Riesz continuity of the Atiyah-Singer Dirac operator under perturbations of the metric

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We say that $\mathrm{g} \sim \mathrm{h}$ if there exists $C \geq 1$ satisfying: for all $x \in \mathcal{M}$ and $u \in \mathrm{T}_x\mathcal{M}$,

$$C^{-1} \left|u\right|_{\mathrm{g}(x)} \leq \left|\zeta_* u\right|_{\mathrm{h}(\zeta(x))} \leq C \left|u\right|_{\mathrm{g}(x)}.$$

The minimal such constant is then $C_L = \inf \{C \geq 1 : \mathrm{g} \sim \mathrm{h}\}$. Then $\rho_M(\mathrm{g}, \zeta^* \mathrm{h}) = \log(C_L)$ is a distance metric.

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Throughout, we assume that $\mathrm{g} \sim \mathrm{h}$.

Spin manifolds and the Dirac operator

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$$\nabla \not e_\alpha = \frac{1}{2} \sum_{b<a} \omega^a_b \otimes (e_b \cdot e_a \cdot \not e_\alpha),$$

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where $e\!\!\!/_\alpha$ is an orthonormal spin frame and $\omega^a_b = w^a_{cb}\ e^b$ is the connection 1-form, is compatible and is a module derivation. The Atiyah-Singer Spin Dirac operator is then defined by

$$D\!\!\!/ \psi = e^j \cdot \nabla_{e_j} \psi$$

for $\psi \in \mathrm{C}^\infty$.

Compatibility and pullback operators

Inside a contractible open set Ω corresponding to a frame, the map U induces a fibrewise unitary map $\not{\mathrm{U}}_\Omega : \not{\Delta}\,\Omega \to \not{\Delta}\,\zeta(\Omega)$ (there are two such choices).

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If this lifts to a global map $\not{\mathrm{U}} : \not{\Delta}\,\mathcal{M} \to \not{\Delta}\,\mathcal{N}$, we say that $\not{\Delta}\,\mathcal{M}$ and $\not{\Delta}\,\mathcal{N}$ are *compatible*. It is readily checked that $\not{\mathrm{U}}$ is $\mathrm{C}^{0,1}$.

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Under the map $\not{\mathrm{U}}$, the Dirac operator $\not{\mathrm{D}}_{\mathrm{h}}$ pulls back to an operator $\not{\mathrm{U}}^{-1}\not{\mathrm{D}}_{\mathrm{h}}\not{\mathrm{U}}$ that is similar to a self-adjoint operator in $\mathrm{L}^2(\not{\Delta}\,\mathcal{M})$.

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Such a global map $\not{\mathrm{U}}$ always exists since we can pullback the Spin structure on $\mathcal{M}$ to a compatible structure on $\mathcal{N}$.

Main Theorem

Theorem

Let $\mathcal{M}$ be a smooth Spin manifold with smooth, complete metric g with Levi-Civita connection ∇^{g}, let $\mathcal{N}$ be a smooth Spin manifold with a $\mathrm{C}^{0,1}$ metric h, and $\zeta : \mathcal{M} \to \mathcal{N}$ a $\mathrm{C}^{1,1}$-diffeomorphism with $\rho_M(\mathrm{g}, \zeta^*\mathrm{h}) \leq 1$. We assume that the spin bundles $\not{\Delta}\mathcal{M}$ and $\not{\Delta}\mathcal{N}$ are compatible. Moreover, suppose that the following hold:

(i) there exists $\kappa > 0$ such that $\mathrm{inj}(\mathcal{M}, \mathrm{g}) \geq \kappa$,

(ii) there exists $C_R > 0$ such that $|\mathrm{Ric}_{\mathrm{g}}| \leq C_R$ and $|\nabla^{\mathrm{g}}\mathrm{Ric}_{\mathrm{g}}| \leq C_R$,

(iii) there exists $C_{\mathrm{h}} > 0$ such that $|\nabla^{\mathrm{g}}(\zeta^*\mathrm{h})| \leq C_{\mathrm{h}}$ almost-everywhere.

Theorem (cont.)

