# First-order elliptic boundary value problems beyond self-adjoint adapted boundary operators

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# Motivation

$M$ Riemannian Spin manifold, smooth compact boundary $\Sigma$,

$$\mathcal{W} := \mathrm{T}M \otimes \not{\Delta}\, M \cong \not{\Delta}\, M \overset{\perp}{\oplus} \not{\Delta}_{\frac{3}{2}}\, M.$$

Induced Dirac operator

$$\not{D}_{\mathcal{W}} = \begin{pmatrix} \not{D} & \mathsf{T}^* \\ \mathsf{T} & \not{D}_{\mathrm{RS}} \end{pmatrix}.$$

Natural questions:

- ⋆ Boundary conditions? Ellipticity? Self-adjointness? Regularity of solutions?
- ⋆ Fredholmness and index theorems?

# Adapted operator

Boundary adapted operator $\mathrm{A}_{\mathrm{RS}}$ to $\not{D}_{\mathrm{RS}}$ on $\not{\Delta}_{\frac{3}{2}} \Sigma := \not{\Delta}_{\frac{3}{2}} M|_{\Sigma}$.

Principal symbol:

$$\sigma_{\mathrm{A}_{\mathrm{RS}}}(x, \xi) = \sigma_{\not{D}_{\mathrm{RS}}}(x, \tau(x))^{-1} \circ \sigma_{\not{D}_{\mathrm{RS}}}(x, \xi).$$

Fundamental assumption in the Bär-Ballmann framework [BB12]:

$\mathrm{A}_{\mathrm{RS}}$ is a symmetric operator.

Boundary value problems for $\not{D}_{\mathrm{RS}}$ "live" in:

$$\check{H}(\mathrm{A}_{\mathrm{RS}}) := \chi_{(-\infty,0]}(\mathrm{A}_{\mathrm{RS}})\mathrm{H}^{\frac{1}{2}}(\not{\Delta}_{\frac{3}{2}} \Sigma) \quad \bigoplus \quad \chi_{(0,\infty)}(\mathrm{A}_{\mathrm{RS}})\mathrm{H}^{-\frac{1}{2}}(\not{\Delta}_{\frac{3}{2}} \Sigma),$$

STOP Rarita-Schwinger $\not{D}_{\mathrm{RS}}$ does not give rise to a symmetric $\mathrm{A}_{\mathrm{RS}}$.

# General setup

(A1) $M$ is a manifold with compact boundary $\Sigma = \partial M$;

(A2) $\tau$ is an interior co-vectorfield along $\partial M$;

(A3) $\mu$ is a smooth volume measure on $M$ and $\nu$ is the induced smooth volume measure on $\Sigma$;

(A4) $(\mathcal{E}, \mathrm{h}^{\mathcal{E}}),\ (\mathcal{F}, \mathrm{h}^{\mathcal{F}}) \to M$ are Hermitian vector bundles over $M$;

(A5) $D : \mathrm{C}^\infty(\mathcal{E}) \to \mathrm{C}^\infty(\mathcal{F})$ is a first-order elliptic differential operator;

(A6) $D$ and $D^*$ are complete (i.e., $\mathrm{C}^\infty_{\mathrm{c}}(E)$ dense in $\mathcal{D}(D_{\max})$).

Consequence: reduce to cylinder $Z_{[0,T)} : [0,T) \times \Sigma$.

$T > 0$ determined by (A1)-(A6).

# Adapted boundary operator

$A$ adapted boundary operator to $D$ if:

$$\sigma_A(x,\xi) = \sigma_D(x,\tau(x))^{-1} \circ \sigma_D(x,\xi).$$

- Exists and are elliptic differential operators of order 1.
- Unique up to an operator of order zero.
- Discrete spectrum, generally non-orthogonal eigenspaces.

Admissible cut $r \in \mathbb{R}$: the line $l_r := \{\zeta \in \mathbb{C} : \mathrm{Re}\ \zeta = r\}$ is *not* in the spectrum of $A$.

Ellipticity $\mathrm{spec}(\sigma_{\mathrm{A}}(x, \xi)) \cap \imath\mathbb{R} = \varnothing$, $\forall (x, \xi) \in \Sigma \times \mathrm{T}^*\Sigma$.

