Index theory and boundary value problems for general first-order elliptic differential operators

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Satisfies:

$$\not{D}^2 f = \sum_{k=1}^{3}\sum_{j=1}^{2} \partial_k^2 f_j e_j = \Delta f.$$

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There exists $B \in \mathcal{L}(\mathrm{L}^2(\partial\Omega; \mathbb{C}^2))$ such that $\mathrm{A} := \mathrm{A}_0 + B$ self-adjoint.

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$$\eta_{\mathrm{A}}(s) := \sum_{\lambda\in\mathrm{spec}(\mathrm{A})\setminus\{0\}} \frac{\mathrm{sgn}(\lambda)}{|\lambda|^s}.$$

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Then,

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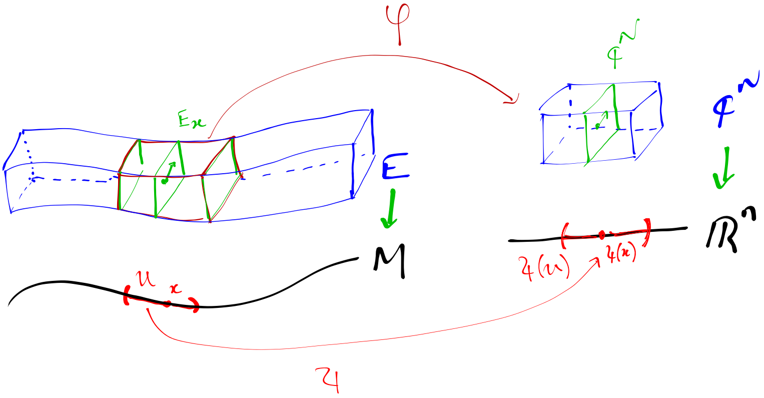
Self-adjointness fundamental in the Bär-Ballmann framework.

Geometric dictionary

$(\Omega, \langle \cdot, \cdot \rangle_{\mathbb{R}^3})$	$\rightsquigarrow$	(M, g) manifold with metric g
$\not{\Delta}\,\Omega$	$\rightsquigarrow$	$\not{\Delta}\,M \to M$ spin bundle
$\mathbb{C}^N$	$\rightsquigarrow$	$E \to M$, vector bundle of rank N.



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$\forall u \in \mathrm{C}^\infty_{\mathrm{cc}}(\mathcal{E}), v \in \mathrm{C}^\infty_{\mathrm{cc}}(\mathcal{F})$,

$$\langle \mathrm{D}u, v \rangle_{\mathrm{L}^2(\mathcal{M};\mathcal{F},\mathrm{h}^{\mathcal{F}})} = \left\langle u, \mathrm{D}^\dagger v \right\rangle_{\mathrm{L}^2(\mathcal{M};\mathcal{E},\mathrm{h}^{\mathcal{E}})}$$

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$T > 0$ determined by (A1)-(A6).

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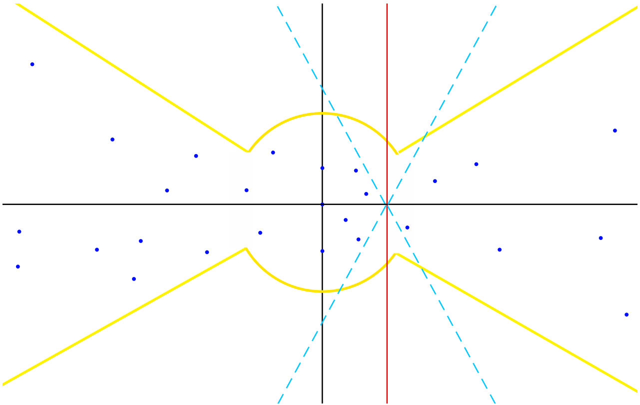
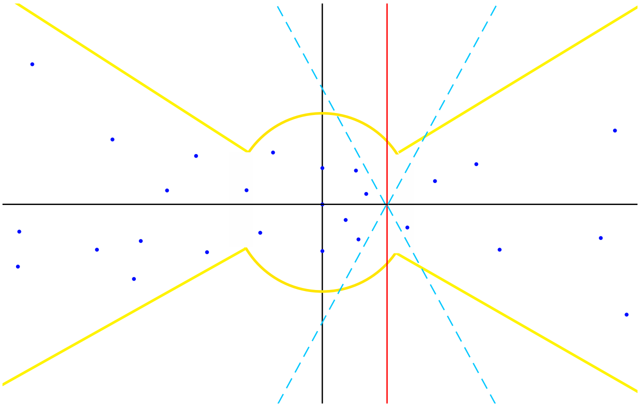
- Theorem of Shubin: there exists $\omega \in [0, \pi/2)$, $R > 0$, $C < \infty$ such that $\operatorname{spec}(\mathrm{A}) \subset S_\omega \cup B_R(0)$ and