Then, we have the perturbation estimate

$$\left\| \frac{\not{D}_{\mathrm{g}}}{\sqrt{1+\not{D}_{\mathrm{g}}^2}} - \frac{\not{U}^{-1}\not{D}_{\mathrm{h}}\not{U}}{\sqrt{1+(\not{U}^{-1}\not{D}_{\mathrm{h}}\not{U})^2}} \right\|_{\mathrm{L}^2\to\mathrm{L}^2} \lesssim \rho_M(\mathrm{g},\zeta^*\mathrm{h}),$$

where the implicit constant depends on $\dim \mathcal{M}$ and the constants appearing in (i)-(iii).

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Motivations come from connections to the *spectral flow* as outlined by Lesch in [L].

Generalised bounded geometry

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We say that $(\mathcal{V}, \mathrm{h})$ satisfies *generalised bounded geometry* if there exists a uniform $\rho > 0$ and $C \geq 1$ such that for each $x \in \mathcal{M}$, there is a continuous local trivialisation $\psi_x : \mathrm{B}_\rho(x) \times \mathbb{C}^N \to \pi_{\mathcal{V}}^{-1}(\mathrm{B}_\rho(x))$ satisfying:

$$C^{-1} \left|u\right|_{\mathbb{C}^N} \leq \left|\psi_x(y)u\right|_{\mathrm{h}(y)} \leq C \left|u\right|_{\mathbb{C}^N}$$

for $u \in \mathbb{C}^N$ and $y \in \mathrm{B}_\rho(x)$.

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The value ρ is called the GBG radius and in application, the GBG trivialisations have higher regularity.

Exponential growth and local Poincaré inequality

We say that $(\mathcal{M}, \mathrm{g}, \mu)$ has exponential volume growth if there exists $c_E \geq 1$, $\kappa, c > 0$ such that

$$0 < \mu(\mathrm{B}(x, tr)) \leq ct^{\kappa} \mathrm{e}^{c_E tr} \mu(\mathrm{B}(x, r)) < \infty, \qquad (\mathsf{E}_{\mathsf{loc}})$$

for every $t \geq 1,\ r > 0$ and $x \in \mathcal{M}$.

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The manifold $\mathcal{M}$ satisfies a *local Poincaré inequality* if there exists $c_P \geq 1$ such that for all $f \in \mathrm{W}^{1,2}(\mathcal{M})$,

$$\|f - f_{\mathrm{B}}\|_{\mathrm{L}^2(\mathrm{B})} \leq c_P \ \mathrm{rad}(\mathrm{B}) \|f\|_{\mathrm{W}^{1,2}(\mathrm{B})} \qquad (\mathsf{P}_{\mathsf{loc}})$$

for all balls B in $\mathcal{M}$ such that $\mathrm{rad}(\mathrm{B}) \leq 1$.

First order differential operators on $\mathcal{V}$

We say that an operator $\mathrm{D} : \mathrm{C}^\infty(\mathcal{V}) \to \mathrm{L}^\infty_{\mathrm{loc}}(\mathcal{V})$ is a *first-order* differential operator if inside each frame $\{e^i\}$ for $\mathcal{V}$ and $\{v_j\}$ for $\mathrm{T}\mathcal{M}$ near x, there exist coefficients α_l^{jk} and terms ω_q^p such that

$$\mathrm{D}u = (\alpha_l^{jk} \nabla_{v_j} u_k + u_i \omega_l^i)\ e^l,$$

where $u = u_i e^i \in \mathrm{C}^\infty(\mathcal{V})$.

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We consider two essentially self-adjoint first-order differential operators D and $\tilde{\mathrm{D}}$ on $\mathrm{C}^\infty_{\mathrm{c}}(\mathcal{V})$, and with slight abuse of notation we use this notation for their self-adjoint extensions.