Theorem of Shubin in [Shu01]: there exists $\omega_r \in [0, \pi/2)$ such that $A_r := A - r$ is $\omega_r$ bi-sectorial.

Theorem of Grubb in [Gru12] (c.f. also Seeley in [See67]): spectral projectors $\chi^{\pm}(\mathrm{A}_r)$ are $\Psi$DOs of order zero.

Space: $\check{H}(A) := \chi^{-}(A_r)\mathrm{H}^{\frac{1}{2}}(E_\Sigma) \oplus \chi^{+}(A_r)\mathrm{H}^{-\frac{1}{2}}(E_\Sigma)$.

Norm: $\|u\|^2_{\check{H}(A)} := \|\chi^{-}(A_r)u\|^2_{\mathrm{H}^{\frac{1}{2}}} + \|\chi^{+}(A_r)u\|^2_{\mathrm{H}^{-\frac{1}{2}}}$.

# Theorem 1: Maximal domains and $\check{H}(A)$, $\check{H}(\tilde{A})$ spaces

(i) $\mathrm{C}^\infty_{\mathrm{c}}(E)$ is dense in $\mathcal{D}(D_{\max})$ and $\mathcal{D}((D^*)_{\max})$ with respect to corresponding graph norms.

(ii) The restriction maps $\mathrm{C}^\infty_{\mathrm{c}}(E) \to \mathrm{C}^\infty_{\mathrm{c}}(E_\Sigma)$ and $\mathrm{C}^\infty_{\mathrm{c}}(F) \to \mathrm{C}^\infty_{\mathrm{c}}(F_\Sigma)$ given by $u \mapsto u|_\Sigma$ extend uniquely to surjective bounded linear maps $\mathcal{D}(D_{\max}) \to \check{H}(A)$ and $\mathcal{D}((D^*)_{\max}) \to \check{H}(\tilde{A})$.

(iii) The spaces

$$\mathcal{D}(D_{\max}) \cap \mathrm{H}^1_{\mathrm{loc}}(E_\Sigma) = \left\{ u \in \mathcal{D}(D_{\max}) : u|_\Sigma \in \mathrm{H}^{\frac{1}{2}}(E_\Sigma) \right\}$$

$$\mathcal{D}((D^*)_{\max}) \cap \mathrm{H}^1_{\mathrm{loc}}(F_\Sigma) = \left\{ u \in \mathcal{D}((D^*)_{\max}) : u|_\Sigma \in \mathrm{H}^{\frac{1}{2}}(F_\Sigma) \right\}.$$

(iv) For all $u \in \mathcal{D}(D_{\max})$ and $v \in \mathcal{D}((D^*)_{\max})$,

$$\langle D_{\max} u, v \rangle_{\mathrm{L}^2(F)} - \langle u, (D^*)_{\max} v \rangle_{\mathrm{L}^2(E)} = - \left\langle \sigma_0 u|_\Sigma, v|_\Sigma \right\rangle_{\mathrm{L}^2(F_\Sigma)}.$$

(v) Higher regularity:

$$\mathcal{D}(D_{\max}) \cap \mathrm{H}^{\mathrm{k}+1}_{\mathrm{loc}}(E)$$
$$= \{ u \in \mathcal{D}(D_{\max}) : Du \in \mathrm{H}^{\mathrm{k}}_{\mathrm{loc}}(F) \text{ and } \chi^+(A_r)(u|_\Sigma) \in \mathrm{H}^{\mathrm{k}+\frac{1}{2}}(E_\Sigma) \}.$$

# Boundary conditions

A *closed* linear subspace $B \subset \check{H}(A)$ is called a *boundary condition* for $D$. Associated operator domains:

$$\mathcal{D}(D_{B,\max}) = \left\{u \in \mathcal{D}(D_{\max}) : u|_{\Sigma} \in B\right\}$$
$$\mathcal{D}(D_B) = \left\{u \in \mathcal{D}(D_{\max}) \cap \mathrm{H}^1_{\mathrm{loc}}(E_\Sigma) : u|_{\Sigma} \in B\right\},$$

and similarly for the formal adjoint $D^*$ with $A$ replaced by $\tilde{A}$.

