Index theory and boundary value problems for general first-order elliptic differential operators

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10 May 2022

Motivation from index theory

D be a differential operator, seen as an unbounded operator on L^2.

D is *Fredholm*: $\operatorname{ran}\mathrm{D}$ closed and $\ker\mathrm{D}$ and $\ker\mathrm{D}^*$ are finite dimensional.

Index (analytic) of D:

$$\operatorname{index}\ \mathrm{D} = \dim\ker\mathrm{D} - \dim\ker\mathrm{D}^*.$$

Invariant, in particular, $\operatorname{index}\ \mathrm{D}_t = \operatorname{index}\ \mathrm{D}$ for continuous deformation of t through Fredholm operators.

Usually, D determined by geometry : $t \to \mathrm{D}_t$ geometric deformation, i.e., evolving time slices in spacetime.

Index formula: relate geometry, topology and boundary.

$$\mathrm{index}\ \mathrm{D} = \int \text{“Curvatures related to } \mathrm{D}\text{”} + \text{“Boundary contribution”}.$$

Atiyah-Patodi-Singer (sufficiently nice D, in particular elliptic):

$$\text{“Boundary contribution”} = \frac{\ker(\mathrm{A}) + \eta(\mathrm{A})}{2},$$

A *adapted operator to the boundary* - determined by D ,

$\eta(\mathrm{A})$ - measuring spectral *asymmetry* of A.

Need: boundary condition to make formula work.

Non-local boundary conditions: topologically obstructed for local boundary conditions.

Motivation: a Euclidean example

$\Omega \subset \mathbb{R}^3$, domain with smooth compact boundary $\partial\Omega \subset \Omega$.

Spin "bundle" $\not\Delta\,\Omega \cong \mathbb{C}^2$.

Representation $\rho(e_i)u := \sigma_i u$, given by Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Spin-Dirac operator:

$$\not{D} f := -\imath \sum_{j=1}^{3} \rho(e_j)\partial_i f = -\imath(\ \sigma_1\partial_1 f + \sigma_2\partial_2 f + \sigma_3\partial_3 f\ ).$$

Satisfies:

$$\not{D}^2 f = \sum_{k=1}^{3}\sum_{j=1}^{2} \partial_k^2 f_j e_j = \Delta f.$$

Adapted operator

$\not{D}_c$, with $\mathrm{dom}(\not{D}_c) = \mathrm{C}^\infty_{\mathrm{cc}}(\Omega) := \left\{ u \in \mathrm{C}^\infty_{\mathrm{c}}(\Omega) : \mathrm{spt}\ u \subset \mathring{\Omega} \right\}$.

$\not{D}_c$ is symmetric. Define:

$$\not{D}_{\max} := \not{D}_c^*, \quad \not{D}_{\min} := \overline{\not{D}_c} = (\not{D}_c^*)^*.$$

Adapted operator on the boundary: fix ν outward pointing normal, $\{\tilde{e}_i\}_{i=1}^2$ orthonormal vectors at $x \in \partial\Omega$. Then,

$$\mathrm{A}_0 u(x) := \rho(\nu)^{-1} \sum_{i=1}^{2} \left(\rho(\tilde{e}_i) \nabla^{\mathbb{C}^2}_{\tilde{e}_i} u \right)(x).$$

Elliptic regularity: $\mathrm{dom}(\mathrm{A}_0) = \mathrm{H}^1(\partial\Omega; \mathbb{C}^2)$.

There exists $B \in \mathcal{L}(\mathrm{L}^2(\partial\Omega; \mathbb{C}^2))$ such that $\mathrm{A} := \mathrm{A}_0 + B$ self-adjoint.

Boundary conditions

Boundary conditions for $\not{\mathrm{D}}$ "live" in:

$$\check{\mathrm{H}}(\mathrm{A}) := \chi_{(-\infty,0]}(\mathrm{A})\mathrm{H}^{\frac{1}{2}}(\partial\Omega;\mathbb{C}^2) \quad \bigoplus \quad \chi_{(0,\infty)}(\mathrm{A})\mathrm{H}^{-\frac{1}{2}}(\partial\Omega;\mathbb{C}^2).$$

That is,

$$u \mapsto u|_{\partial\Omega} : \mathrm{dom}(\not{\mathrm{D}}_{\max}) \to \check{\mathrm{H}}(\mathrm{A})$$

bounded surjection with kernel $\mathrm{dom}(\not{\mathrm{D}}_{\min})$.

