Square roots of perturbed sub-elliptic operators on Lie groups

Lashi Bandara

(Joint work with Tom ter Elst, Auckland and Alan McIntosh, ANU)

Centre for Mathematics and its Applications
Australian National University

August 13, 2012

POSTI/Mprime Seminar

University of Calgary

Lie groups

Let $\mathcal{G}$ be a Lie group of dimension n and $\mathfrak{g}$ is Lie algebra.

We let $d\mu$ denote the left invariant *Haar* measure.

Algebraic basis and vectorfields

A set $\{a_1, \ldots, a_k\} \subset \mathfrak{g}$ is an *algebraic basis* if we can recover a basis for $\mathfrak{g}$ by multi-commutation.

We assume that the $\{a_i\}$ are linearly independent.

Let A_i denote the left translation of a_i.

The vectorfields $\{A_i\}$ are linearly independent and *global*.

Distance

Theorem of Carathéodory-Chow tells us that for any two points $x, y \in \mathcal{G}$, we can find a curve $\gamma : [0,1] \to \mathcal{G}$ such that

$$\dot{\gamma}(t) = \sum_i \dot{\gamma}^i(t) A_i(\gamma(t)) \quad \in \mathrm{span}\left\{A_i(\gamma(t))\right\}.$$

The length of such a curve then is given by

$$\ell(\gamma) = \int_0^1 (\sum_i \left|\dot{\gamma}^i(t)\right|^2)^{\frac{1}{2}} \, dt$$

Define distance $d(x, y)$ as the infimum over the length of all such curves.

The measure $d\mu$ is Borel-regular with respect to d and we consider $(\mathcal{G}, d, d\mu)$ as a measure metric space.

Sub-Laplacian

Define an associated *sub-Laplacian* by:

$$\Delta = -\sum_i A_i^2.$$

This is a densely-defined, self-adjoint operator on $\mathrm{L}^2(\mathcal{G})$.

We say that a Lie group is *nilpotent* if

$$\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}],\ \mathfrak{g}_2 = [\mathfrak{g}, \mathfrak{g}_1],\ \mathfrak{g}_3 = [\mathfrak{g}_1, \mathfrak{g}_2], \ldots$$

is eventually 0. That is, there is a k such that $\mathfrak{g}_k = 0$.

On such spaces, we consider the uniformly elliptic second order operator

$$\mathrm{D}_H = -b \sum_{i,j} A_i b_{ij} A_j$$

where $b, b_{ij} \in \mathrm{L}^\infty(\mathcal{G})$.

The main theorem for nilpotent Lie groups

Theorem (B.-E.-Mc)

Let $\mathcal{G}$ be a connected nilpotent and suppose there exist $\kappa_1, \kappa_2 > 0$ such that

$$\mathrm{Re}\, b(x) \geq \kappa_1 \quad \textit{and} \quad \mathrm{Re} \int_{\mathcal{G}} \sum_{i,j} b_{ij} A_i u \overline{A_j u} \geq \kappa_2 \sum_i \|A_i u\|^2$$

for almost all $x \in \mathcal{G}$ and $u \in \mathrm{H}^1(\mathcal{G})$. Then,

(i) $\mathcal{D}(\sqrt{\mathrm{D}_H}) = \cap_{i=1}^m \mathcal{D}(A_i) = \mathrm{H}^1(\mathcal{G})$, *and*

(ii) $\|\sqrt{\mathrm{D}_H} u\| \simeq \sum_{i=1}^m \|A_i u\|$ *for all $u \in \mathrm{H}^1(\mathcal{G})$.*

Stability

Theorem (B.-E.-Mc)

Let $0 < \eta_i < \kappa_i$ *and suppose that* $\tilde{b}, \tilde{b}_{ij} \in \mathrm{L}^\infty(\mathcal{G})$ *such that* $\|\tilde{b}\|_\infty \leq \eta_1$ *and* $\|(\tilde{b}_{ij})\|_\infty \leq \eta_2$*. Then,*

$$\|\sqrt{\mathrm{D}_H}u - \sqrt{\tilde{\mathrm{D}}_H}u\| \lesssim (\|\tilde{b}\|_\infty + \|(\tilde{b}_{ij})\|_\infty) \sum_{i=1}^{k} \|A_i u\|,$$

for $u \in \mathrm{H}^1(\mathcal{G})$ *and where*

$$\tilde{\mathrm{D}}_H = (b + \tilde{b}) \sum_{i,j=1}^{k} A_i(b_{ij} + \tilde{b}_{ij})A_j.$$

Operator theory

Procedure in [AKMc]. Let $\mathscr{H}$ be a Hilbert space.