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(A4) $\mathrm{T}^*\mathcal{M}$ has $\mathrm{C}^{0,1}$ GBG frames ν_j quantified by $\rho_{\mathrm{T}^*\mathcal{M}} > 0$ and $C_{\mathrm{T}^*\mathcal{M}} < \infty$, with regularity $|\nabla \nu_j| < C_{G,\mathrm{T}^*\mathcal{M}}$ with $C_{G,\mathrm{T}^*\mathcal{M}} < \infty$ almost-everywhere,

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(A5) $\mathcal{V}$ has $\mathrm{C}^{0,1}$ GBG frames e_j quantified by $\rho_{\mathcal{V}} > 0$ and $C_{\mathcal{V}} < \infty$, with regularity $|\nabla e_j| < C_{G,\mathcal{V}}$ with $C_{G,\mathcal{V}} < \infty$ almost-everywhere, (A6) D is a first-order PDO with L^∞ coefficients. In particular, $[\mathrm{D}, \eta]$ is a pointwise multiplication operator on almost-every fibre $\mathcal{V}_x$, and there exists $c_{\mathrm{D}} > 0$ such that

$$|[\mathrm{D}, \eta]\, u(x)| \leq c_{\mathrm{D}} \operatorname{Lip} \eta(x)\, |u(x)|$$

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(A7) D satisfies $|\mathrm{D} e_j| \leq C_{D,\mathcal{V}}$ with $C_{D,\mathcal{V}} < \infty$ almost-everywhere inside each GBG frame $\{e_j\}$,

(A8) D and $\tilde{\mathrm{D}}$ both have domains $\mathrm{W}^{1,2}(\mathcal{V})$ with $\mathrm{C} \geq 1$ the smallest constants satisfying

$$\mathrm{C}^{-1}\|u\|_{\mathrm{D}} \leq \|u\|_{\mathrm{W}^{1,2}} \leq \mathrm{C}\|u\|_{\mathrm{D}} \quad \text{and}$$
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(A9) D satisfies the Riesz-Weitzenböck condition

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The implicit constants in our perturbation estimates will be allowed to depend on $\mathrm{C}(\mathcal{M}, \mathcal{V}, \mathrm{D}, \tilde{\mathrm{D}})$ which is the maximum of the constants appearing in (A1)-(A9).

The general theorem

Theorem

Let $(\mathcal{M}, \mathrm{g})$ be a smooth Riemannian manifold with g that is $\mathrm{C}^{0,1}$, complete, and satisfying ($\mathsf{E}_{\mathsf{loc}}$) and ($\mathsf{P}_{\mathsf{loc}}$). Let $(\mathcal{V}, \mathrm{h}, \nabla)$ be a smooth vector bundle with $\mathrm{C}^{0,1}$ metric h and connection ∇ that are compatible almost-everywhere.

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Let D, $\tilde{\mathrm{D}}$ be self-adjoint operators on $\mathrm{L}^2(\mathcal{V})$ and assume the hypotheses (A1)-(A9) on $\mathcal{M}$, $\mathcal{V}$, D and $\tilde{\mathrm{D}}$. Let

$$A_1 \in \mathrm{L}^\infty(\mathcal{L}(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V}, \mathcal{V})),$$

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and let $\|A\|_\infty = \|A_1\|_\infty + \|A_2\|_\infty + \|A_3\|_\infty$.

Assume that

$$\tilde{\mathrm{D}}\psi = \mathrm{D}\psi + A_1 \nabla \psi + \operatorname{div} A_2 \psi + A_3 \psi,$$

holds in a distributional sense for $\psi \in \mathrm{W}^{1,2}(\mathcal{V})$.

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Then, for each $\omega \in (0, \pi/2)$ and $\sigma \in (0, \infty]$, whenever $f \in \mathrm{Hol}^\infty(\mathrm{S}^{\mathrm{o}}_{\omega,\sigma})$, we have the perturbation estimate

$$\|f(\tilde{\mathrm{D}}) - f(\mathrm{D})\|_{\mathrm{L}^2(\mathcal{V}) \to \mathrm{L}^2(\mathcal{V})} \lesssim \|f\|_{\mathrm{L}^\infty(\mathrm{S}_{\omega,\sigma})} \|A\|_\infty,$$

where the implicit constant depends on $\mathrm{C}(\mathcal{M}, \mathcal{V}, \mathrm{D}, \tilde{\mathrm{D}})$.

Assume that

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holds in a distributional sense for $\psi \in \mathrm{W}^{1,2}(\mathcal{V})$.

