}^{\frac{1}{2}}(E_\Sigma)$ and $B^{\mathrm{ad}} \subset \mathrm{H}^{\frac{1}{2}}(F_\Sigma)$ implies B elliptic.

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Key estimate:

$$\|u\|_{\mathrm{H}^{\frac{1}{2}}} \simeq \|u\|_{\check{H}(A)} \lesssim \|\chi^-(A_r)u\|_{\mathrm{H}^{\frac{1}{2}}} + \|\chi^+(A_r)u\|_{\mathrm{H}^{-\frac{1}{2}}}$$

for all $u \in B$.

Spaces:

$$
\begin{aligned}
W_-^* &:= \chi^-(A_r^*)\mathrm{L}^2(E_\Sigma) \cap \sigma_0^* B^{\mathrm{ad}} & \quad W_- &:= \chi^-(A_r) W_-^* \\
W_+ &:= \chi^+(A_r)\mathrm{L}^2(E_\Sigma) \cap B & \quad W_+^* &:= \chi^+(A_r^*) W_+ \\
V_-^* &:= \chi^-(A_r^*)\mathrm{L}^2(E_\Sigma) \cap (W_-^*)^\perp & \quad V_- &:= \chi^-(A_r) V_-^* \\
V_+ &:= \chi^+(A_r)\mathrm{L}^2(E_\Sigma) \cap W_+^\perp & \quad V_+^* &:= \chi^+(A_r^*) V_+.
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V_+ &:= \chi^+(A_r)\mathrm{L}^2(E_\Sigma) \cap W_+^\perp & V_+^* &:= \chi^+(A_r^*) V_+.
\end{aligned}
$$

Splitting:

$$\mathrm{L}^2(E_\Sigma) = V_- \oplus W_- \oplus V_+ \oplus W_+ = V_-^* \oplus W_-^* \oplus V_+^* \oplus W_+^*.$$

$$X_- = \chi^-(A_r)|_{B \cap W_+^\perp} : B \cap W_+^\perp \to \chi^-(A_r)B, \text{ and}$$

$$X_+^* = \chi^+(A_r^*)|_{\sigma_0^* B^{\mathrm{ad}} \cap (W_-^*)^\perp} : \sigma_0^* B^{\mathrm{ad}} \cap (W_-^*)^\perp \to \chi^+(A_r^*)\sigma_0^* B^{\mathrm{ad}}.$$

are isomorphisms with their ranges.

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$$g_0 = P_{V_+}(X_-)^{-1} \quad \text{and} \quad h_0 = P_{V_-^*}(X_+^*)^{-1}.$$

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$$g_0 = P_{V_+}(X_-)^{-1} \quad \text{and} \quad h_0 = P_{V_-^*}(X_+^*)^{-1}.$$

Obtain:

$$B = W_+ \oplus \left\{ v \in V_-^{\frac{1}{2}} : v + g_0 v \right\}$$
$$B^{\mathrm{ad}} = W_-^* \oplus \left\{ u \in (V_+^*)^{\frac{1}{2}} : u + h_0 u \right\}.$$

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