(H1) The operator $\Gamma : \mathcal{D}(\Gamma) \subset \mathscr{H} \to \mathscr{H}$ is closed, densely-defined and *nilpotent* ($\Gamma^2 = 0$).

(H2) The operators $B_1, B_2 \in \mathcal{L}(\mathscr{H})$ satisfy

$$\mathrm{Re}\,\langle B_1 u, u\rangle \geq \kappa_1 \|u\| \qquad \text{whenever } u \in \mathcal{R}(\Gamma^*)$$
$$\mathrm{Re}\,\langle B_2 u, u\rangle \geq \kappa_2 \|u\| \qquad \text{whenever } u \in \mathcal{R}(\Gamma)$$

where $\kappa_1, \kappa_2 > 0$ are constants.

(H3) The operators B_1, B_2 satisfy $B_1 B_2(\mathcal{R}(\Gamma)) \subset \mathcal{N}(\Gamma)$ and $B_2 B_1(\mathcal{R}(\Gamma^*)) \subset \mathcal{N}(\Gamma^*)$.

Let $\Gamma_B^* = B_1 \Gamma^* B_2$, $\Pi_B = \Gamma + \Gamma_B^*$, and $\Pi = \Gamma + \Gamma^*$.

Harmonic analysis and Kato square root type estimates

Theorem (Kato square root type estimate)

Suppose that (Γ, B_1, B_2) *satisfy (H1)-(H3) and*

$$\int_0^\infty \|t\Pi_B(1 + t^2\Pi_B^2)^{-1}u\|^2 \ \frac{dt}{t} \simeq \|u\|^2$$

for all $u \in \overline{\mathcal{R}(\Pi_B)} \subset \mathscr{H}$. *Then,*

(i) *There is a spectral decomposition* $\mathscr{H} = \mathcal{N}(\Pi_B) \oplus E_B^+ \oplus E_B^-$, *where* $E_B^\pm$ *are spectral subspaces and the sum is in general non-orthogonal, and*

(ii) $\mathcal{D}(\Gamma) \cap \mathcal{D}(\Gamma_B^*) = \mathcal{D}(\Pi_B) = \mathcal{D}(\sqrt{\Pi_B^2})$ *with*

$$\|\Gamma u\| + \|\Gamma_B u\| \simeq \|\Pi_B u\| \simeq \|\sqrt{\Pi_B^2} u\| \text{ for all } u \in \mathcal{D}(\Pi_B).$$

Homogeneous conditions

(H4) Let $\mathcal{X}$ be a complete, connected metric space and μ a Borel-regular measure on $\mathcal{X}$ that is *doubling*. Then set $\mathscr{H} = \mathrm{L}^2(\mathcal{X}, \mathbb{C}^N; d\mu)$.

(H5) The operators B_i are matrix-valued pointwise multiplication operators such that the function $x \mapsto B_i(x)$ are $\mathrm{L}^\infty(\mathcal{X}, \mathcal{L}(\mathbb{C}^N))$.

(H6) For every bounded Lipschitz function $\xi : \mathcal{X} \to \mathbb{C}$, multiplication by ξ preserves $\mathcal{D}(\Gamma)$ and $\mathrm{M}_\xi = [\Gamma, \xi I]$ is a multiplication operator. Furthermore, there exists a constant $m > 0$ such that $|\mathrm{M}_\xi(x)| \leq m\, |\mathrm{Lip}\, \xi(x)|$ for almost all $x \in \mathcal{X}$.

(H7) For each open ball B, we have

$$\int_B \Gamma u \; d\mu = 0 \quad \text{and} \quad \int_B \Gamma^* v \; d\mu = 0$$

for all $u \in \mathcal{D}(\Gamma)$ with $\mathrm{spt}\ u \subset B$ and for all $v \in \mathcal{D}(\Gamma^*)$ with $\mathrm{spt}\ v \subset B$.

