Rough metrics, the Kato square root problem, and the continuity of a flow tangent to the Ricci flow

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Motivation: the flow of Gigli-Mantegazza

Let $\mathcal{M}$ be a compact manifold with a smooth metric g. Let Δ_{g} be its Laplacian (on functions) and $\rho^{\mathrm{g}}_{\cdot}(\cdot, \cdot) \in \mathrm{C}^{\infty}(\mathbb{R}_{+} \times \mathcal{M} \times \mathcal{M})$ denote the heat kernel.

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Fix a point $x \in \mathcal{M}$ and a time $t > 0$, and two tangent vectors $u, v \in \mathrm{T}_x\mathcal{M}$. Let $\varphi_{t,x,v} \in \mathrm{L}^2(\mathcal{M})$ with $\int_{\mathcal{M}} \varphi_{t,x,v}\, d\mu_{\mathrm{g}} = 0$ be the solution to the PDE:

$$-\operatorname{div}_{\mathrm{g},y} \rho^{\mathrm{g}}_t(x,y)\nabla\varphi_{t,x,v}(y) = \mathrm{d}_x(\rho^{\mathrm{g}}_t(x,y))(v), \qquad \text{(GMC)}$$

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Gigli and Mantegazza in [GM] define $\mathrm{g}_t(x)$ on tangent vectors $u, v \in \mathrm{T}_x\mathcal{M}$ by the expression:

$$\mathrm{g}_t(x)(u,v) = \int_{\mathcal{M}} \mathrm{g}(y)(\nabla\varphi_{t,x,u}(y), \nabla\varphi_{t,x,v}(y))\ \rho^{\mathrm{g}}_t(x,y)\ d\mu_{\mathrm{g}}(y). \tag{GM}$$

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Moreover, Gigli and Mantegazza show that:

$$\partial_t \mathrm{g}_t(\dot\gamma(s), \dot\gamma(s))\big|_{t=0} = -2\,\mathrm{Ric}_{\mathrm{g}}(\dot\gamma(s), \dot\gamma(s)),$$

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In fact, the fact that (GMC) can be given meaning in Wasserstein space means exactly that the flow of distance metrics d_t can be defined for an RCD-space $(\mathcal{X}, \mathrm{d}, \mu)$ (a measure metric space with a notion of Ricci curvature bounded from below and with a Hilbertian Sobolev space).

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In particular, this allows us to flow spaces containing singularities. Given that there are few tools to consider regularity questions in the RCD setting, we consider this problem on a *smooth* manifold $\mathcal{M}$ but with low-regularity metrics.

Rough metrics

Assume that $\mathcal{M}$ is a manifold (possibly noncompact).

Definition (Rough metric)

Let $\tilde{\mathrm{g}}$ be a $(2,0)$ symmetric tensor field with measurable coefficients and that for each $x \in \mathcal{M}$, there is some chart (U,ψ) near x and a constant $C \geq 1$ such that

$$C^{-1} \left|u\right|_{\psi^*\delta(y)} \leq \left|u\right|_{\tilde{\mathrm{g}}(y)} \leq C \left|u\right|_{\psi^*\delta(y)},$$

for almost-every $y \in U$ and where δ is the Euclidean metric in $\psi(U)$. Then we say that $\tilde{\mathrm{g}}$ is a rough metric, and such a chart (U,ψ) is said to satisfy the *local comparability condition*.

Induced measure

Define $\mu_{\tilde{g}}$ for a rough metric $\tilde{g}$ by writing

$$d\mu_{\tilde{g}}(x) = \sqrt{\det \tilde{g}(x)}\, d\mathscr{L}(x)$$

inside charts satisfying the local comparability condition and then patching them together via a partition of unity.

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This measure is Borel-regular and finite on compact sets. It is unknown whether they are generally Radon. However, if $\mathcal{M}$ is compact, then it is.

Moreover, L^p theory exists (trivial) and ∇ on $\mathrm{C}^\infty \cap \mathrm{L}^2$ is a closable, densely-defined operator which gives Sobolev spaces $\mathrm{W}^{1,2}(\mathcal{M})$ and $\mathrm{W}^{1,2}_0(\mathcal{M})$.

