Stability of quadratic estimates and manifolds with non-smooth metrics

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Setup

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Consider the following *uniformly elliptic* second order differential operator $\mathrm{L}_A : \mathcal{D}(\mathrm{L}_A) \subset \mathrm{L}^2(\mathcal{M}) \to \mathrm{L}^2(\mathcal{M})$ defined by

$$\mathrm{L}_A u = aS^* ASu = -a \,\mathrm{div}(A_{11} \nabla u) - a \,\mathrm{div}(A_{10} u) + aA_{01} \nabla u + aA_{00} u.$$

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That is, that there exist $\kappa_1, \kappa_2 > 0$ such that

$$\mathrm{Re}\,\langle av, v\rangle \geq \kappa_1 \|v\|^2,\ v \in \mathrm{L}^2$$
$$\mathrm{Re}\,\langle ASu, Su\rangle \geq \kappa_2(\|u\|^2 + \|\nabla u\|^2),\ u \in \mathrm{W}^{1,2}$$

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Theorem (B.-Mc [3])

Let $(\mathcal{M}, \mathrm{g})$ *be a smooth, complete Riemannian manifold* $|\mathrm{Ric}| \leq C$ *and* $\mathrm{inj}(M) \geq \kappa > 0$*. Suppose there exist* $\kappa_1, \kappa_2 > 0$ *such that*

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for $v \in \mathrm{L}^2(\mathcal{M})$ *and* $u \in \mathrm{W}^{1,2}(\mathcal{M})$*. Then,* $\mathcal{D}(\sqrt{\mathrm{L}_A}) = \mathcal{D}(\nabla) = \mathrm{W}^{1,2}(\mathcal{M})$ *and* $\|\sqrt{\mathrm{L}_A} u\| \simeq \|\nabla u\| + \|u\| = \|u\|_{\mathrm{W}^{1,2}}$ *for all* $u \in \mathrm{W}^{1,2}(\mathcal{M})$*.*

Stability

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for $v \in \mathrm{L}^2(\mathcal{M})$ *and* $u \in \mathrm{W}^{1,2}(\mathcal{M})$. *Then for every* $\eta_i < \kappa_i$, *whenever* $\|\tilde{a}\|_\infty \leq \eta_1$, $\|\tilde{A}\|_\infty \leq \eta_2$, *the estimate*

$$\left\| \sqrt{\mathrm{L}_A}\, u - \sqrt{\mathrm{L}_{A+\tilde{A}}}\, u \right\| \lesssim (\|\tilde{a}\|_\infty + \|\tilde{A}\|_\infty) \|u\|_{\mathrm{W}^{1,2}}$$

holds for all $u \in \mathrm{W}^{1,2}(\mathcal{M})$. *The implicit constant depends in particular on* A, a *and* η_i.

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(H2) Suppose that $B_1, B_2 \in \mathcal{L}(\mathscr{H})$ such that here exist $\kappa_1, \kappa_2 > 0$ satisfying

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(H3) The operators B_1, B_2 satisfy $B_1 B_2 \mathcal{R}(\Gamma) \subset \mathcal{N}(\Gamma)$ and $B_2 B_1 \mathcal{R}(\Gamma^*) \subset \mathcal{N}(\Gamma^*)$.

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Let $\Gamma_B^* = B_1 \Gamma^* B_2$, $\Pi_B = \Gamma + \Gamma_B^*$ and $\Pi = \Gamma + \Gamma^*$.

Quadratic estimates and Kato type problems

Proposition

If (H1)-(H3) are satisfied and

$$\int_0^\infty \left\| t\Pi_B(\mathrm{I} + t^2\Pi_B^2)^{-1}u \right\|^2 \ \frac{dt}{t} \simeq \|u\|$$

for $u \in \overline{\mathcal{R}(\Pi_B)}$*, then*

(i) $\mathcal{D}(\Gamma) \cap \mathcal{D}(\Gamma_B^*) = \mathcal{D}(\Pi_B) = \mathcal{D}(\sqrt{\Pi_B^2})$*, and*

(ii) $\|\Gamma u\| + \|\Gamma_B u\| \simeq \|\Pi_B u\| \simeq \|\sqrt{\Pi_B^2} u\|$*, for all* $u \in \mathcal{D}(\Pi_B)$*.*

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This result has been at the heart of the work of Axelsson (Rosén), Keith, McIntosh in [2] and [1], as well as the work of Morris in [5] and B. in [4].