(H8) -1 (Poincaré hypothesis)
There exists $C' > 0$, $c > 0$ and an operator
$\Xi : \mathcal{D}(\Xi) \subset \mathrm{L}^2(\mathcal{X}, \mathbb{C}^N) \to \mathrm{L}^2(\mathcal{X}, \mathbb{C}^M)$ such that $\mathcal{D}(\Pi) \cap \mathcal{R}(\Pi) \subset \mathcal{D}(\Xi)$ and

$$\int_B |u - u_B|^2 \, d\mu \leq C' r^2 \int_B |\Xi u|^2 \, d\mu$$

for all balls $B = B(x, r)$ and $u \in \mathcal{D}(\Pi) \cap \mathcal{R}(\Pi)$.

-2 (Coercivity hypothesis)
There exists $\tilde{C} > 0$ such that for all $u \in \mathcal{D}(\Pi) \cap \mathcal{R}(\Pi)$,

$$\|\Xi u\| \leq \tilde{C} \|\Pi u\|.$$

This is slightly different from (H8) in [Bandara].

Theorem (B.)

Let $\mathcal{X}$, (Γ, B_1, B_2) *satisfy (H1)-(H8). Then,* Π_B *satisfies the quadratic estimate*

$$\int_0^\infty \|t\Pi_B(1 + t^2\Pi_B^2)^{-1}u\|^2 \ \frac{dt}{t} \simeq \|u\|^2$$

for all $u \in \overline{\mathcal{R}(\Pi_B)} \subset \mathrm{L}^2(\mathcal{X}, \mathbb{C}^N)$.

Geometric setup

Define the bundle $\mathcal{W} = \operatorname{span}\{A_i\} \subset \mathrm{T}\mathcal{G}$ and complexify it.

Equip $\mathcal{W}$ with the inner product $\mathrm{h}(A_i, A_j) = \delta_{ij}$.

Equip $\mathcal{G}$ with the *sub-connection*

$$\nabla f = A_k f A^k$$

where $A^k = {A_k}^* \in \mathcal{W}^*$.

Equip $\mathcal{W}$ with the sub-connection

$$\tilde{\nabla}(u^i A_i) = (\nabla u_i) \otimes A_i$$

We have that $\mathcal{W} \cong \mathbb{C}^k$ and $\mathrm{L}^2(\mathcal{G}) \oplus \mathrm{L}^2(\mathcal{W}) \cong \mathrm{L}^2(\mathbb{C}^{k+1})$.

Operator setup

Define: $\Gamma : \mathcal{D}(\Gamma) \subset \mathrm{L}^2(\mathcal{G}) \oplus \mathrm{L}^2(\mathcal{W}^*) \to \mathrm{L}^2(\mathcal{G}) \oplus \mathrm{L}^2(\mathcal{W}^*)$ by

$$\Gamma = \begin{pmatrix} 0 & 0 \\ \nabla & 0 \end{pmatrix}.$$

Then,

$$\Gamma^* = \begin{pmatrix} 0 & -\operatorname{div} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \Pi = \begin{pmatrix} 0 & -\operatorname{div} \\ \nabla & 0 \end{pmatrix},$$

where we define $\operatorname{div} = -\nabla^*$.

Let $B = (b_{ij})$. Then, define

$$B_1 = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}.$$

Proof of the homogeneous problem

Set $\mathcal{X} = \mathcal{G}$ and $\mathscr{H} = \mathrm{L}^2(\mathcal{G}) \oplus \mathrm{L}^2(\mathcal{W})$.

(H1) The sub-connection ∇ is densely-defined and closed and so is Γ. Nilpotency is by construction.

(H2) By accretivity assumptions.

(H3) By construction.

Proof (cont.)

(H4) The measure $d\mu$ is Borel-regular and the nilpotency of $\mathcal{G}$ implies that it is doubling.

(H5) By assumption.

(H6) It is an easy fact that for all bounded Lipschitz $\xi : \mathcal{G} \to \mathbb{C}$,

$$|\mathrm{M}_\xi(x)| = |[\Gamma, \xi(x) I]| = |\nabla \xi(x)| \leq k \,\mathrm{Lip}\, \xi(x)$$

for for almost all $x \in \mathcal{G}$.

(H7) By the left invariance of the measure $d\mu$.

Proof (cont.)