Metric perturbations

Definition

We say that two rough metrics g and $\tilde{\mathrm{g}}$ are C-close if

$$C^{-1} |u|_{\tilde{\mathrm{g}}(x)} \leq |u|_{\mathrm{g}(x)} \leq C |u|_{\tilde{\mathrm{g}}(x)}$$

for almost-every $x \in \mathcal{M}$ where $C \geq 1$. Two such metrics are said to be C-close everywhere if this inequality holds for every $x \in \mathcal{M}$.

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Note: on a compact manifold, there is always a C-close *smooth* metric g given a rough metric $\tilde{\mathrm{g}}$.

Proposition

Let g *and* $\tilde{\mathrm{g}}$ *be two rough metrics that are* C*-close. Then, there exists* $\mathrm{B} \in \mathbf{\Gamma}(\mathrm{T}^*\mathcal{M} \otimes \mathrm{T}\mathcal{M})$ *such that it is symmetric, almost-everywhere positive and invertible, and*

$$\tilde{\mathrm{g}}_x(\mathrm{B}(x)u, v) = \mathrm{g}_x(u, v)$$

for almost-every $x \in \mathcal{M}$*. Furthermore, for almost-every* $x \in \mathcal{M}$*,*

$$C^{-2} \left|u\right|_{\tilde{\mathrm{g}}(x)} \leq \left|\mathrm{B}(x)u\right|_{\tilde{\mathrm{g}}(x)} \leq C^2 \left|u\right|_{\tilde{\mathrm{g}}(x)},$$

and the same inequality with $\tilde{\mathrm{g}}$ *and* g *interchanged. If* $\tilde{\mathrm{g}} \in \mathrm{C}^k$ *and* $\mathrm{g} \in \mathrm{C}^l$ *(with* $k, l \geq 0$*), then the properties of* B *are valid for all* $x \in \mathcal{M}$ *and* $\mathrm{B} \in \mathrm{C}^{\min\{k,l\}}(\mathrm{T}^*\mathcal{M} \otimes \mathrm{T}\mathcal{M})$.

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(i) whenever $p \in [1, \infty)$, $\mathrm{L}^p(\mathcal{T}^{(r,s)}\mathcal{M}, \mathrm{g}) = \mathrm{L}^p(\mathcal{T}^{(r,s)}\mathcal{M}, \tilde{\mathrm{g}})$ with

$$C^{-\left(r+s+\frac{n}{2p}\right)} \|u\|_{p,\tilde{\mathrm{g}}} \leq \|u\|_{p,\mathrm{g}} \leq C^{r+s+\frac{n}{2p}} \|u\|_{p,\tilde{\mathrm{g}}},$$

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(ii) for $p = \infty$, $\mathrm{L}^\infty(\mathcal{T}^{(r,s)}\mathcal{M}, \mathrm{g}) = \mathrm{L}^\infty(\mathcal{T}^{(r,s)}\mathcal{M}, \tilde{\mathrm{g}})$ with

$$C^{-(r+s)} \|u\|_{\infty,\tilde{\mathrm{g}}} \leq \|u\|_{\infty,\mathrm{g}} \leq C^{r+s} \|u\|_{\infty,\tilde{\mathrm{g}}},$$

(iii) the Sobolev spaces $\mathrm{W}^{1,p}(\mathcal{M}, \mathrm{g}) = \mathrm{W}^{1,p}(\mathcal{M}, \tilde{\mathrm{g}})$ and $\mathrm{W}_0^{1,p}(\mathcal{M}, \mathrm{g}) = \mathrm{W}_0^{1,p}(\mathcal{M}, \tilde{\mathrm{g}})$ with

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(iii) the Sobolev spaces $\mathrm{W}^{1,p}(\mathcal{M},\mathrm{g}) = \mathrm{W}^{1,p}(\mathcal{M},\tilde{\mathrm{g}})$ and $\mathrm{W}^{1,p}_0(\mathcal{M},\mathrm{g}) = \mathrm{W}^{1,p}_0(\mathcal{M},\tilde{\mathrm{g}})$ with

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(v) the divergence operators satisfy $\mathrm{div}_{D,\mathrm{g}} = \theta^{-1}\,\mathrm{div}_{D,\tilde{\mathrm{g}}}\,\theta\mathrm{B}$ and $\mathrm{div}_{N,\mathrm{g}} = \theta^{-1}\,\mathrm{div}_{N,\tilde{\mathrm{g}}}\,\theta\mathrm{B}$.

