

L^∞ coefficient operators and non-smooth Riemannian metrics

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Setup

Let \mathcal{M} be a smooth, complete Riemannian manifold with metric g , Levi-Civita connection ∇ , and volume measure $d\mu$.

Write $\operatorname{div} = -\nabla^*$ in L^2 and let $S = (I, \nabla)$.

Consider the following *uniformly elliptic* second order differential operator $L_A : \mathcal{D}(L_A) \subset L^2(\mathcal{M}) \rightarrow L^2(\mathcal{M})$ defined by

$$L_A u = a S^* A S u = -a \operatorname{div}(A_{11} \nabla u) - a \operatorname{div}(A_{10} u) + a A_{01} \nabla u + a A_{00} u.$$

where a and $A = (A_{ij})$ are L^∞ multiplication operators.

That is, that there exist $\kappa_1, \kappa_2 > 0$ such that

$$\operatorname{Re} \langle av, v \rangle \geq \kappa_1 \|v\|^2, \quad v \in L^2$$

$$\operatorname{Re} \langle ASu, Su \rangle \geq \kappa_2 (\|u\|^2 + \|\nabla u\|^2), \quad u \in H^1$$

The problem

The *Kato square root problem on manifolds* is to determine when the following holds:

$$\left\{ \begin{array}{l} \mathcal{D}(\sqrt{L_A}) = H^1(\mathcal{M}) \\ \|\sqrt{L_A}u\| \simeq \|\nabla u\| + \|u\| = \|u\|_{H^1}, \quad u \in H^1(\mathcal{M}) \end{array} \right.$$

Theorem (B.-Mc [BMc])

Let (\mathcal{M}, g) be a smooth, complete Riemannian manifold $|\text{Ric}| \leq C$ and $\text{inj}(\mathcal{M}) \geq \kappa > 0$. Suppose there exist $\kappa_1, \kappa_2 > 0$ such that

$$\begin{aligned}\text{Re} \langle av, v \rangle &\geq \kappa_1 \|v\|^2 \\ \text{Re} \langle ASu, Su \rangle &\geq \kappa_2 \|u\|_{\text{H}^1}^2\end{aligned}$$

for $v \in L^2(\mathcal{M})$ and $u \in \text{H}^1(\mathcal{M})$. Then, $\mathcal{D}(\sqrt{L_A}) = \mathcal{D}(\nabla) = \text{H}^1(\mathcal{M})$ and $\|\sqrt{L_A}u\| \simeq \|\nabla u\| + \|u\| = \|u\|_{\text{H}^1}$ for all $u \in \text{H}^1(\mathcal{M})$.

Stability

Theorem (B.-Mc [BMc])

Let (\mathcal{M}, g) be a smooth, complete Riemannian manifold with $|\text{Ric}| \leq C$ and $\text{inj}(\mathcal{M}) \geq \kappa > 0$. Suppose that there exist $\kappa_1, \kappa_2 > 0$ such that

$$\begin{aligned}\text{Re} \langle av, v \rangle &\geq \kappa_1 \|v\|^2 \\ \text{Re} \langle ASu, Su \rangle &\geq \kappa_2 \|u\|_{H^1}^2\end{aligned}$$

for $v \in L^2(\mathcal{M})$ and $u \in H^1(\mathcal{M})$. Then for every $\eta_i < \kappa_i$, whenever $\|\tilde{a}\|_\infty \leq \eta_1$, $\|\tilde{A}\|_\infty \leq \eta_2$, the estimate

$$\left\| \sqrt{L_A} u - \sqrt{L_{A+\tilde{A}}} u \right\| \lesssim (\|\tilde{a}\|_\infty + \|\tilde{A}\|_\infty) \|u\|_{H^1}$$

holds for all $u \in H^1(\mathcal{M})$. The implicit constant depends in particular on A, a and η_i .

History of the problem

In the 1960's, Kato considered the following *abstract evolution equation*

$$\frac{du}{dt} + A(t)u = f(t), \quad t \in [0, T].$$

on a Hilbert space \mathcal{H} .