Then, for each $\omega \in (0, \pi/2)$ and $\sigma \in (0, \infty]$, whenever $f \in \mathrm{Hol}^\infty(\mathrm{S}^{\mathrm{o}}_{\omega,\sigma})$, we have the perturbation estimate

$$\|f(\tilde{\mathrm{D}}) - f(\mathrm{D})\|_{\mathrm{L}^2(\mathcal{V}) \to \mathrm{L}^2(\mathcal{V})} \lesssim \|f\|_{\mathrm{L}^\infty(\mathrm{S}_{\omega,\sigma})} \|A\|_\infty,$$

where the implicit constant depends on $\mathrm{C}(\mathcal{M}, \mathcal{V}, \mathrm{D}, \tilde{\mathrm{D}})$.

Note, $\mathrm{S}^{\mathrm{o}}_{\omega,\sigma} := \left\{ x + iy : y^2 < \tan^2 \omega x^2 + \sigma^2 \right\}$.

Obtaining the main theorem from the general

- Note that g is smooth, h is locally Lipschitz, and set $\mathcal{V} = \Delta\!\!\!/\,\mathcal{M}$ to a spin bundle of $(\mathcal{M}, \mathrm{g})$ along with a compatible spin bundle $\Delta\!\!\!/\,\mathcal{N}$ on $(\mathcal{N}, \mathrm{h})$. Let $\mathrm{D} = \not{\mathrm{D}}_{\mathrm{g}}$ and $\tilde{\mathrm{D}} = \not{\mathrm{U}}^{-1} \not{\mathrm{D}}_{\mathrm{h}} \not{\mathrm{U}}$.

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- The ellipticity $\mathcal{D}(\mathrm{D}\!\!\!/_{\mathrm{g}}) = \mathrm{W}^{1,2}(\Delta\!\!\!/\,\mathcal{M})$ can be seen immediately from the fact that $\mathrm{Ric} \geq -C_R \mathrm{g}$ which implies $\mathcal{R}_S \geq -C_R$ and by invoking the Bochner formula. For the other operator, we need the following Lemma.

Lemma

Under the geometric assumptions: $\mathrm{inj}(\mathcal{M}, \mathrm{g}) \geq \kappa$, $|\mathrm{Ric}| \leq C_R$ *, there exists a sequence of points* x_i *and a smooth partition of unity* $\{\eta_i\}$ *uniformly locally finite and subordinate to* $\{B(x_i, r_H)\}$ *satisfying* $\sum_i |\nabla^j \eta_i|) \leq C_H$ *for* $j = 0, ..., 2$. *Moreover, there exists* $M > 0$ *such that* $1 \leq M \sum_i \eta_i^2$.

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- The Riesz-Weitzenböck condition (A9) is obtained by a similar localisation along with the addition assumption $|\nabla^{\mathrm{g}}\mathrm{Ric}| \leq C_R$ which yields $|\partial_l \partial_k \mathrm{g}_{ij}| \lesssim 1$ inside harmonic balls.

The operator decomposition

- The decomposition $\mathrm{D} - \mathrm{D}' = A_1^{\Omega} \nabla + \operatorname{div} A_2^{\Omega} + A_3^{\Omega}$ inside a local trivialisation Ω is a matter of calculation and does not require curvature assumptions.

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- To obtain a global decomposition, we require a Lipschitz partition of unity $\{\eta_j\}$ subordinate to local trivialisations on balls $\{\mathrm{B}_j\}$ satisfying: there exists $C_1, C_2, C_3 > 0$ such that
  (i) $|\nabla e_{j,i}| \leq C_1$,
  (ii) $\left|\partial_{e_{j,k}} \tilde{\mathrm{g}}(e_{j,i}, e_{j,l})\right| \leq C_2$, where $\tilde{\mathrm{g}} = \zeta^* \mathrm{h}$, and
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  for all $i, k, l = 1, \ldots, n = \dim(\mathcal{M})$ and all $j = 1, \ldots$.
- It is easy to see our previous Lemma and gradient bound $|\nabla^{\mathrm{g}}(\zeta^* \mathrm{h})| \lesssim 1$ imply (i)-(iii).