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$$\Gamma_{\mathrm{g}} = \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix}, \ B_1 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \text{ and } B_2 = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}.$$

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Then,

$$\Gamma_{\mathrm{g}}^* = \begin{pmatrix} 0 & S_{\mathrm{g}}^* \\ 0 & 0 \end{pmatrix} \text{ and } \Pi_B^2 = \begin{pmatrix} \mathrm{L}_A & 0 \\ 0 & * \end{pmatrix}$$

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and

$$\Pi_{B,\mathrm{g}}(u,0) = (0,u,\nabla u) \text{ and } \sqrt{\Pi_{B,\mathrm{g}}^2}(u,0) = (\sqrt{\mathrm{L}_A}u, 0).$$

Measure notions on manifolds independent of a metric

Definition (Notions of measure)

We say that:

(i) a set $A \subset \mathcal{M}$ is measurable if whenever (U, ψ) is a chart satisfying $U \cap A \neq \emptyset$, then $\varphi(U \cap A) \subset \mathbb{R}^n$ is $\mathscr{L}$-measurable,

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(iv) a set Z is a *null set* or set of *null measure* if requiring $\mathscr{L}(\varphi(U \cap Z)) = 0$ for each chart (U, ψ),

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(v) a property P is valid almost-everywhere if it is valid $\mathscr{L}$-a.e. in each coordinate chart (U, ψ).

Rough metrics

Definition (Rough metric)

Suppose that $\mathrm{g} \in \mathbf{\Gamma}(\mathcal{T}^{(2,0)}\mathcal{M})$ is symmetric and satisfies the following *local comparability condition*: for every $x \in \mathcal{M}$, there exists a chart (U, ψ) near x and constant $C \geq 1$ such that

$$C^{-1} \left|u\right|_{\psi^*\delta(y)} \leq \left|u\right|_{\mathrm{g}(y)} \leq C \left|u\right|_{\psi^*\delta(y)}$$

for $u \in \mathrm{T}_y\mathcal{M}$ and for almost-every $y \in U$.

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(vi) g is Borel and finite on compact sets.

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As a consequence, we define

$$\mathrm{d}_{\mathrm{g}}(x, y) = \inf \{\ell_{\mathrm{g}}(\gamma) : \gamma(0) = x, \gamma(1) = y, \ \gamma \text{ abs. cts.}\}$$

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The map $\mathrm{d}_{\mathrm{g}} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_+$ is a pseudo-metric and the induced topology is coarser than the topology of $\mathcal{M}$.

L^p spaces, Sobolev spaces

$\mathrm{L}^p(\mathcal{T}^{(r,s)}\mathcal{M}, \mathrm{g})$ space defined as $f \in \mathbf{\Gamma}(\mathcal{T}^{(r,s)}\mathcal{M})$ such that

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The Sobolev space $\mathrm{W}^{1,p}(\mathcal{M}, \mathrm{g})$ is the defined as the set $u \in \mathrm{C}^\infty \cap \mathrm{L}^2(\mathcal{M})$ with $\nabla u \in \mathrm{C}^\infty \cap \mathrm{L}^2(\mathcal{M})$ under the norm $\|\cdot\|_{\mathrm{W}^{1,p}} = \|\cdot\|_p + \|\nabla \cdot\|_p$.