(Generalised) Atiyah-Patodi-Singer BC:

$$\mathrm{B}_{\mathrm{APS}} := \chi_{(-\infty,0]}(\mathrm{A})\mathrm{H}^{\frac{1}{2}}(\partial\Omega;\mathbb{C}^2).$$

Eta-invariant: $\eta(\mathrm{A}) := \eta_{\mathrm{A}}(0)$ where

$$\eta_{\mathrm{A}}(s) := \sum_{\lambda\in\mathrm{spec}(\mathrm{A})\setminus\{0\}} \frac{\mathrm{sgn}(\lambda)}{|\lambda|^s}.$$

Rarita-Schwinger Operator

Twisted bundle $\mathrm{T}^*\Omega \otimes \not{\Delta}\,\Omega \cong \mathbb{C}^3 \otimes \mathbb{C}^2$.

Let $\iota : \not{\Delta}\,\Omega \to \mathrm{T}^*\Omega \otimes \not{\Delta}\,\Omega$ given by

$$\iota(\psi) = -\frac{1}{3}\sum_{j=1}^{3} e_j \otimes \rho(e_j)\psi.$$

$\frac{3}{2}$-Spin bundle given by

$$\not{\Delta}_{\frac{3}{2}}\,\Omega := \iota(\not{\Delta}\,\Omega)^{\perp} = \ker\gamma,$$

where

$$\gamma(v \otimes \psi) := \rho(v)\psi.$$

Then,

$$\mathrm{T}^*\Omega \otimes \not{\Delta}\,\Omega = \iota(\not{\Delta}\,\Omega) \overset{\perp}{\oplus} \not{\Delta}_{\frac{3}{2}}\,\Omega.$$

Induced Dirac operator

$$\not{D}^{\mathrm{T}^*\Omega \otimes \not{\Delta}\,\Omega} f = \sum_{i=1}^{3} \rho(e_i)(\partial_i f) = \sum_{i=1}^{3} \sum_{j,k=1}^{2} (\partial_i f_{jk})\; e_j \otimes \sigma_i e_k.$$

Orthogonal projection $\mathbf{P}_{\frac{3}{2}} : \mathrm{T}^*\Omega \otimes \not{\Delta}\,\Omega \to \not{\Delta}_{\frac{3}{2}}\,\Omega$.

Rarita-Schwinger is then:

$$\not{D}_{\mathrm{RS}} := \mathbf{P}_{\frac{3}{2}} \circ \not{D}^{\mathrm{T}^*\Omega \otimes \not{\Delta}\,\Omega}|_{\not{\Delta}_{\frac{3}{2}}\,\Omega}, \qquad \not{D}^{\mathrm{T}^*\Omega \otimes \not{\Delta}\,\Omega} = \begin{pmatrix} \iota \not{D} \iota^* & \mathrm{T}^* \\ \mathrm{T} & \not{D}_{\mathrm{RS}} \end{pmatrix}.$$

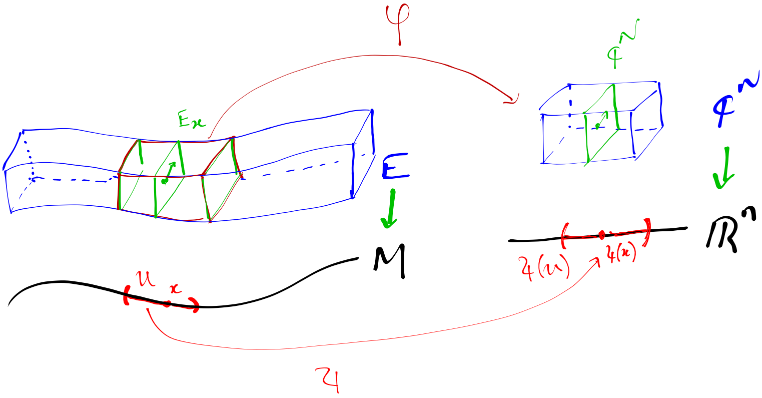
Extract boundary adapted operator A_{RS} as before.

There is no B such that $\mathrm{A}_{\mathrm{RS}} + B$ is self-adjoint.

Self-adjointness fundamental in the Bär-Ballmann framework.

Geometric dictionary

$(\Omega, \langle \cdot, \cdot \rangle_{\mathbb{R}^3})$	$\rightsquigarrow$	(M, g) manifold with metric g
$\not{\Delta}\,\Omega$	$\rightsquigarrow$	$\not{\Delta}\,M \to M$ spin bundle
$\mathbb{C}^N$	$\rightsquigarrow$	$E \to M$, vector bundle of rank N.