$$|\zeta| \|(\zeta - \mathrm{A})^{-1}\| \leq C,$$

for all $\zeta \notin S_\omega \cup B_R(0)$.





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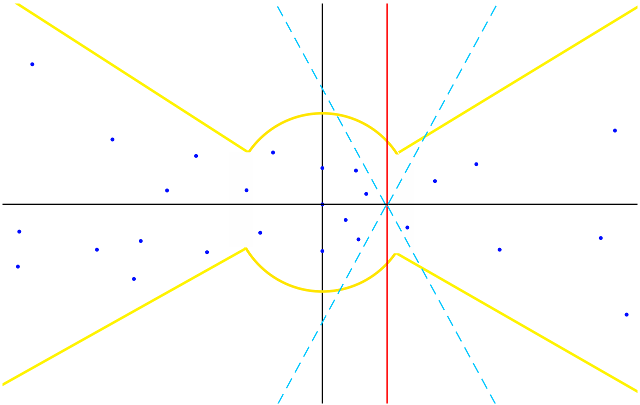

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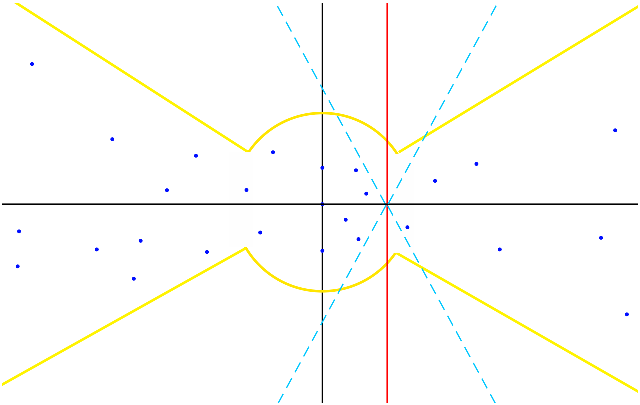

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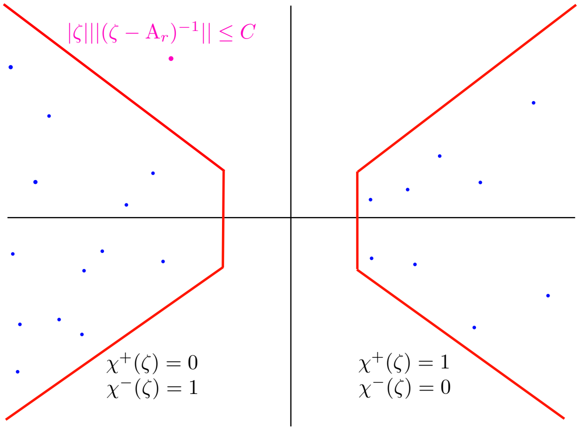
- For such r, there exist $\omega_r \in [0, \pi/2)$ such that $\mathrm{A}_r := \mathrm{A} - r$ is invertible ω_r bi-sectorial.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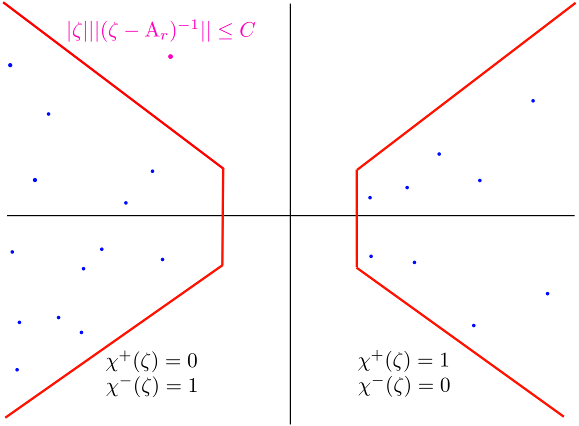