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Divergence

Proposition

The space $C_c^\infty(\mathcal{T}^{(r,s)}\mathcal{M})$ *is dense in* $L^p(\mathcal{T}^{(r,s)}\mathcal{M}, g)$. *The operators* $\nabla_p : C^\infty \cap L^p(\mathcal{M}) \to C^\infty \cap L^p(T^*\mathcal{M})$ *and* $\nabla_c : C_c^\infty(\mathcal{M}) \to C_c^\infty(T^*\mathcal{M})$ *are closable, densely-defined operators. Furthermore,* $W^{1,p}(\mathcal{M}) = \mathcal{D}(\overline{\nabla_p})$ *and* $W_0^{1,p} = \mathcal{D}(\overline{\nabla_c})$.

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For the case $p = 2$, we define

$$\mathrm{div}_g = -\nabla_2{}^*, \text{ and } \mathrm{div}_{0,g} = -\nabla_0{}^*,$$

which operator theory guarantees are closed, densely-defined.

Uniformly close geometries

Definition (Uniformly close metrics)

Let g and $\tilde{\mathrm{g}}$ be two rough metrics and support there exists $C \geq 1$ such that

$$C^{-1} \left|u\right|_{\tilde{\mathrm{g}}(x)} \leq \left|u\right|_{\mathrm{g}(x)} \leq C \left|u\right|_{\tilde{\mathrm{g}}(x)},$$

for $u \in \mathrm{T}_x\mathcal{M}$ and almost-every x in $\mathcal{M}$. Then, we say that g and $\tilde{\mathrm{g}}$ are *uniformly close* or C*-close*. If the inequality holds everywhere, then we say that the two metrics are C*-close everywhere*.

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for $u \in \mathrm{T}_x\mathcal{M}$ and almost-every x in $\mathcal{M}$. Then, we say that g and $\tilde{\mathrm{g}}$ are *uniformly close* or C*-close*. If the inequality holds everywhere, then we say that the two metrics are C*-close everywhere*.

If g and $\tilde{\mathrm{g}}$ are both at least continuous, then C-close and C-close everywhere are equivalent.

For any two metrics, there exists a.e. symmetric positive $\mathrm{B} \in \mathbf{\Gamma}(\mathrm{T}^*\mathcal{M} \otimes \mathrm{T}\mathcal{M})$ such that

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The volume measure is then $d\mu_{\mathrm{g}} = \theta \ d\mu_{\tilde{\mathrm{g}}}$ where

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Furthermore,

$$C^{-\frac{n}{2}} \mu_{\tilde{\mathrm{g}}} \leq \mu_{\mathrm{g}} \leq C^{\frac{n}{2}} \mu_{\mathrm{g}}.$$

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$$C^{-\left(r+s+\frac{n}{2p}\right)} \|u\|_{p,\tilde{\mathrm{g}}} \leq \|u\|_{p,\mathrm{g}} \leq C^{r+s+\frac{n}{2p}} \|u\|_{p,\tilde{\mathrm{g}}},$$

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(ii) for $p = \infty$, $\mathrm{L}^\infty(\mathcal{T}^{(r,s)}, \mathrm{g}) = \mathrm{L}^\infty(\mathcal{T}^{(r,s)}, \tilde{\mathrm{g}})$ with

$$C^{-(r+s)} \|u\|_{\infty,\tilde{\mathrm{g}}} \leq \|u\|_{\infty,\mathrm{g}} \leq C^{r+s} \|u\|_{\infty,\tilde{\mathrm{g}}},$$

Preservation of Sobolev spaces

(i) the Sobolev spaces $\mathrm{W}^{1,p}(\mathcal{M}, \mathrm{g}) = \mathrm{W}^{1,p}(\mathcal{M}, \tilde{\mathrm{g}})$ and $\mathrm{W}^{1,p}_0(\mathcal{M}, \mathrm{g}) = \mathrm{W}^{1,p}_0(\mathcal{M}, \tilde{\mathrm{g}})$ with

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(iii) the divergence operators satisfy $\mathrm{div}_{\mathrm{g}} = \theta^{-1} \mathrm{div}_{\tilde{\mathrm{g}}} \theta \mathrm{B}$ and $\mathrm{div}_{0,\mathrm{g}} = \theta^{-1} \mathrm{div}_{0,\tilde{\mathrm{g}}} \theta \mathrm{B}$.