Broad aim

Hermitian vector bundles $(\mathcal{E}, \mathrm{h}^{\mathcal{E}})$, $(\mathcal{F}, \mathrm{h}^{\mathcal{F}}) \to (\mathcal{M}, \mu)$ meas. manifold.

Diff op $\mathrm{D} : \mathrm{C}^\infty(\mathcal{E}) \to \mathrm{C}^\infty(\mathcal{F}) \rightsquigarrow \exists! \mathrm{D}^\dagger : \mathrm{C}^\infty(\mathcal{F}) \to \mathrm{C}^\infty(\mathcal{E})$ formal adjoint.I.e., $\forall u \in \mathrm{C}^\infty_{\mathrm{cc}}(\mathcal{E}), v \in \mathrm{C}^\infty_{\mathrm{cc}}(\mathcal{F})$,

$$\langle \mathrm{D}u, v \rangle_{\mathrm{L}^2(\mathcal{M};\mathcal{F},\mathrm{h}^{\mathcal{F}})} = \left\langle u, \mathrm{D}^\dagger v \right\rangle_{\mathrm{L}^2(\mathcal{M};\mathcal{E},\mathrm{h}^{\mathcal{E}})}$$

Define: $\mathrm{D}_{\max} := (\mathrm{D}^\dagger)^*, \quad \mathrm{D}_{\min} := \overline{\mathrm{D}|_{\mathrm{C}^\infty_{\mathrm{cc}}(\mathcal{E})}}$.

Understand *all* closed extensions of $\mathrm{D}_{\min}$ sitting in $\mathrm{D}_{\max}$. I.e., control

$$\mathrm{dom}(\mathrm{D}_{\max}) / \mathrm{dom}(\mathrm{D}_{\min}).$$

$\partial\mathcal{M} \neq \varnothing$ want map $\gamma : \mathrm{dom}(\mathrm{D}_{\max}) \to \check{\mathrm{H}}$ built out of boundary trace map, bounded surjection with $\ker(\gamma) = \mathrm{dom}(\mathrm{D}_{\min})$. **Compute topology of $\check{\mathrm{H}}$ purely in terms of data on $\partial\mathcal{M}$.**

General setup

(A1) $\mathcal{M}$ is a manifold with compact boundary $\partial\mathcal{M} \subset \mathcal{M}$;

(A2) τ is an interior pointing co-vectorfield along $\partial\mathcal{M}$;

(A3) μ is a smooth volume measure on $\mathcal{M}$ and ν is the induced smooth volume measure on $\partial\mathcal{M}$;

(A4) $(\mathcal{E}, \mathrm{h}^{\mathcal{E}}),\ (\mathcal{F}, \mathrm{h}^{\mathcal{F}}) \to \mathcal{M}$ are Hermitian vector bundles over $\mathcal{M}$;

(A5) $\mathrm{D} : \mathrm{C}^\infty(\mathcal{M};\mathcal{E}) \to \mathrm{C}^\infty(\mathcal{M};\mathcal{F})$ is a first-order elliptic differential operator;

(A6) D and $\mathrm{D}^\dagger$ (formal adjoint of D) are complete (i.e., $\mathrm{C}^\infty_{\mathrm{c}}(\mathcal{M};\mathcal{E})$ dense in $\mathrm{dom}(\mathrm{D}_{\max})$).

Consequence: reduce to cylinder $Z_{[0,T)} : [0,T) \times \partial\mathcal{M}$.

$T > 0$ determined by (A1)-(A6).

Adapted boundary operator

A adapted boundary operator to D if:

$$\sigma_{\mathrm{A}}(x,\xi) = \sigma_{\mathrm{D}}(x,\tau(x))^{-1} \circ \sigma_{\mathrm{D}}(x,\xi).$$