In 1962, Kato showed in [Kato61] that for $0 \leq \alpha < 1/2$ and $A(t)$ *maximal accretive* that

$$\begin{aligned} \mathcal{D}(A(t)^\alpha) &= \mathcal{D}(A(t)^{* \alpha}) = \mathcal{D}_\alpha = \text{const}, \text{ and} \\ \|A(t)^\alpha u\| &\simeq \|A(t)^{* \alpha} u\|, \quad u \in \mathcal{D}_\alpha. \end{aligned} \tag{K_\alpha}$$

Counter examples were known for $\alpha > 1/2$.

Kato asked two questions:

(K1) Does (K_α) hold for $\alpha = 1/2$?

(K2) For the case $\omega = 0$, we know (K1) is automatically true, but is

$$\left\| \frac{d}{dt} \sqrt{A(t)} u \right\| \lesssim \|u\|$$

for $u \in \mathcal{W}$?

In 1972, McIntosh provided a counter example in [Mc72] demonstrating that (K1) is false in such generality.

In 1982, McIntosh showed that (K2) also did not hold in general in [Mc82].

The *Kato square root problem* then became the following.

Suppose $A \in L^\infty$ is a pointwise matrix multiplication operator satisfying the following ellipticity condition:

$$\operatorname{Re} \langle A \nabla u, \nabla u \rangle \geq \kappa \|\nabla u\|^2, \quad \text{for some } \kappa > 0.$$

Is it then true that

$$\begin{aligned} \mathcal{D}(\sqrt{\operatorname{div} A \nabla}) &= H^1(\mathbb{R}^n) \\ \left\| \sqrt{\operatorname{div} A \nabla} u \right\| &\simeq \|\nabla u\| \end{aligned} \tag{K1}$$

This was answered in the positive in 2002 by Pascal Auscher, Steve Hofmann, Michael Lacey, Alan McIntosh and Phillippe Tchamitchian in [AHLMcT].

Axelsson (Rosén)-Keith-McIntosh framework

In [AKMc], the authors created a *first-order* framework using the language of Dirac type operators to study such problems.

- (H1) Let Γ be a densely-defined, closed, nilpotent operator on a Hilbert space \mathcal{H} ,
- (H2) Suppose that $B_1, B_2 \in \mathcal{L}(\mathcal{H})$ such that there exist $\kappa_1, \kappa_2 > 0$ satisfying

$$\operatorname{Re} \langle B_1 u, u \rangle \geq \kappa_1 \|u\|^2 \quad \text{and} \quad \operatorname{Re} \langle B_2 v, v \rangle \geq \kappa_2 \|v\|^2$$

for $u \in \mathcal{R}(\Gamma^*)$ and $v \in \mathcal{R}(\Gamma)$,

- (H3) The operators B_1, B_2 satisfy $B_1 B_2 \mathcal{R}(\Gamma) \subset \mathcal{N}(\Gamma)$ and $B_2 B_1 \mathcal{R}(\Gamma^*) \subset \mathcal{N}(\Gamma^*)$.

Let $\Gamma_B^* = B_1 \Gamma^* B_2$, $\Pi_B = \Gamma + \Gamma_B^*$ and $\Pi = \Gamma + \Gamma^*$.

Quadratic estimates and Kato type problems

Proposition

If (H1)-(H3) are satisfied and

$$\int_0^\infty \|t\Pi_B(\mathbf{I} + t^2\Pi_B^2)^{-1}u\|^2 \frac{dt}{t} \simeq \|u\|^2$$

for $u \in \overline{\mathcal{R}(\Pi_B)}$, then

- (i) $\mathcal{D}(\Gamma) \cap \mathcal{D}(\Gamma_B^*) = \mathcal{D}(\Pi_B) = \mathcal{D}(\sqrt{\Pi_B^2})$, and
- (ii) $\|\Gamma u\| + \|\Gamma_B u\| \simeq \|\Pi_B u\| \simeq \|\sqrt{\Pi_B^2} u\|$, for all $u \in \mathcal{D}(\Pi_B)$.

Set $\mathcal{H} = L^2(\mathcal{M}) \oplus (L^2(\mathcal{M}) \oplus L^2(T^*\mathcal{M}))$.