- For boundary condition $B$, the operator $D_B$ closed and between $D_{cc}$ (on $\mathrm{C}^\infty_{\mathrm{cc}}(E)$) and $D_{\max}$.
- $D_c$ closed extension of $D_{cc}$, then $B := \left\{u|_{\Sigma} : u \in \mathcal{D}(D_c)\right\}$ is a boundary condition and $D_c = D_{B,\max}$.
- Boundary condition $B \subset \mathrm{H}^{\frac{1}{2}}(E_\Sigma)$ if and only if $D_B = D_{B,\max}$.
- Adjoint boundary condition $B^{\mathrm{ad}}$ so that $D_B^{\mathrm{ad}} = D_{B^{\mathrm{ad}}}$:

$$B^{\mathrm{ad}} := \left\{v \in \check{H}(-\tilde{A}) : \langle \sigma_0 u, v\rangle_{\mathrm{L}^2(F_\Sigma)} = 0 \quad \forall u \in B\right\}.$$

# Elliptic boundary conditions

$B \subset \mathrm{H}^{\frac{1}{2}}(E_\Sigma)$ boundary condition is called *elliptic* if there exists an admissible cut $r \in \mathbb{R}$ and:

(i) $W_\pm$, $V_\pm$ are mutually complementary subspaces such that

$$V_\pm \oplus W_\pm = \chi^\pm(A_r)\mathrm{L}^2(E_\Sigma),$$

(ii) $W_\pm$ are finite dimensional with $W_\pm, W_\pm^* \subset \mathrm{H}^{\frac{1}{2}}(E_\Sigma)$, and

(iii) $g : V_- \to V_+$ bounded linear map with $g(V_-^{\frac{1}{2}}) \subset V_+^{\frac{1}{2}}$ and $g^*((V_+^*)^{\frac{1}{2}}) \subset (V_-^*)^{\frac{1}{2}}$ such that

$$B = W_+ \oplus \left\{ v + gv : v \in V_-^{\frac{1}{2}} \right\}.$$

# Theorem 2: Characterisation of elliptic boundary conditions

$B \subset \mathrm{H}^{\frac{1}{2}}(E_\Sigma)$ be a subspace, then the following are equivalent:

(i) $B$ a boundary condition and $B^{\mathrm{ad}} \subset \mathrm{H}^{\frac{1}{2}}(F_\Sigma)$,

(ii) the definition is satisfied for any admissible spectral cut $r \in \mathbb{R}$,

(iii) $B$ an elliptic boundary condition.

Elliptic boundary condition $B$ for $D \iff B^{\mathrm{ad}}$ elliptic boundary condition for $D^*$ and

$$\sigma_0^*(B^{\mathrm{ad}}) = W_-^* \oplus \left\{ u - g^* u : u \in (V_+^*)^{\frac{1}{2}} \right\}.$$

# Pseudo-local and local boundary conditions

- Classical pseudo-differential projector $P$ of order zero (not necessarily orthogonal), the space

$$B = P(\mathrm{H}^{\frac{1}{2}}(E_\Sigma))$$

is called a *pseudo-local boundary condition*.

- Boundary condition $B \subset \mathrm{H}^{\frac{1}{2}}(E_\Sigma)$ a *local boundary condition* if there exists a sub-bundle $E' \subset E_\Sigma$ such that

$$B = \mathrm{H}^{\frac{1}{2}}(E').$$

# Theorem 3: Pseudo-local boundary conditions

Given a pseudo-local boundary condition $B = P(\mathrm{H}^{\frac{1}{2}}(E_\Sigma))$, the following are equivalent:

(i) $B$ an elliptic boundary condition,

(ii) for admissible cut $r \in \mathbb{R}$, the operator

$$P - \chi^+(A_r) : \mathrm{L}^2(E_\Sigma) \to \mathrm{L}^2(E_\Sigma)$$

is Fredholm,

(iii) for admissible cut $r \in \mathbb{R}$, the operator

$$P - \chi^+(A_r) : \mathrm{L}^2(E_\Sigma) \to \mathrm{L}^2(E_\Sigma)$$

is elliptic classical pseudo of order zero.

In particular, if $D_B u$ is smooth, then $u$ is smooth up to the boundary.