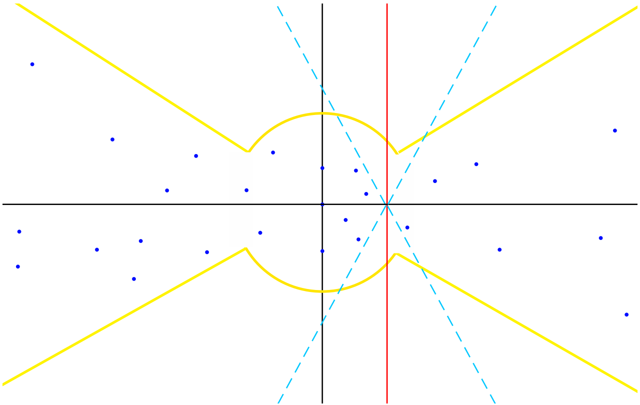
- Exists and are elliptic differential operators of order 1.
- Unique up to an operator of order zero.
- Ellipticity of D $\implies$ for all $(x,\xi) \in \partial\mathcal{M} \times \mathrm{T}^*\partial\mathcal{M}$ and $\xi \neq 0$,

$$\mathrm{spec}(\sigma_{\mathrm{A}}(x,\xi)) \cap \mathbb{R} = \varnothing.$$

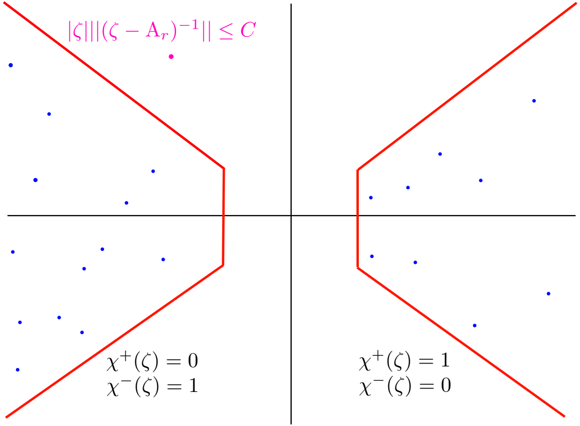
- Theorem of Shubin: there exists $\omega \in [0,\pi/2)$, $R > 0$, $C < \infty$ such that $\mathrm{spec}(\mathrm{A}) \subset S_\omega \cup B_R(0)$ and

$$|\zeta| \|(\zeta - \mathrm{A})^{-1}\| \leq C,$$

for all $\zeta \notin S_\omega \cup B_R(0)$.



- Discrete spectrum, generally *non-orthogonal* generalised eigenspaces.
- *Admissible spectral cut* $r \in \mathbb{R}$: the line $l_r := \{\zeta \in \mathbb{C} : \mathrm{Re}\ \zeta = r\}$ is *not* in the spectrum of A.
- For such r, there exist $\omega_r \in [0, \pi/2)$ such that $\mathrm{A}_r := \mathrm{A} - r$ is invertible ω_r bi-sectorial.



- Theorem of Grubb: $\chi^{\pm}(\mathrm{A}_r)$ are ΨDO projectors of order zero.
- Space: $\check{\mathrm{H}}(\mathrm{A}) := \chi^-(\mathrm{A}_r)\mathrm{H}^{\frac{1}{2}}(\partial\mathcal{M};\mathcal{E}) \oplus \chi^+(\mathrm{A}_r)\mathrm{H}^{-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})$.

  Norm: $\|u\|^2_{\check{\mathrm{H}}(\mathrm{A})} := \|\chi^-(\mathrm{A}_r)u\|^2_{\mathrm{H}^{\frac{1}{2}}} + \|\chi^+(\mathrm{A}_r)u\|^2_{\mathrm{H}^{-\frac{1}{2}}}$.

Theorem 1: Maximal domain and the $\check{\mathrm{H}}(\mathrm{A})$ space

(i) $u \mapsto u|_{\partial\mathcal{M}} : \mathrm{C}_\mathrm{c}^\infty(\mathcal{M};\mathcal{E}) \to \mathrm{C}_\mathrm{c}^\infty(\partial\mathcal{M};\mathcal{E})$ extends uniquely to a bounded surjection $\mathrm{dom}(\mathrm{D}_{\max}) \to \check{\mathrm{H}}(\mathrm{A})$.

(ii) The space

$$\mathrm{dom}(\mathrm{D}_{\max}) \cap \mathrm{H}^1_{\mathrm{loc}}(\mathcal{M};\mathcal{E}) = \left\{ u \in \mathrm{dom}(\mathrm{D}_{\max}) : u|_{\partial\mathcal{M}} \in \mathrm{H}^{\frac{1}{2}}(\partial\mathcal{M};\mathcal{E}) \right\}$$

(iii) The $\mathrm{L}^2(\partial\mathcal{M};\mathcal{E})$ inner product extends to a perfect pairing

$$\langle \cdot, \cdot \rangle : \check{\mathrm{H}}(\mathrm{A}) \times \check{\mathrm{H}}(-\mathrm{A}^*) \to \mathbb{C}.$$