Reduction of the problem

As before, let $\mathscr{H} = \mathrm{L}^2(\mathcal{M}) \oplus \mathrm{L}^2(\mathcal{M}) \oplus \mathrm{L}^2(\mathrm{T}^*\mathcal{M})$ with two inner products $\langle \cdot \, , \cdot \rangle_{\mathrm{g}}$ and $\langle \cdot \, , \cdot \rangle_{\tilde{\mathrm{g}}}$ induced by by g and $\tilde{\mathrm{g}}$ respectively.

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Let $S = (I, \overline{\nabla_2})$ and

$$\Gamma = \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix},\ \Gamma_{\mathrm{g}}^* = \begin{pmatrix} 0 & S_{\mathrm{g}}^* \\ 0 & 0 \end{pmatrix},\ \Gamma_{\tilde{\mathrm{g}}}^* = \begin{pmatrix} 0 & S_{\tilde{\mathrm{g}}}^* \\ 0 & 0 \end{pmatrix}.$$

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Let $\mathrm{E}(u, v, w) = (\theta u, \theta v, \theta \mathrm{B} w)$ so that $\langle u, v \rangle_{\mathrm{g}} = \langle Eu, v \rangle_{\tilde{\mathrm{g}}}$ for all $u, v \in \mathscr{H}$.

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Let $\mathrm{E}(u, v, w) = (\theta u, \theta v, \theta \mathrm{B} w)$ so that $\langle u, v \rangle_{\mathrm{g}} = \langle Eu, v \rangle_{\tilde{\mathrm{g}}}$ for all $u, v \in \mathscr{H}$.

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Reduction of the problem (cont.)

As before, let $a \in \mathrm{L}^\infty(\mathcal{M})$ and $A \in \mathrm{L}^\infty(\mathcal{L}(\mathrm{L}^2(\mathcal{M}) \oplus \mathrm{L}^2(\mathrm{T}^*\mathcal{M}))$ with constants $\kappa_1, \kappa_2 > 0$ such that

$$\mathrm{Re}\,\langle au, u\rangle_{\mathrm{g}} \geq \kappa_1 \|u\|_{\mathrm{g}}^2$$
$$\mathrm{Re}\,\langle Av, v\rangle_{\mathrm{g}} \geq \kappa_2 \|v\|_{\mathrm{W}^{1,2},\mathrm{g}}^2$$

for $u \in \mathrm{L}^2(\mathcal{M})$ and $v \in \mathrm{W}^{1,2}(\mathcal{M})$.

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Then, writing

$$B_1 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix},$$

let

$$\Pi_{B,\mathrm{g}} = \Gamma + B_1 \Gamma_{\mathrm{g}}^* B_2 = \Gamma + B_1 \mathrm{E}^{-1} \Gamma_{\tilde{\mathrm{g}}}^* \mathrm{E} B_2 = \Gamma + \tilde{B}_1 \Gamma_{\tilde{\mathrm{g}}}^* \tilde{B}_2 = \Pi_{\tilde{B},\tilde{\mathrm{g}}}.$$

Change of Ellipticity

We have that

$$\operatorname{Re} \langle B_1 u, u \rangle_{\mathrm{g}} \geq \kappa_1 \|u\|_{\mathrm{g}}^2$$
$$\operatorname{Re} \langle B_2 u, u \rangle_{\mathrm{g}} \geq \kappa_2 \|v\|_{\mathrm{g}}^2$$

for $u \in \mathrm{L}^2(\mathcal{M}) \oplus 0 \oplus 0 \supset \mathcal{R}(\Gamma_{\tilde{\mathrm{g}}}^*)$ and $v \in 0 \oplus \mathrm{L}^2(\mathcal{M}) \oplus \mathrm{L}^2(\mathrm{T}^*\mathcal{M}) \supset \mathcal{R}(\Gamma)$.