Define

$$\Gamma_g = \begin{bmatrix} 0 & 0 \\ S & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}.$$

Then,

$$\Gamma_g^* = \begin{bmatrix} 0 & S^* \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \Pi_B^2 = \begin{bmatrix} L_A & 0 \\ 0 & * \end{bmatrix}$$

and

$$\Pi_{B,g}(u, 0) = (0, u, \nabla u) \quad \text{and} \quad \sqrt{\Pi_{B,g}^2}(u, 0) = (\sqrt{L_A}u, 0).$$

Theorem 1 and 2 are a direct consequence of the following.

Proposition

Let \mathcal{M} be a smooth, complete manifold with smooth metric g . Suppose there exist $\eta, \kappa > 0$ such that $|\text{Ric}_g| \leq \eta$ and $\text{inj}(\mathcal{M}, g) \geq \kappa$. Furthermore, suppose that there exist κ_1, κ_2 such that

$$\text{Re} \langle av, v \rangle \geq \kappa_1 \|v\|^2, \quad v \in L^2(\mathcal{M}, g)$$

$$\text{Re} \langle ASu, Su \rangle \geq \kappa_2 \|u\|_{H^1}^2, \quad u \in H^1(\mathcal{M}, g).$$

Then,

$$\int_0^\infty \|t\Pi_B(I + t^2\Pi_B^2)^{-1}u\|^2 \frac{dt}{t} \simeq \|u\|^2$$

for $u \in \overline{\mathcal{R}(\Pi_B)}$.

Rough metrics

We let \mathcal{M} be a smooth, complete manifold as before but now let g be a C^0 metric. Let μ_g denote the volume measure with respect to g .

Definition

Let $h \in C^0(\mathcal{T}^{(2,0)}\mathcal{M})$. Then, define

$$\|h\|_{\text{op},g} = \text{esssup}_{\mu_g(x)\text{-a.e.}} \sup_{|u|_g=|v|_g=1} |h_x(u, v)|,$$

and

$$\|h\|_{\text{op},s,g} = \sup_{x \in \mathcal{M}} \sup_{|u|_g=|v|_g=1} |h_x(u, v)|.$$

We say that two C^0 metrics g and \tilde{g} are δ -close if $\|g - \tilde{g}\|_{\text{op},g} < \delta$ for $\delta > 0$.

Translating from one metric to another

Suppose that we now have a smooth metric \tilde{g} . Write $f_u(v) = \tilde{g}(u, v) - g(u, v)$. By Riesz-Representation theorem, we find that there exists B such that

$$\tilde{g}((I + B)u, v) = g(u, v).$$

That is, exactly, we can absorb the lack of regularity of g in terms of a symmetric, bounded operator in \tilde{g} . The operator $\|B\| = \|\tilde{g} - g\|_{\text{op},g}$. The change of measures

$$\theta = \frac{d\mu_g}{d\mu_{\tilde{g}}}$$

is given in terms of $I + B$.

Stability of function spaces

Proposition

Let g and \tilde{g} be two C^0 metrics on \mathcal{M} . If there exists $\delta \in [0, 1)$ such that $\|g - \tilde{g}\|_{\text{op},g} < \delta$, then

(i) the spaces $L^2(\mathcal{M}, g) = L^2(\mathcal{M}, \tilde{g})$ with

$$(1 - \delta)^{\frac{n}{4}} \|\cdot\|_{2,g} \leq \|\cdot\|_{2,\tilde{g}} \leq (1 + \delta)^{\frac{n}{4}} \|\cdot\|_{2,g},$$

(ii) the Sobolev spaces $H^1(\mathcal{M}, \tilde{g}) = H^1(\mathcal{M}, g)$ with

$$\frac{(1 - \delta)^{\frac{n}{4}}}{1 + \delta} \|\cdot\|_{H^1,g} \leq \|\cdot\|_{H^1,\tilde{g}} \leq \frac{(1 + \delta)^{\frac{n}{4}}}{1 - \delta} \|\cdot\|_{H^1,g}.$$

Π_B under a change of metric

The operator Γ_g does not change under the change of metric. However,

Proposition

$\Gamma_g^* = \Theta \Gamma_{g_1}^* C$ where Θ is a bounded multiplication operator on $L^2(\mathcal{M})$ and C is a bounded multiplication operator on $L^2(\mathcal{M}) \oplus L^2(T^*\mathcal{M})$.