Reduction to quadratic estimates

For $t > 0$, let us define operators

$$\mathrm{P}_t = \frac{1}{\mathrm{I} + t^2\mathrm{D}^2},\ \tilde{\mathrm{P}}_t = \frac{1}{\mathrm{I} + t^2\tilde{\mathrm{D}}^2},\ \mathrm{Q}_t = t\mathrm{D}\mathrm{P}_t, \quad \text{and} \quad \tilde{\mathrm{Q}}_t = t\tilde{\mathrm{D}}\tilde{\mathrm{P}}_t.$$

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The fact that D and $\tilde{\mathrm{D}}$ are self adjoint gives

$$\int_0^\infty \|\tilde{\mathrm{Q}}_t u\|^2\ \frac{dt}{t} \leq \frac{1}{2}\|u\|^2 \quad \text{and} \quad \int_0^\infty \|\mathrm{Q}_t u\|^2\ \frac{dt}{t} \leq \frac{1}{2}\|u\|^2,$$

and also

$$\sup_t \|\mathrm{P}_t\|,\ \sup_t \|\tilde{\mathrm{P}}_t\|,\ \sup_t \|\mathrm{Q}_t\|,\ \sup_t \|\tilde{\mathrm{Q}}_t\| \leq \frac{1}{2}.$$

Proposition

Suppose that

$$\int_0^1 \|\tilde{\mathrm{Q}}_t A_1 \nabla (\imath \mathrm{I} + \mathrm{D})^{-1} \mathrm{P}_t f\|^2 \frac{dt}{t} \leq C_1 \|A\|_\infty^2 \|f\|^2, \text{ and}$$
$$\int_0^1 \|t \tilde{\mathrm{P}}_t \operatorname{div} A_2 \mathrm{P}_t f\|^2 \frac{dt}{t} \leq C_2 \|A\|_\infty^2 \|f\|^2$$

for all $u \in \mathrm{L}^2(\mathcal{V})$. *Then, for* $\omega \in (0, \pi/2)$ *and* $\sigma \in (0, \infty)$, *whenever* $f \in \mathrm{Hol}^\infty(\mathrm{S}^o_{\omega,\sigma})$, *we obtain that*

$$\|f(\tilde{\mathrm{D}}) - f(\mathrm{D})\| \lesssim \|f\|_\infty \|A\|_\infty$$

where the implicit constant depends on $C_1,\ C_2$ *and* $\mathrm{C}(\mathcal{M}, \mathcal{V}, \mathrm{D}, \tilde{\mathrm{D}})$.

Prelude to the harmonic analysis

- By the proposition, we consider quadratic estimates of the general form
$$\int_0^1 \|\mathbf{Q}_t S \mathrm{P}_t f\|^2 \ \frac{dt}{t} \lesssim \|A\|_\infty^2 \|f\|^2,$$
where $S : \mathrm{L}^2(\mathcal{V}) \to \mathrm{L}^2(\mathcal{W})$ and $\mathbf{Q}_t : \mathrm{L}^2(\mathcal{W}) \to \mathrm{L}^2(\mathcal{V})$, where $\mathcal{W}$ is an auxiliary vector bundle and $\mathbf{Q}_t$ is a family of operators with sufficient decay.

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- Attack this via Euclidean harmonic analysis techniques. Need *dyadic structure*, sufficiently "good" notion of *integration* (via some sort of fixed system of trivialisations), *averaging*, etc. to import these techniques as in [BMc].

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- For $Q \in \mathscr{Q}$, the GBG coordinates of Q are the GBG coordinates of the GBG cube $\widehat{Q}$.

Cube integration and dyadic averaging

- Define *cube integration*, as a map $\mathrm{B}(x_{\widehat{Q}}, \rho) \times \mathscr{Q} \ni (x, Q) \mapsto (\int_Q \cdot\,)(x)$. For $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{V})$, and $y \in \mathrm{B}(x_{\widehat{Q}}, \rho)$ we write

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- Define the *cube average* $u_Q \in \mathrm{L}^\infty(\mathcal{V})$ of some $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{V})$ as

$$u_Q(y) = \begin{cases} \fint_Q u \; d\mu & y \in \mathrm{B}(x_{\widehat{Q}}, \rho) \\ 0 & y \not\in \mathrm{B}(x_{\widehat{Q}}, \rho). \end{cases}$$

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- For each $t > 0$, define the *dyadic averaging operator* $\mathbb{E}_t : \mathrm{L}^1_{\mathrm{loc}}(\mathcal{V}) \to \mathrm{L}^1_{\mathrm{loc}}(\mathcal{V})$ by

$$\mathbb{E}_t u(x) = u_Q(x)$$

  where $Q \in \mathscr{Q}_t$ and $x \in Q$.