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The ellipticity for $\tilde{B}_i$ in terms of $\tilde{\mathrm{g}}$ then becomes

$$\mathrm{Re} \left\langle \tilde{B}_1 u, u \right\rangle_{\tilde{\mathrm{g}}} \geq \frac{\kappa_1}{C^{\frac{n}{2}}} \|u\|_{\tilde{\mathrm{g}}}^2$$
$$\mathrm{Re} \left\langle \tilde{B}_2 u, u \right\rangle_{\tilde{\mathrm{g}}} \geq \frac{\kappa_2}{C^{1+\frac{n}{2}}} \|v\|_{\tilde{\mathrm{g}}}^2$$

for similar u and v.

The reduction of quadratic estimates

Proposition

Let g *and* $\tilde{\mathrm{g}}$ *be two* C*-close metrics. Then, the quadratic estimates*

$$\int_0^\infty \left\| t\Pi_{B,\mathrm{g}}(1 + t^2\Pi_{B,\mathrm{g}}^2)^{-1}u \right\|_{\mathrm{g}}^2 \frac{dt}{t} \simeq \|u\|_{\mathrm{g}}^2$$

is satisfies for all $u \in \overline{\mathcal{R}(\Pi_{B,\mathrm{g}})}$ *if and only if*

$$\int_0^\infty \left\| t\Pi_{\tilde{B},\tilde{\mathrm{g}}}(1 + t^2\Pi_{B,\tilde{\mathrm{g}}}^2)^{-1}u \right\|_{\tilde{\mathrm{g}}}^2 \frac{dt}{t} \simeq \|u\|_{\tilde{\mathrm{g}}}^2$$

is satisfied for all $u \in \overline{\mathcal{R}(\Pi_{\tilde{B},\tilde{\mathrm{g}}})}$.

The Kato square root problem for rough metrics

Theorem

Let g *be a rough metric and suppose there exists a* C*-close metric* $\tilde{\mathrm{g}}$ *that is smooth, complete and satisfying:*

(i) *there exists* $\eta > 0$ *such that* $|\mathrm{Ric}_{\tilde{\mathrm{g}}}|_{\tilde{\mathrm{g}}} \leq \eta$,

(ii) *there exists* $\kappa > 0$ *such that* $\mathrm{inj}(\mathcal{M}, \tilde{\mathrm{g}}) \geq \kappa$,

(iii) $\|\mathrm{g} - \tilde{\mathrm{g}}\|_{\mathrm{op},\mathrm{g}} < 1$.

Then, $\mathcal{D}(\sqrt{aS_{\mathrm{g}}^* AS}) = \mathrm{W}^{1,2}(\mathcal{M})$ *and* $\left\|\sqrt{aS_{\mathrm{g}}^* ASu}\right\|_{\mathrm{g}} \simeq \|u\|_{\mathrm{W}^{1,2},\mathrm{g}}$ *for all* $u \in \mathrm{W}^{1,2}(\mathcal{M})$.

Application to compact manifolds with continuous metrics

Given a C^0 metric g, we can always find a C^∞ metric $\tilde{g}$ that is C-close for any $C > 1$ norm by pasting together Euclidean metrics via a partition of unity.

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Further if we assume that $\mathcal{M}$ is compact, then automatically $|\mathrm{Ric}_{\tilde{\mathrm{g}}}| \leq C_{\tilde{\mathrm{g}}}$ and $\mathrm{inj}(\mathcal{M}, \tilde{\mathrm{g}}) \geq \kappa_{\tilde{\mathrm{g}}} > 0$.

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Theorem

Let $\mathcal{M}$ be a smooth, compact Riemannian manifold and let g be a C^0 metric on $\mathcal{M}$. Then, the quadratic estimate

$$\int_0^\infty \left\| t\Pi_{B,\mathrm{g}}(\mathrm{I} + t\Pi_{B,\mathrm{g}}^2)^{-1} u \right\|^2 \ \frac{dt}{t} \simeq \|u\|^2$$

is satisfied for all $u \in \overline{\mathcal{R}(\Pi_{B,\mathrm{g}})}$.

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