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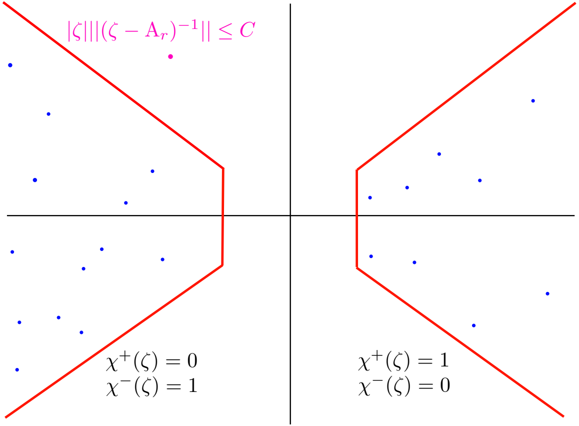

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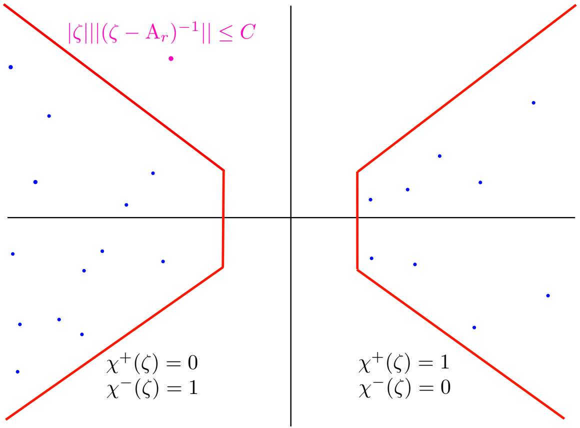

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(iii) The $\mathrm{L}^2(\partial\mathcal{M};\mathcal{E})$ inner product extends to a perfect pairing

$$\langle \cdot, \cdot \rangle : \check{\mathrm{H}}(\mathrm{A}) \times \check{\mathrm{H}}(-\mathrm{A}^*) \to \mathbb{C}.$$

(iv) For all $u \in \mathrm{dom}(\mathrm{D}_{\max})$ and $v \in \mathrm{dom}(\mathrm{D}^{\dagger}_{\max})$,

$$\langle \mathrm{D}_{\max} u, v\rangle_{\mathrm{L}^2(\mathcal{M};\mathcal{F})} - \langle u, \mathrm{D}^{\dagger}_{\max} v\rangle_{\mathrm{L}^2(\mathcal{M};\mathcal{E})} = -\left\langle u|_{\partial\mathcal{M}}, \sigma_0^{*}\, v|_{\partial\mathcal{M}}\right\rangle_{\check{\mathrm{H}}(\mathrm{A})\times\check{\mathrm{H}}(-\mathrm{A}^{*})}.$$

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(v) Higher regularity:

$$\mathrm{dom}(\mathrm{D}_{\max}) \cap \mathrm{H}^{k+1}_{\mathrm{loc}}(\mathcal{M};\mathcal{E}) = \Big\{ u \in \mathrm{dom}(\mathrm{D}_{\max}) : \mathrm{D}u \in \mathrm{H}^{k}_{\mathrm{loc}}(\mathcal{M};\mathcal{F}) \text{ and } \chi^+(\mathrm{A}_r)(u|_{\partial\mathcal{M}}) \in \mathrm{H}^{k+\frac{1}{2}}(\partial\mathcal{M};\mathcal{E}) \Big\}.$$

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Similarly for the formal adjoint $\mathrm{D}^\dagger$ with A replaced by $\tilde{\mathrm{A}}$.

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- $B \subset \mathrm{H}^{\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})$ boundary condition if and only if $\mathrm{D}_B = \mathrm{D}_{B,\max}$.