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- For $x \in Q \in \mathscr{Q}$ and $w \in \mathcal{V}_x \cong \mathbb{C}^N$, and write $w = w_i\ e^i(x)$ in the GBG frame $\left\{e^i(x)\right\}$ associated to Q.
- Define the *constant extension* of w by

$$w^c(y) = \begin{cases} w_i\ e^i(y) & y \in \mathrm{B}(x_{\widehat{Q}}, \rho) \\ 0 & y \not\in \mathrm{B}(x_{\widehat{Q}}, \rho), \end{cases}$$

  and we note that $w^c \in \mathrm{L}^\infty(\mathcal{V})$.
- For $x \in Q \in \mathscr{Q}$, and $w \in \mathcal{V}_x$, define the *principal part* of $\mathbf{Q}_t$ by

$$\gamma_t^{\mathbf{Q}}(x)w = (\mathbf{Q}_t w^c)(x).$$

The estimate break-up

Break up the required estimate via the "Kato square root estimate paradigm":

$$\begin{aligned}\int_0^1 \|\mathbf{Q}_t S \mathrm{P}_t f\|^2 \ \frac{dt}{t} &\lesssim \int_0^1 \|(\mathbf{Q}_t - \gamma_t \mathbb{E}_t) S \mathrm{P}_t f\|^2 \ \frac{dt}{t} \\ &\quad + \int_0^1 \|\gamma_t \mathbb{E}_t S (\mathrm{I} - \mathrm{P}_t) f\|^2 \ \frac{dt}{t} \\ &\quad + \int_0^1 \|\gamma_t \mathbb{E}_t S f\|^2 \ \frac{dt}{t}. \\ &=: I + II + III\end{aligned}$$

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This decomposition is the one that is motivated by the solution of the Kato square root problem ([AHLMcT], [AKMc]).

Off-diagonal decay and quadratic estimates

- Defining $\langle a \rangle = \max\{1, a\}$, we assume that $\mathbf{Q}_t$ satisfies the following *off-diagonal estimates*: there exists $C_{\mathbf{Q}} > 0$ such that, for each $M > 0$, there exists a constant $C_{\Delta,M} > 0$ satisfying:

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$$\|\chi_E \mathbf{Q}_t(\chi_F u)\|_{\mathrm{L}^2(\mathcal{V})} \leq C_{\Delta,M} \|A\|_\infty^2 \left\langle \frac{\rho(E,F)}{t} \right\rangle^{-M} \exp\left(-C_{\mathbf{Q}} \frac{\rho(E,F)}{t}\right) \|\chi_F u\|_{\mathrm{L}^2(\mathcal{W})}$$

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- Assume that $\mathbf{Q}_t$ satisfies quadratic estimates: there exists $C'_{\mathbf{Q}} > 0$ so that

$$\int_0^1 \|\mathbf{Q}_t u\|^2 \ \frac{dt}{t} \leq C'_{\mathbf{Q}} \|A\|_\infty^2 \|u\|^2$$

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$$\int_{\mathrm{B}} |u - u_Q|^2 \ d\mu \leq C_P r^{\kappa} \mathrm{e}^{c_E r t} (rt)^2 \int_{\mathrm{B}} \left(|\nabla u|^2 + |u|^2 \right) \ d\mu$$

for $u \in \mathrm{W}^{1,2}(\mathcal{V})$, for all balls $\mathrm{B} = \mathrm{B}(x_Q, rt)$ with $r \geq C_1/\delta$ where $Q \in \mathscr{Q}_t$ with $t \leq \mathrm{t_S}$.

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- Decompose I into annuli, and using this bundle Poincaré inequality along with the off diagonal decay and assuming $\|\nabla^W S u\| \lesssim \|u\|_{\mathrm{W}^{1,2}(\mathcal{V})}$, obtain the desired bound for I.