Thus,

$$\Pi_{B,g} = \Gamma_g + B_1 \Gamma_g^* B_2 = \Gamma_{\tilde{g}} + B_1 \Theta \Gamma_{g_1}^* C B_2.$$

The idea is to reduce the Kato problem for $\Pi_{B,g}$ to a Kato problem for $\Pi_{\tilde{B},\tilde{g}} = \Gamma_{\tilde{g}} + \tilde{B}_1 \Gamma_{\tilde{g}}^* \tilde{B}_2$ where $\tilde{B}_1 = B_1 \Theta$ and $\tilde{B}_2 = C B_2$ but now with a smooth metric \tilde{g} .

Ellipticity

Let us define ellipticity of L_A with respect to a metric h by

$$\operatorname{Re} \langle av, v \rangle_h \geq \kappa_{1,h} \|v\|_{2,h}^2, \quad v \in L^2(\mathcal{M}) \quad (\text{E})$$

$$\operatorname{Re} \langle ASu, Su \rangle_h \geq \kappa_{2,h} \|u\|_{H^1,h}^2, \quad u \in H^1(\mathcal{M}).$$

Ellipticity (cont.)

Then the loss in the ellipticity constants by transferring from one operator to another is as follows.

Proposition

If $\|g - \tilde{g}\|_{\text{op},g} \leq \delta < 1$, then assuming (E) with respect to g implies (E) with respect to \tilde{g} with an appropriate change of the coefficients a and A with ellipticity constants

$$\kappa_{1,\tilde{g}} = \frac{\kappa_{1,g}}{(1 + \delta)^{\frac{n}{2}}} \quad \text{and} \quad \kappa_{2,\tilde{g}} = \kappa_{2,g} \frac{(1 - \delta)}{(1 + \delta)^{\frac{n}{2}}}.$$

Theorem (Reduction of the rough problem to the smooth)

Let \mathcal{M} be a smooth manifold with a C^0 metric g and suppose that the ellipticity condition (E) is satisfied for (\mathcal{M}, g) . Suppose further that there exists a smooth, complete metric \tilde{g} satisfying:

- (i) there exists $\eta > 0$ such that $|\text{Ric}_{\tilde{g}}|_{\tilde{g}} \leq \eta$,
- (ii) there exists $\kappa > 0$ such that $\text{inj}(\mathcal{M}, \tilde{g}) \geq \kappa$,
- (iii) $\|g - \tilde{g}\|_{\text{op},g} < 1$,

Then, the quadratic estimate

$$\int_0^\infty \|t\Pi_{B,g}(I + t^2\Pi_{B,g}^2)^{-1}u\|_g^2 \frac{dt}{t} \simeq \|u\|_{2,g}^2$$

holds for all $u \in \overline{\mathcal{R}(\Pi_{B,g})}$.

Compact manifolds

Given a C^0 metric g , we can always find a C^∞ metric \tilde{g} that is as close as we would like in the $\|\cdot\|_{\text{op},s,g}$ norm by pasting together Euclidean metrics via a partition of unity.

Further if we assume that \mathcal{M} is compact, then automatically $|\text{Ric}_{\tilde{g}}| \leq C_{\tilde{g}}$ and $\text{inj}(\mathcal{M}, \tilde{g}) \geq \kappa_{\tilde{g}} > 0$.

Theorem

Let \mathcal{M} be a smooth, compact Riemannian manifold and let g be a C^0 metric on \mathcal{M} . Then, the quadratic estimate

$$\int_0^\infty \|t\Pi_{B,g}(\mathbf{I} + t\Pi_{B,g}^2)^{-1}u\|^2 \frac{dt}{t} \simeq \|u\|^2$$

is satisfied for all $u \in \overline{\mathcal{R}(\Pi_{B,g})}$.

The noncompact case

- Although we can obtain arbitrarily close smooth metrics via a partition of unity in the noncompact setting, the desired geometric properties are not automatic.
- Intuition is that the smoothing properties of geometric flows could be utilised to find good close smooth metrics.
- The non-uniqueness of existence of solutions to geometric flows are not an issue since we only require *one* good metric near the initial one.
- A good place to start may be the mean curvature flow since such flows have been studied with rough initial data.
- The first task is to understand the backward behaviour of this flow and the relationship it has to $\|\cdot\|_{\text{op},g}$.

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References II

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