(H8) -1 The nilpotency of $\mathcal{G}$ implies the following Poincaré inequality

$$\int_B |f - f_B|^2 \ d\mu \lesssim r^2 \int_B |\nabla f|^2 \ d\mu$$

for all balls B, and $f \in \mathrm{C}^\infty(B)$. See [SC, (P.1), p118].
Define $\Xi u = (\nabla u_1, \tilde{\nabla} u_2)$.

-2 The crucial fact needed here is the regularity result [ERS, Lemma 4.2] which gives

$$\|A_i A_j f\| \lesssim \|\Delta f\|$$

for $f \in \mathrm{H}^2(\mathcal{G}) = \mathcal{D}(\Delta)$.

Inhomogeneous problem

For general Lie groups, we need to consider operators with lower order terms.

Let $b, b_{ij}, c_k, d_k, e \in \mathrm{L}^{\infty}(\mathcal{G})$. Define the following uniformly elliptic second order operator

$$\mathrm{D}_I = -b \sum_{ij=1}^{m} A_i b_{ij} A_j u - b \sum_{i=1}^{m} A_i c_i u - b \sum_{i=1}^{m} d_i A_i u - beu.$$

Theorem (B.-E.-Mc)

Let $\mathcal{G}$ be a connected Lie group and suppose there exists $\kappa_1, \kappa_2 > 0$ such that

$$\operatorname{Re} b(x) \geq \kappa_1,$$

$$\operatorname{Re} \int_{\mathcal{G}} \left(eu + \sum_{i=1}^{m} d_i A_i u \right) \overline{u} + \sum_{i=1}^{m} \left(c_i u + \sum_{j=1}^{m} b_{ij} A_j u \right) \overline{A_i u} \, d\mu \geq \kappa_2 \left(\|u\|^2 + \sum_{i=1}^{m} \|A_i u\|^2 \right)$$

for almost all $x \in \mathcal{G}$ and $u \in \mathrm{H}^1(\mathcal{G})$. Then,

(i) $\mathcal{D}(\sqrt{\mathrm{D}_I}) = \cap_{i=1}^{m} \mathcal{D}(A_i) = \mathrm{H}^1(\mathcal{G})$, *and*

(ii) $\|\sqrt{\mathrm{D}_I} u\| \simeq \|u\| + \sum_{i=1}^{m} \|A_i u\|$ *for all $u \in \mathrm{H}^1(\mathcal{G})$.*

Spaces of exponential growth

$(\mathcal{X}, d, \mu)$ an exponentially locally doubling measure metric space. That is: there exist $\kappa, \lambda \geq 0$ and constant $C \geq 1$ such that

$$0 < \mu(B(x, tr)) \leq C t^{\kappa} e^{\lambda t r} \mu(B(x, r))$$

for all $x \in \mathcal{X}$, $r > 0$ and $t \geq 1$.

Changes to (H7) and (H8)

The following (H7) from [Morris]:

(H7) There exist $c > 0$ such that for all open balls $B \subset \mathcal{X}$ with $r \leq 1$,

$$\left| \int_B \Gamma u \; d\mu \right| \leq c\mu(B)^{\frac{1}{2}} \|u\| \quad \text{and} \quad \left| \int_B \Gamma^* v \; d\mu \right| \leq c\mu(B)^{\frac{1}{2}} \|v\|$$

for all $u \in \mathcal{D}(\Gamma),\ v \in \mathcal{D}(\Gamma^*)$ with $\mathrm{spt}\ u,\ \mathrm{spt}\ v \subset B$.

We introduce the following *local* (H8):

(H8) -1 (Local Poincaré hypothesis)
There exists $C' > 0$, $c > 0$ and an operator $\Xi : \mathcal{D}(\Xi) \subset \mathrm{L}^2(\mathcal{X}, \mathbb{C}^N) \to \mathrm{L}^2(\mathcal{X}, \mathbb{C}^M)$ such that $\mathcal{D}(\Pi) \cap \mathcal{R}(\Pi) \subset \mathcal{D}(\Xi)$ and

$$\int_B |u - u_B|^2 \, d\mu \leq C' r^2 \int_B (|\Xi u|^2 + |u|^2) \, d\mu$$

for all balls $B = B(x, r)$ and for $u \in \mathcal{D}(\Pi) \cap \mathcal{R}(\Pi)$.