# Ingredients of the proof

Geometric reduction to the "model" operator $D_0 = \sigma_0(\partial_t + A)$:

**Lemma (Lemma 4.1 in [BB12])**

*On the cylinder* $Z_{[0,T)}$,

$$D = \sigma_t(\partial_t + B + R_t),$$
$$D^* = -\sigma_t^*(\partial_t + \tilde{B} + \tilde{R}_t),$$

*for any pair of adapted boundary operators* $B$ *and* $\tilde{B}$ *to* $D$ *and* $D^*$*. Remainder terms* $R_t$ *and* $\tilde{R}_t$ *are* $\Psi$*DO's of order at most one, their coefficients depend smoothly on* $t$*, and*

$$\|R_t u\|_{L^2(\Sigma)} \lesssim t\|Bu\|_{L^2(\Sigma)} + \|u\|_{L^2(\Sigma)}, \text{ and}$$
$$\|\tilde{R}_t v\|_{L^2(\Sigma)} \lesssim t\|\tilde{B}v\|_{L^2(\Sigma)} + \|v\|_{L^2(\Sigma)}.$$

*for* $u \in C^\infty(E_\Sigma)$ *and* $v \in C^\infty(F_\Sigma)$.

# Associated sectorial operators and functional calculus

- Let $\mathrm{sgn}(\mathrm{A}_r) := \chi^+(\mathrm{A}_r) - \chi^-(\mathrm{A}_r)$.
- Define $|\mathrm{A}_r| := \mathrm{A}_r \,\mathrm{sgn}(\mathrm{A}_r)$.
- $|\mathrm{A}_r|$ is invertible $\omega_r$-sectorial.
- $\Psi$DOdifferential calculus: $\mathcal{D}(|\mathrm{A}_r|) = \mathcal{D}(|\mathrm{A}_r|^*)$.
- Theorem of Auscher-McIntosh-Nahmod from [AMN97]: $|\mathrm{A}_r|$ has a $H^\infty$ functional calculus.
- Define $\mathrm{H}^1_{\mathrm{D}}(E) = \mathcal{D}(D_{\max}) \cap \mathrm{H}^1_{\mathrm{loc}}(E)$ with

$$\|u\|^2_{\mathrm{H}^1_{\mathrm{D}}} := \|\eta u\|^2_{\mathrm{H}^1} + \|Du\|^2 + \|u\|,$$

where $\eta$ is a compactly supported cutoff near the boundary.

## Lemma

*For* $\theta \in (\omega_r, \pi/2)$ *fixed, there exists an inner product* $\langle \cdot, \cdot \rangle_{N,\theta}$ *such that* $|A_r|$ *is m-*$\theta$*-accretive and for which the estimate*

$$\|(\partial_t + \mathrm{A})u\|^2_{\mathrm{L}^2(Z_{[0,\infty)})} \simeq \|u'\|^2_{\mathrm{L}^2(Z_{[0,\infty)})} + \|Au\|^2_{\mathrm{L}^2(Z_{[0,\infty)})} - \mathrm{Re}\,\langle |A_r| \,\mathrm{sgn}(A_r)\, u_0, u_0 \rangle_{N,\theta} - r\|u_0\|^2_{N,\theta},$$

*holds for* $u \in \mathrm{C}^\infty_{\mathrm{c}}(Z_{[0,\infty)}; E)$ *where* $u_0 = u|_\Sigma$*.*

Consequences:

- $\chi^+(A_r)u_0 = 0 \implies \|u\|_{\mathrm{H}^1_{\mathrm{D}}} \lesssim \|u\|_{D_0}$,
- relative boundedness of $D$ and $D_0$.

For any equivalent norm $\|\cdot\|_\star \simeq \|\cdot\|_{\mathrm{L}^2(E_\Sigma)}$,

$$\|(\partial_t + \mathrm{A})u\|_\star^2 = \|u'\|_\star^2 + \|\mathrm{A}u\|_\star^2 + \mathrm{Re}\,\langle |\mathrm{A}_r|\,\mathrm{sgn}(A_r)u, u\rangle_\star' + r\,\langle u, u\rangle_\star' \\ + \mathrm{Re}(\langle u', |\mathrm{A}_r|\,\mathrm{sgn}(A_r)u\rangle_\star - \langle |\mathrm{A}_r|\,\mathrm{sgn}(A_r)u', u\rangle_\star)$$

(i) Via Similarity Theorem due to Callier–Grabowski–Le Merdy (c.f. [Haa06] by Haase), there exists $\langle\cdot,\cdot\rangle_\theta$ so that $|\mathrm{A}_r|$ is m-$\theta$-accretive.