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- D_c closed extension of $\mathrm{D}_{\min}$, then

$$B := \{u|_{\partial\mathcal{M}} : u \in \mathrm{dom}(\mathrm{D}_c)\}$$

  boundary condition and $\mathrm{D}_c = \mathrm{D}_{B,\max}$.
- $B \subset \mathrm{H}^{\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})$ boundary condition if and only if $\mathrm{D}_B = \mathrm{D}_{B,\max}$.
- Adjoint boundary condition $B^\dagger$ so that $(\mathrm{D}_B)^* = \mathrm{D}^\dagger_{B^\dagger}$:

$$B^\dagger := \left\{v \in \check{\mathrm{H}}(\tilde{\mathrm{A}}) : \langle u, \sigma_0^*\, v\rangle_{\check{\mathrm{H}}(\mathrm{A})\times\check{\mathrm{H}}(-\mathrm{A}^*)} = 0 \quad \forall u \in B\right\}.$$

- Classical pseudo-differential projector P of order zero (not necessarily orthogonal in L^2), the space

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A boundary condition B is *elliptic* if:

$$\mathrm{dom}(\mathrm{D}_{B,\max}) \subset \mathrm{H}^1_{\mathrm{loc}}(\mathcal{M};\mathcal{E}) \quad \text{and} \quad \mathrm{dom}(\mathrm{D}^\dagger_{B^\dagger,\max}) \subset \mathrm{H}^1_{\mathrm{loc}}(\mathcal{M};\mathcal{F})$$

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In particular, if $\mathrm{D}_B u$ is smooth, then u is smooth up to the boundary.

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Index formula? - Big open question.

Ingredients of the proof

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Lemma (Lemma 4.1 (Bär-Ballmann))

On the cylinder $Z_{[0,T)}$,

$$\mathrm{D} = \sigma_t(\partial_t + \mathrm{A} + R_t),$$

for any adapted boundary operator A *for* D. *The remainder term* R_t *is a* Ψ*DO of order at most one and its coefficients depend smoothly on* t. *Moreover,*

$$\|R_t u\|_{\mathrm{L}^2(\partial\mathcal{M};\mathcal{E})} \lesssim t\|\mathrm{A}u\|_{\mathrm{L}^2(\partial\mathcal{M};\mathcal{E})} + \|u\|_{\mathrm{L}^2(\partial\mathcal{M};\mathcal{E})}$$

for $u \in \mathrm{C}^\infty(\partial\mathcal{M};\mathcal{E})$.

Associated sectorial operators and functional calculus

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for some (and equivalently, all) holomorphic $0 \not\equiv \psi : S_\mu^o \to \mathbb{C}$ with some $C, \alpha > 0$ such that

$$|\psi(\zeta)| \leq C \min\left\{|\zeta|^\alpha, |\zeta|^{-\alpha}\right\}.$$

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$$\begin{aligned} \|\mathrm{D}\mathscr{E}v_-\|^2_{\mathrm{L}^2(\mathcal{M};\mathcal{E})} &= 4\int_0^\infty \|\sigma_0|\mathrm{A}|\mathrm{e}^{-t|\mathrm{A}|}v_-\|^2_{\mathrm{L}^2(\partial\mathcal{M};\mathcal{E})}\ dt \\ &\simeq \int_0^\infty \|t^{\frac{1}{2}}|\mathrm{A}|^{\frac{1}{2}}\mathrm{e}^{-t|\mathrm{A}|}|\mathrm{A}|^{\frac{1}{2}}v_-\|^2_{\mathrm{L}^2(\partial\mathcal{M};\mathcal{E})}\ \frac{dt}{t} \\ &\simeq \||\mathrm{A}|^{\frac{1}{2}}v_-\|^2_{\mathrm{L}^2(\partial\mathcal{M};\mathcal{E})} \simeq \|v_-\|^2_{\mathrm{H}^{\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})}. \end{aligned}$$

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Combining with $\mathrm{D}\mathscr{E}v_+ = \sigma_0(\partial_t + \mathrm{A})v_+ = 0$, obtain:

$$\|\mathscr{E}v\|^2_D \lesssim \|v_-\|^2_{\mathrm{H}^{\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})} + \|v_+\|^2_{\mathrm{H}^{-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})} = \|v\|^2_{\check{\mathrm{H}}(\mathrm{A})}.$$