-2 (Coercivity hypothesis)
There exists $\tilde{C} > 0$ such that for all $u \in \mathcal{D}(\Pi) \cap \mathcal{R}(\Pi)$,

$$\|\Xi u\| + \|u\| \leq \tilde{C} \|\Pi u\|.$$

Theorem (Morris)

Let $\mathcal{X}$, (Γ, B_1, B_2) satisfy (H1)-(H8). Then, Π_B satisfies the quadratic estimate

$$\int_0^\infty \|t\Pi_B(1 + t^2\Pi_B^2)^{-1}u\|^2 \ \frac{dt}{t} \simeq \|u\|^2$$

for all $u \in \overline{\mathcal{R}(\Pi_B)} \subset \mathrm{L}^2(\mathcal{X}, \mathbb{C}^N)$.

Setup

Set $\mathcal{X} = \mathcal{G}$ and $\mathscr{H} = \mathrm{L}^2(\mathcal{G}) \oplus \mathrm{L}^2(\mathcal{G}) \oplus \mathrm{L}^2(\mathcal{W}) \cong \mathrm{L}^2(\mathbb{C}^{k+2})$.

Let $S = (I, \nabla)$, $S^* = [I - \mathrm{div}]$.

Let

$$\Gamma = \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix},\ \Gamma^* = \begin{pmatrix} 0 & S^* \\ 0 & 0 \end{pmatrix},\ \text{and } \Pi^* = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}.$$

Let $\tilde{B}_{00} = e,\ \tilde{B}_{10} = (c_1, \ldots, c_m),\ \tilde{B}_{01} = (d_1, \ldots, d_m)^{\mathrm{tr}},\ \tilde{B}_{11} = (b_{ij})$, and $B = (\tilde{B}_{ij})$.

Then, we can write

$$B_1 = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}.$$

Proof

The proofs of (H1)-(H6) are similar to the homogeneous situation.

(H7) The proof is the same as the homogeneous situation except the lower order term introduces the term $\mu(B)^{\frac{1}{2}} \|u\|$ on the right.

(H8) -1 The existence of a local Poincaré inequality is guaranteed by [ER2, Proposition 2.4]:

$$\int_B |f - f_B|^2 \ d\mu \lesssim r^2 \int_B (|\nabla f|^2 + |f|^2) \ d\mu$$

for all balls $B = B(x, r)$ and where $f \in \mathrm{C}^\infty(B)$.
Define $\Xi u = (\nabla u_1, \nabla u_2, \tilde{\nabla} u_3)$.

-2 The crucial fact needed here is the regularity result in [ER, Theorem 7.2],

$$\|A_i A_j u\|^2 \lesssim \|\Delta u\|^2 + \|u\|^2$$

for $u \in \mathrm{H}^2(\mathcal{G}) = \mathcal{D}(\Delta)$.

References I

[AKMc] Andreas Axelsson, Stephen Keith, and Alan McIntosh, *Quadratic estimates and functional calculi of perturbed Dirac operators*, Invent. Math. **163** (2006), no. 3, 455–497.

[Bandara] L. Bandara, *Quadratic estimates for perturbed Dirac type operators on doubling measure metric spaces*, ArXiv e-prints (2011).

[Morris] Andrew Morris, *Local hardy spaces and quadratic estimates for Dirac type operators on Riemannian manifolds*, Ph.D. thesis, Australian National University, 2010.

[SC] L. Saloff-Coste and D. W. Stroock, *Opérateurs uniformément sous-elliptiques sur les groupes de Lie*, J. Funct. Anal. **98** (1991), no. 1, 97–121. MR 1111195 (92k:58264)

[ER] A. F. M. ter Elst and Derek W. Robinson, *Subelliptic operators on Lie groups: regularity*, J. Austral. Math. Soc. Ser. A **57** (1994), no. 2, 179–229. MR 1288673 (95e:22019)

References II

[ER2] A. F. M. Ter Elst and Derek W. Robinson, *Second-order subelliptic operators on Lie groups. I. Complex uniformly continuous principal coefficients*, Acta Appl. Math. **59** (1999), no. 3, 299–331. MR 1744755 (2001a:35043)

[ERS] A. F. M. ter Elst, Derek W. Robinson, and Adam Sikora, *Heat kernels and Riesz transforms on nilpotent Lie groups*, Colloq. Math. **74** (1997), no. 2, 191–218. MR 1477562 (99a:35029)