(iv) For all $u \in \mathrm{dom}(\mathrm{D}_{\max})$ and $v \in \mathrm{dom}(\mathrm{D}^{\dagger}_{\max})$,

$$\langle \mathrm{D}_{\max} u, v\rangle_{\mathrm{L}^2(\mathcal{M};\mathcal{F})} - \langle u, \mathrm{D}^{\dagger}_{\max} v\rangle_{\mathrm{L}^2(\mathcal{M};\mathcal{E})} = -\left\langle u|_{\partial\mathcal{M}}, \sigma_0^* \, v|_{\partial\mathcal{M}} \right\rangle_{\check{\mathrm{H}}(\mathrm{A})\times\check{\mathrm{H}}(-\mathrm{A}^*)}.$$

(v) Higher regularity:

$$\mathrm{dom}(\mathrm{D}_{\max}) \cap \mathrm{H}^{k+1}_{\mathrm{loc}}(\mathcal{M};\mathcal{E}) = \Big\{ u \in \mathrm{dom}(\mathrm{D}_{\max}) : \mathrm{D}u \in \mathrm{H}^{k}_{\mathrm{loc}}(\mathcal{M};\mathcal{F}) \text{ and } \chi^{+}(\mathrm{A}_r)(u|_{\partial\mathcal{M}}) \in \mathrm{H}^{k+\frac{1}{2}}(\partial\mathcal{M};\mathcal{E}) \Big\}.$$

Boundary conditions

$B \subset \check{\mathrm{H}}(\mathrm{A})$ is a *boundary condition* for D if it is a closed, linear subspace.

Associated operator domains:

$$\begin{aligned} \mathrm{dom}(\mathrm{D}_{B,\max}) &= \left\{ u \in \mathrm{dom}(\mathrm{D}_{\max}) : u|_{\partial\mathcal{M}} \in B \right\} \\ \mathrm{dom}(\mathrm{D}_{B}) &= \left\{ u \in \mathrm{dom}(\mathrm{D}_{\max}) \cap \mathrm{H}^1_{\mathrm{loc}}(\partial\mathcal{M};\mathcal{E}) : u|_{\partial\mathcal{M}} \in B \right\}. \end{aligned}$$

Similarly for the formal adjoint $\mathrm{D}^\dagger$ with A replaced by $\tilde{\mathrm{A}}$.

- $\mathrm{D}_{B,\max}$ closed and between $\mathrm{D}_{\min}$ and $\mathrm{D}_{\max}$.
- D_c closed extension of $\mathrm{D}_{\min}$, then

$$B := \left\{ u|_{\partial\mathcal{M}} : u \in \mathrm{dom}(\mathrm{D}_c) \right\}$$

  boundary condition and $\mathrm{D}_c = \mathrm{D}_{B,\max}$.
- $B \subset \mathrm{H}^{\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})$ boundary condition if and only if $\mathrm{D}_B = \mathrm{D}_{B,\max}$.
- Adjoint boundary condition $B^\dagger$ so that $(\mathrm{D}_B)^* = \mathrm{D}^\dagger_{B^\dagger}$:

$$B^\dagger := \left\{ v \in \check{\mathrm{H}}(\tilde{\mathrm{A}}) : \langle u, \sigma_0^* \, v \rangle_{\check{\mathrm{H}}(\mathrm{A}) \times \check{\mathrm{H}}(-\mathrm{A}^*)} = 0 \quad \forall u \in B \right\}.$$

- Classical pseudo-differential projector P of order zero (not necessarily orthogonal in L^2), the space

$$B := \overline{P\,\mathrm{H}^{\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})}^{\check{\mathrm{H}}(\mathrm{A})}$$

is called a *pseudo-local boundary condition*.