(ii) Define $\|\cdot\|_{N,\theta}^2 := \|\chi^+(\mathrm{A}_r)\cdot\|_\theta^2 + \|\chi^-(\mathrm{A}_r)\cdot\|_\theta^2$.

(iii) $|\mathrm{A}_r|$ is still m-$\theta$-accretive with respect to $\langle\cdot,\cdot\rangle_{N,\theta}$.

(iv) $\chi^\pm(\mathrm{A}_r)$ are self-adjoint in $\langle\cdot,\cdot\rangle_{N,\theta}$ as is $\mathrm{sgn}(\mathrm{A}_r)$.

(v) $a(u,v) = \langle |\mathrm{A}_r| u, v\rangle_{N,\theta}$ is m-$\theta$-accretive form.

$$\langle u', |\mathrm{A}_r|\,\mathrm{sgn}(A_r)u\rangle_{N,\theta} - \langle |\mathrm{A}_r|\,\mathrm{sgn}(A_r)u', u\rangle_{N,\theta} \\ = a(\mathrm{sgn}(\mathrm{A}_r)u', u)^{\mathrm{conj}} - a(u, \mathrm{sgn}(\mathrm{A}_r)u') \in \mathrm{Im}\,\mathbb{R}$$

# Higher regularity

Banach-valued Cauchy problem: $f \in \mathrm{L}^2(Z_{[0,\rho]}, E)$,

$$\partial_t W(t;f) + |A_r| W(t;f) = f(t), \qquad \lim_{t \to 0} W(t;f) = 0.$$

Solution given by:

$$W(t;f) = \int_0^t e^{-(t-s)|A_r|} f(s)\, ds.$$

Define:

$$S_{0,r} u(t) = \int_0^t e^{-(t-s)|A_r|} \sigma_0^{-1} \chi^+(A_r) u(s)\, ds \\ - \int_t^\rho e^{-(s-t)|A_r|} \sigma_0^{-1} \chi^-(A_r) u(s)\, ds$$

Let $(C_\rho u)(s) = u(\rho - s)$,

(i) $S_{0,r} u(t) = W(t; \sigma_0^{-1} \chi^+(A_r) u) - W(\rho - t; \sigma_0^{-1} \chi^-(A_r) C_\rho u)$

(ii) $\chi^+(A_r)(S_{0,r} u)(0) = \chi^-(A_r)(S_{0,r} u)(\rho) = 0$.

(iii) $D_{0,r} S_{0,r} = \mathrm{I}$, where $D_{0,r} = \sigma_0(\partial_t + \mathrm{A}_r)$.

(iv) $S_{0,r} : \mathrm{H}^{\mathrm{k}}(Z_{[0,\rho]}, E) \to \mathrm{H}^{\mathrm{k}+1}(Z_{[0,\rho]}, E)$ bounded.

(v) $D_{0,r} : \mathrm{H}^{\mathrm{k}+1}(Z_{[0,\rho]}, E; B_0) \to \mathrm{H}^{\mathrm{k}}(Z_{[0,\rho]}, E)$ isomorphism, where $B_0 = \chi^-(A_r)\mathrm{H}^{\frac{1}{2}}(E_\Sigma) \oplus \chi^+(A_r)\mathrm{H}^{\frac{1}{2}}(E_\Sigma)$

(vi) $(I - S_{0,r} D_{0,r}) u = e^{-t|A_r|}(\chi^+(A_r) u(0))$ whenever $\chi^-(A_r)(u(\rho)) = 0$,

Key estimate:

$$\int_0^\rho \|\partial_t W(t; f)\|^2_{\mathrm{L}^2(E_\Sigma)} dt + \int_0^\rho \||A_r| W(t; f)\|^2_{\mathrm{L}^2(E_\Sigma)} \lesssim \int_0^\rho \|f(t)\|^2_{\mathrm{L}^2(E_\Sigma)}.$$

# Elliptic boundary conditions

Prove: $B \subset \mathrm{H}^{\frac{1}{2}}(E_\Sigma)$ and $B^{\mathrm{ad}} \subset \mathrm{H}^{\frac{1}{2}}(F_\Sigma)$ implies $B$ elliptic.