Maximal regularity

Banach-valued Cauchy problem: $f \in \mathrm{L}^2(Z_{[0,\rho]}; \mathcal{E})$,

$$\partial_t W(t; f) + |\mathrm{A}_r| W(t; f) = f(t)$$
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Key estimate - maximal regularity:

$$\int_0^\rho \|\partial_t W(t;f)\|^2_{\mathrm{L}^2(\partial\mathcal{M};\mathcal{E})} dt + \int_0^\rho \||\mathrm{A}_r| W(t;f)\|^2_{\mathrm{L}^2(\partial\mathcal{M};\mathcal{E})} \lesssim \int_0^\rho \|f(t)\|^2_{\mathrm{L}^2(\partial\mathcal{M};\mathcal{E})}.$$

Define:

$$S_{0,r}u(t) = \int_0^t \mathrm{e}^{-(t-s)|\mathrm{A}_r|}\chi^+(A_r)u(s)\ ds - \int_t^\rho \mathrm{e}^{-(s-t)|A_r|}\chi^-(A_r)u(s)\ ds$$

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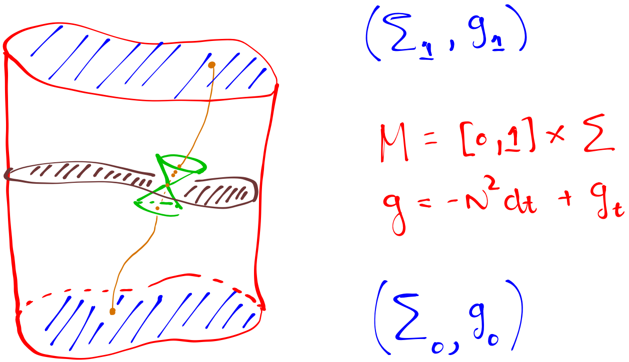
  H^∞-functional calculus and harmonic analysis: alternative perspective of Atiyah-Patodi-Singer.

- *Lorentzian manifolds with spacelike boundary*



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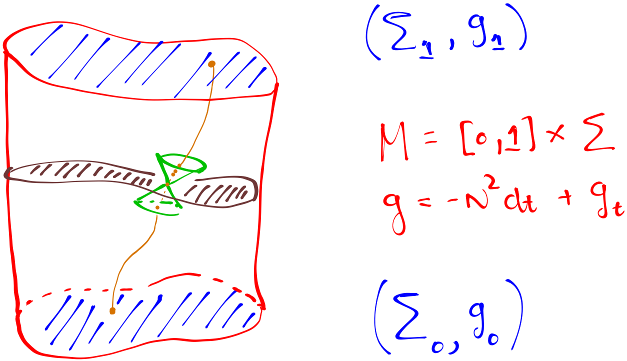

Extension operator: wave propagation operator.

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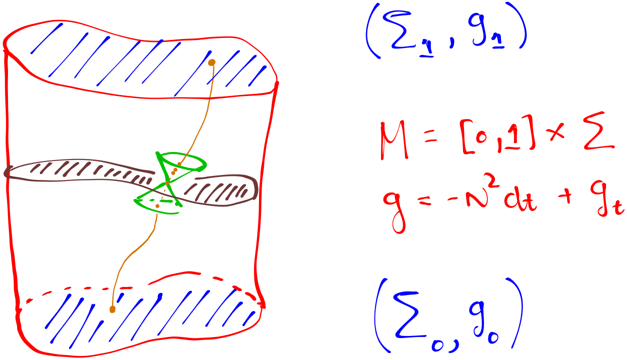

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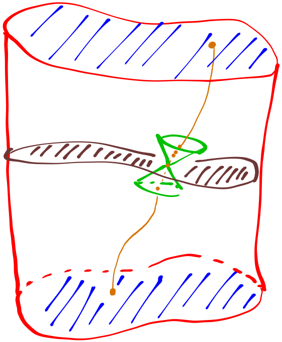

(Σ_1, g_1)

$$M = [0,1] \times \Sigma$$

$$g = -N^2 dt + g_t$$

(Σ_0, g_0)

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Key idea: identify the right function spaces.