- $B \subset \mathrm{H}^{\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})$ a *local boundary condition* if there exists a smooth sub-bundle $E' \subset E_{\partial\mathcal{M}}$ such that

$$B = \overline{\mathrm{H}^{\frac{1}{2}}(\partial\mathcal{M};E')}^{\check{\mathrm{H}}(\mathrm{A})}.$$

A boundary condition B is *elliptic* if:

$$\mathrm{dom}(\mathrm{D}_{B,\max}) \subset \mathrm{H}^1_{\mathrm{loc}}(\mathcal{M};\mathcal{E}) \quad \text{and} \quad \mathrm{dom}(\mathrm{D}^{\dagger}_{B^{\dagger},\max}) \subset \mathrm{H}^1_{\mathrm{loc}}(\mathcal{M};\mathcal{F})$$

Theorem 2: Pseudo-local boundary conditions

Given a pseudo-local boundary condition $B = \overline{P\,\mathrm{H}^{\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})}^{\check{\mathrm{H}}(\mathrm{A})}$, the following are equivalent:

(i) B an elliptic boundary condition and $B = P\,\mathrm{H}^{\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})$,

(ii) there exists an admissible spectral cut $r \in \mathbb{R}$ and

$$P - \chi^{+}(\mathrm{A}_r) : \mathrm{L}^2(\partial\mathcal{M};\mathcal{E}) \to \mathrm{L}^2(\partial\mathcal{M};\mathcal{E})$$

is Fredholm,

(iii) there exists an admissible spectral cut $r \in \mathbb{R}$ and

$$P - \chi^{+}(\mathrm{A}_r) : \mathrm{L}^2(\partial\mathcal{M};\mathcal{E}) \to \mathrm{L}^2(\partial\mathcal{M};\mathcal{E})$$

is an elliptic classical pseudo of order zero.

In particular, if $\mathrm{D}_B u$ is smooth, then u is smooth up to the boundary.

APS in the general setting

Given an invertible adapted boundary operator A, the boundary condition

$$B_{\mathrm{APS}} := \chi^{-}(\mathrm{A})\mathrm{H}^{\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})$$

is elliptic and pseudo-local.

If $\mathcal{M}$ is compact, then $\mathrm{D}_{B_{\mathrm{APS}}}$ is Fredholm.

Index formula? - Big open question.

Ingredients of the proof

Geometric reduction to the "model" operator $\mathrm{D}_0 = \sigma_0(\partial_t + \mathrm{A})$:

Lemma (Lemma 4.1 (Bär-Ballmann))

On the cylinder $Z_{[0,T)}$,

$$\mathrm{D} = \sigma_t(\partial_t + \mathrm{A} + R_t),$$

for any adapted boundary operator A *for* D. *The remainder term* R_t *is a* Ψ*DO of order at most one and its coefficients depend smoothly on* t. *Moreover,*

$$\|R_t u\|_{\mathrm{L}^2(\partial\mathcal{M};\mathcal{E})} \lesssim t\|\mathrm{A}u\|_{\mathrm{L}^2(\partial\mathcal{M};\mathcal{E})} + \|u\|_{\mathrm{L}^2(\partial\mathcal{M};\mathcal{E})}$$

for $u \in \mathrm{C}^\infty(\partial\mathcal{M};\mathcal{E})$.

Associated sectorial operators and functional calculus

- Let $\mathrm{sgn}(\mathrm{A}_r) := \chi^+(\mathrm{A}_r) - \chi^-(\mathrm{A}_r)$.
- Define $|\mathrm{A}_r| := \mathrm{A}_r \,\mathrm{sgn}(\mathrm{A}_r)$.
- $|\mathrm{A}_r|$ is invertible ω_r-sectorial.
- ΨDOdifferential calculus: $\mathrm{dom}(|\mathrm{A}_r|) = \mathrm{dom}(|\mathrm{A}_r|^*) = \mathrm{H}^1(\partial\mathcal{M};\mathcal{E})$.
- Theorem of Auscher-McIntosh-Nahmod:$|\mathrm{A}_r|$ has an H^∞ functional calculus. I.e.,

$$\int_0^\infty \|\psi(t|\mathrm{A}_r|)u\|^2 \,\frac{dt}{t} \simeq \|u\|^2, \quad \forall u \in \mathrm{L}^2(\partial\mathcal{M};\mathcal{E})$$

for some (and equivalently, all) holomorphic $0 \not\equiv \psi : S^o_\mu \to \mathbb{C}$ with some $C, \alpha > 0$ such that

$$|\psi(\zeta)| \leq C \min\left\{|\zeta|^\alpha, |\zeta|^{-\alpha}\right\}.$$

Role of the H^∞ calculus

Assume $M := [0,\infty) \times \partial\mathcal{M}$, $\mathrm{D} := \sigma_0(\partial_t + \mathrm{A})$. Extension operator $\mathscr{E} : \mathrm{C}_c^\infty(\partial\mathcal{M};\mathcal{E}) \to \mathrm{dom}(\mathrm{D}_{\max})$

$$\mathscr{E}v := \mathrm{e}^{-t|\mathrm{A}|}v \quad = \mathrm{e}^{-t|\mathrm{A}|}v_+ + \mathrm{e}^{-t|\mathrm{A}|}v_-, \quad v_\pm := \chi^\pm(\mathrm{A})v.$$