Estimating term II

- On each dyadic cube Q, and for each $u \in \mathrm{W}^{1,2}(\mathcal{V})$ with $\mathrm{spt}\ u \subset Q$ and $v \in \mathcal{D}(\mathrm{div})$ with $\mathrm{spt}\ v \subset Q$, we have that

$$\left|\int_Q \mathrm{D}u\ d\mu\right|,\ \left|\int_Q \nabla u\ d\mu\right| \lesssim \mu(Q)^{\frac{1}{2}} \|u\| \quad \text{and}$$

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- For Υ one of D, $\tilde{\mathrm{D}}$, ∇, or div,

$$\left|\fint_Q \Upsilon u\ d\mu\right|^2 \lesssim \frac{1}{\ell(Q)^\eta} \left(\fint_Q |u|^2\ d\mu\right)^{\frac{\eta}{2}} \left(\fint_Q |\Upsilon u|^2\right)^{1-\frac{\eta}{2}} + \fint_Q |u|^2,$$

for all $u \in \mathcal{D}(\Upsilon)$, $Q \in \mathscr{Q}$, $t \in (0, \mathrm{t_S}]$,

- For U_t one of $\mathrm{R}_t = (\mathrm{I} + \imath t\mathrm{D})^{-1}$, $\mathrm{P}_t = (\mathrm{I} + t^2\mathrm{D}^2)^{-1}$, $\mathrm{Q}_t = t\mathrm{D}(\mathrm{I} + t^2\mathrm{D}^2)^{-1}$, $t\nabla\mathrm{P}_t$, $\tilde{\mathrm{P}}_t t \,\mathrm{div}$, and $\tilde{\mathrm{Q}}_t$, there exists $C_U > 0$ such that, for each $M > 0$, there exists a constant $C_\Delta > 0$ so that

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for every Borel set $E,\ F \subset \mathcal{M}$ and $u \in \mathrm{L}^2(\mathcal{V})$.
- The estimate is then a Schur-type estimate, i.e., the required estimate follows from showing that:

$$\|\mathbb{E}_t S(\mathrm{I} - \mathrm{P}_t)\mathrm{Q}_s\| \lesssim \min\left\{ \left(\frac{s}{t}\right)^\alpha, \left(\frac{t}{s}\right)^\alpha \right\}$$

Estimating term III

- The measure ν is a *local Carleson measure* on $\mathcal{M} \times (0, t']$ (for some $t' \in (0, \mathrm{t_S}])$ if

$$\|\nu\|_{\mathcal{C}} = \sup_{t \in (0,t']} \sup_{Q \in \mathscr{Q}_t} \frac{\nu(\mathrm{R}(Q))}{\mu(Q)} < \infty,$$

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- For a Carleson measure ν, Carleson's inequality yields

$$\iint_{\mathcal{M} \times (0,t']} |\mathbb{E}_t(x)|^2 \, d\nu(x,t) \lesssim \|\nu\|_{\mathcal{C}} \|u\|^2$$

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  for all $u \in \mathrm{L}^2(\mathcal{V})$.
- Reduce the estimate of III to showing that

$$d\nu(x,t) = |\gamma_t(x)|^2 \, \frac{d\mu(x)dt}{t}.$$

- Note that

$$\iint_{\mathrm{R}(Q)} |\gamma_t(x)|^2 \ \frac{d\mu(x)dt}{t} \lesssim \sup_{|w|_{\mathbb{C}^N}=1} \int_0^{\ell(Q)} \int_Q |\gamma_t \mathbb{E}_t w_Q|^2 \frac{d\mu dt}{t}.$$

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- Further split the right hand side:

$$\int_0^{\ell(Q)} \int_Q |\gamma_t \mathbb{E}_t w_Q|^2 \frac{d\mu dt}{t} \lesssim \int_0^{\ell(Q)} \int_Q |(\gamma_t \mathbb{E}_t - \mathbf{Q}_t) w_Q|^2 \frac{d\mu dt}{t} + \int_0^{\ell(Q)} \int_Q |\mathbf{Q}_t w_Q|^2 \frac{d\mu dt}{t}$$

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- The required estimates follow immediately from off-diagonal estimates due to the smoothness of the coefficients A.

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