**Lemma**

*$W_+ = \mathcal{R}(\chi^+(A_r)) \cap B$ and $W_-^* = \mathcal{R}(\chi^-(A_r^*)) \cap \sigma_0^* B^{\mathrm{ad}}$ are finite dimensional subspaces of $\mathrm{H}^{\frac{1}{2}}(E_\Sigma)$ and $\chi^-(A_r)B$ and $\chi^+(A_r^*)\sigma_0^* B^{\mathrm{ad}}$ are closed subspaces of $\mathrm{H}^{\frac{1}{2}}(E_\Sigma)$.*

Key estimate:

$$\|u\|_{\mathrm{H}^{\frac{1}{2}}} \simeq \|u\|_{\check{H}(A)} \lesssim \|\chi^-(A_r)u\|_{\mathrm{H}^{\frac{1}{2}}} + \|\chi^+(A_r)u\|_{\mathrm{H}^{-\frac{1}{2}}}$$

for all $u \in B$.

Spaces:

$$
\begin{aligned}
W_-^* &:= \chi^-(A_r^*)\mathrm{L}^2(E_\Sigma) \cap \sigma_0^* B^{\mathrm{ad}} & W_- &:= \chi^-(A_r)W_-^* \\
W_+ &:= \chi^+(A_r)\mathrm{L}^2(E_\Sigma) \cap B & W_+^* &:= \chi^+(A_r^*)W_+ \\
V_-^* &:= \chi^-(A_r^*)\mathrm{L}^2(E_\Sigma) \cap (W_-^*)^\perp & V_- &:= \chi^-(A_r)V_-^* \\
V_+ &:= \chi^+(A_r)\mathrm{L}^2(E_\Sigma) \cap W_+^\perp & V_+^* &:= \chi^+(A_r^*)V_+.
\end{aligned}
$$

Splitting:

$$\mathrm{L}^2(E_\Sigma) = V_- \oplus W_- \oplus V_+ \oplus W_+ = V_-^* \oplus W_-^* \oplus V_+^* \oplus W_+^*.$$

$$X_- = \chi^-(A_r)|_{B \cap W_+^\perp} : B \cap W_+^\perp \to \chi^-(A_r)B, \text{ and}$$
$$X_+^* = \chi^+(A_r^*)|_{\sigma_0^* B^{\mathrm{ad}} \cap (W_-^*)^\perp} : \sigma_0^* B^{\mathrm{ad}} \cap (W_-^*)^\perp \to \chi^+(A_r^*)\sigma_0^* B^{\mathrm{ad}}.$$

are isomorphisms with their ranges.

$$g_0 = P_{V_+}(X_-)^{-1} \quad \text{and} \quad h_0 = P_{V_-^*}(X_+^*)^{-1}.$$

Obtain:

$$B = W_+ \oplus \left\{ v \in V_-^{\frac{1}{2}} : v + g_0 v \right\}$$
$$B^{\mathrm{ad}} = W_-^* \oplus \left\{ u \in (V_+^*)^{\frac{1}{2}} : u + h_0 u \right\}.$$

# References I


[AMN97] Pascal Auscher, Alan McIntosh, and Andrea Nahmod.
Holomorphic functional calculi of operators, quadratic estimates and interpolation.
*Indiana Univ. Math. J.*, 46(2):375–403, 1997.

[BB12] Christian Bär and Werner Ballmann.
Boundary value problems for elliptic differential operators of first order.
17:1–78, 2012.

[Gru12] Gerd Grubb.
The sectorial projection defined from logarithms.
*Math. Scand.*, 111(1):118–126, 2012.

[Haa06] Markus Haase.
*The functional calculus for sectorial operators*, volume 169 of *Operator Theory: Advances and Applications*.
Birkhäuser Verlag, Basel, 2006.

[See67] R. T. Seeley.
Complex powers of an elliptic operator.
pages 288–307, 1967.

[Shu01] M. A. Shubin.
*Pseudodifferential operators and spectral theory*.
Springer-Verlag, Berlin, second edition, 2001.
Translated from the 1978 Russian original by Stig I. Andersson.