Show:

$$\|\mathscr{E}v\|_D^2 = \|\mathscr{E}v\|^2_{\mathrm{L}^2(\mathcal{M};\mathcal{E})} + \|\mathrm{D}\mathscr{E}v\|^2_{\mathrm{L}^2(\mathcal{M};\mathcal{E})} \lesssim \|v\|^2_{\check{\mathrm{H}}(\mathrm{A})} = \|v_-\|^2_{\mathrm{H}^{\frac{1}{2}}} + \|v_+\|^2_{\mathrm{H}^{-\frac{1}{2}}}.$$

Inhomogeneous part:

$$\begin{aligned}\|\mathscr{E}v\|^2_{\mathrm{L}^2(\mathcal{M};\mathcal{E})} &= \int_0^\infty \|\mathrm{e}^{-t|\mathrm{A}|}v\|^2_{\mathrm{L}^2(\partial\mathcal{M};\mathcal{E})}\, dt \\ &= \int_0^\infty \|t^{\frac{1}{2}}|\mathrm{A}|^{\frac{1}{2}}\mathrm{e}^{-t|\mathrm{A}|}|\mathrm{A}|^{-\frac{1}{2}}v\|^2_{\mathrm{L}^2(\partial\mathcal{M};\mathcal{E})}\frac{dt}{t} \\ &\simeq \||\mathrm{A}|^{-\frac{1}{2}}v\|^2_{\mathrm{L}^2(\partial\mathcal{M};\mathcal{E})} \simeq \|v\|^2_{\mathrm{H}^{-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})}.\end{aligned}$$

Firstly,

$$\begin{aligned}
\mathrm{D}\mathscr{E}v_- &= \sigma_0(\partial_t + \mathrm{A})\mathscr{E}v_- \\
&= \sigma_0(\partial_t + |\mathrm{A}|\,\mathrm{sgn}(\mathrm{A}))\chi^-(\mathrm{A})\mathscr{E}v_- \\
&= \sigma_0(\partial_t - |\mathrm{A}|)\mathrm{e}^{-t|\mathrm{A}|}v_- \\
&= -2\sigma_0|\mathrm{A}|\mathrm{e}^{-t|\mathrm{A}|}v_-.
\end{aligned}$$

Then,

$$\begin{aligned}
\|\mathrm{D}\mathscr{E}v_-\|^2_{\mathrm{L}^2(\mathcal{M};\mathcal{E})} &= 4\int_0^\infty \|\sigma_0|\mathrm{A}|\mathrm{e}^{-t|\mathrm{A}|}v_-\|^2_{\mathrm{L}^2(\partial\mathcal{M};\mathcal{E})}\ dt \\
&\simeq \int_0^\infty \|t^{\frac{1}{2}}|\mathrm{A}|^{\frac{1}{2}}\mathrm{e}^{-t|\mathrm{A}|}|\mathrm{A}|^{\frac{1}{2}}v_-\|^2_{\mathrm{L}^2(\partial\mathcal{M};\mathcal{E})}\ \frac{dt}{t} \\
&\simeq \||\mathrm{A}|^{\frac{1}{2}}v_-\|^2_{\mathrm{L}^2(\partial\mathcal{M};\mathcal{E})} \simeq \|v_-\|^2_{\mathrm{H}^{\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})}.
\end{aligned}$$

Combining with $\mathrm{D}\mathscr{E}v_+ = \sigma_0(\partial_t + \mathrm{A})v_+ = 0$, obtain:

$$\|\mathscr{E}v\|^2_D \lesssim \|v_-\|^2_{\mathrm{H}^{\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})} + \|v_+\|^2_{\mathrm{H}^{-\frac{1}{2}}(\partial\mathcal{M};\mathcal{E})} = \|v\|^2_{\check{\mathrm{H}}(\mathrm{A})}.$$

Maximal regularity

Banach-valued Cauchy problem: $f \in \mathrm{L}^2(Z_{[0,\rho]}; \mathcal{E})$,

$$\partial_t W(t;f) + |\mathrm{A}_r| W(t;f) = f(t)$$
$$\lim_{t \to 0} W(t;f) = 0.$$

Solution given by:

$$W(t;f) = \int_0^t \mathrm{e}^{-(t-s)|\mathrm{A}_r|} f(s)\ ds.$$

Key estimate - maximal regularity:

$$\int_0^\rho \|\partial_t W(t;f)\|^2_{\mathrm{L}^2(\partial\mathcal{M};\mathcal{E})} dt + \int_0^\rho \||\mathrm{A}_r| W(t;f)\|^2_{\mathrm{L}^2(\partial\mathcal{M};\mathcal{E})} \lesssim \int_0^\rho \|f(t)\|^2_{\mathrm{L}^2(\partial\mathcal{M};\mathcal{E})}.$$

Define:

$$S_{0,r}u(t) = \int_0^t \mathrm{e}^{-(t-s)|\mathrm{A}_r|}\chi^+(A_r)u(s)\ ds$$
$$- \int_t^\rho \mathrm{e}^{-(s-t)|A_r|}\chi^-(A_r)u(s)\ ds$$

Let $(C_\rho u)(s) = u(\rho - s)$,

(i) $(\partial_t + \mathrm{A}_r)\, S_{0,r} = \mathrm{I}$.

(ii) $S_{0,r} : \mathrm{H}^{\mathrm{k}}(Z_{[0,\rho]};\mathcal{E}) \to \mathrm{H}^{\mathrm{k}+1}(Z_{[0,\rho]};\mathcal{E})$ bounded.

(iii) whenever $u(\rho) = 0$ (or $\mathrm{spt}\ u \subset Z_{[0,\rho)}$),

$$\big(\ \mathrm{I} - S_{0,r}\,(\partial_t + \mathrm{A}_r)\ \big)u = \mathrm{e}^{-t|A_r|}\big(\ \chi^+(A_r)u(0)\ \big).$$

Future program

Using current viewpoint as a *template*:

- *General order case* [Magnus Goffeng (Lund), Hemanth Saratchandran (Adelaide)]

  Seeley (1965) gives a "Czech" space: mixed-order Sobolev spaces via Calderón projectors.

- *Lipschitz boundary* [Andreas Rosén (Gothenburg) and Magnus Goffeng (Lund)]

  Quadratic estimates to be proved directly - methods from the Kato square root problem:
  dyadic decomposition, off-diagonal estimates (automatic for first-order), reduce to local $T(b)$ theorem and Carleson measure estimate.

- *Nonlinear problems -* L^p *estimates*

$$\|\mathscr{E}u\|^p_{\mathrm{L}^p(\mathcal{M};\mathcal{E})} = \int_0^\infty \int_{\partial\mathcal{M}} |\mathrm{e}^{-t|A|}u|^p \; d\mu_{\partial\mathcal{M}} \; dt = \int_0^\infty \|t^{\frac{1}{p}}\mathrm{e}^{-t|A|}u\|^p \; \frac{dt}{t}.$$

  Leads to: Besov space data on the boundary.
  Guess: $\check{\mathrm{H}}_p := \chi^-(\mathrm{A})\mathbb{B}^{1-\frac{1}{p}}_{p,p}(\partial\mathcal{M};\mathcal{E}) \oplus \chi^+(\mathrm{A})\mathbb{B}^{-\frac{1}{p}}_{p,p}(\partial\mathcal{M};E)$.

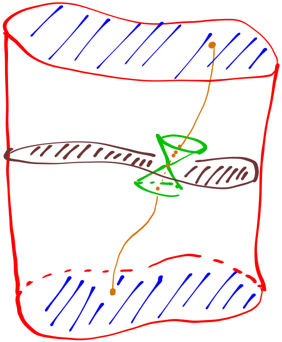
- η*-invariants for non-Dirac type operators*

  Recall:

$$\eta_{\mathrm{A}}(s) := \sum_{\lambda\in\mathrm{spec}(\mathrm{A})\setminus\{0\}} \frac{\mathrm{sgn}(\lambda)}{|\lambda|^s}.$$

  $\eta_{\mathrm{A}}(0)$ defined? $s \mapsto \eta_{\mathrm{A}}(s)$ analytic?
  H^∞-functional calculus and harmonic analysis: alternative perspective of Atiyah-Patodi-Singer.

- *Lorentzian manifolds with spacelike boundary*



(Σ_1, g_1)

$M = [0,1] \times \Sigma$

$g = -N^2 dt + g_t$

(Σ_0, g_0)

Extension operator: wave propagation operator.
Bisectoriality is a problem: need strip type or similar.
Key idea: identify the right function spaces.