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Note: Rough metrics are natural geometric invariances of the Kato square root problem. See [B2].

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Concern: regularity of the metric

$$x \mapsto \mathrm{g}_t(x)(u,v) = \langle \rho_t^{\mathrm{g}}(x,\cdot)\nabla\varphi_{t,x,u}, \nabla\varphi_{t,x,v}\rangle_{\mathrm{L}^2(\mathrm{g})}.$$

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Let $\mathcal{M}$ be a smooth, compact manifold and g a rough metric. Let $\varnothing \neq \mathcal{N} \subset \mathcal{M}$ be an open set.

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(ii) *If the heat kernel $(x, y) \mapsto \rho_t^{\mathrm{g}}(x, y) \in \mathrm{C}^1(\mathcal{M}^2)$ and $(x, y) \mapsto \rho_t^{\mathrm{g}}(x, y) \in \mathrm{C}^k(\mathcal{N}^2)$ where $k \geq 1$. Then, for $t > 0$, g_t is a Riemannian metric on $\mathcal{N}$ of regularity C^{k-1}.*

Standing question: What happens if we *only* assume that $(x, y) \mapsto \rho_t^{\mathrm{g}}(x, y) \in \mathrm{C}^1(\mathcal{N}^2)$ (i.e., no $\mathrm{C}^{0,1}$ or C^1 assumptions on global regularity).

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Theorem (B. ([B], 2015))

Let $\mathcal{M}$ be a smooth, compact manifold, and $\varnothing \neq \mathcal{N} \subset \mathcal{M}$, an open set. Suppose that g is a rough metric and that $\rho_t^{\mathrm{g}} \in \mathrm{C}^1(\mathcal{N}^2)$. Then, g_t as defined by (GM) *exists on $\mathcal{N}$ and it is continuous.*

The equation (GMC) is a specific case of *pointwise* linear problems of the form:

$$\mathrm{L}_x u_x = \eta_x \tag{PE}$$

for suitable source data $\eta_x \in \mathrm{L}^2(\mathcal{M})$ and where $\mathrm{L}_x = -\operatorname{div} A_x \nabla$ is a family of divergence form operators.

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Theorem (B. ([B], 2015))

Let $\mathcal{M}$ be a smooth manifold and g a smooth metric. At $x \in \mathcal{M}$ suppose that $x \mapsto A_x$ are real, symmetric, elliptic, bounded measurable coefficients that are L^∞-continuous at x, and that $x \mapsto \eta_x$ is L^2-continuous at x. If $x \mapsto u_x$ solves (PE) *at x with $\int_{\mathcal{M}} \eta_x \, d\mu_{\mathrm{g}} = 0$, then $x \mapsto u_x$ is L^2-continuous at x.*

Representation of solutions to the PDE

The equation (PE) can be further reduced to studying elliptic problems of the form

$$\mathrm{L}_A u = -\operatorname{div}_{\mathrm{g}} A \nabla u = f, \tag{E}$$

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By the operator theory of self-adjoint operators, we obtain that $\mathrm{L}^2(\mathcal{M}) = \mathcal{N}(\mathrm{L}_A) \oplus^{\perp} \overline{\mathcal{R}(\mathrm{L}_A)}$.

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Then, for $f \in \mathcal{R}$, $u = \mathrm{L}_A^{-1} f$ is a solution to (E) satisfying $\int_{\mathcal{M}} u \; d\mu_{\mathrm{g}} = 0$.

Back to the pointwise elliptic linear equation

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Let $T_x = \sqrt{\mathrm{L}_x} = \sqrt{-\operatorname{div} A_x \nabla}$, and let $u_x, u_y \in \mathrm{L}^2(\mathcal{M})$ such that $\int_{\mathcal{M}} u_x \, d\mu_{\mathrm{g}} = \int_{\mathcal{M}} u_y \, d\mu_{\mathrm{g}} = 0$.

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Then,

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where $v_x = T_x^{-1} u_x$ and $v_y = T_y^{-1} u_y$.

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$$\|\mathrm{L}_x^{-1} u_x - \mathrm{L}_y^{-1} u_y\| = \|T_x^{-1} v_x - T_y^{-1} v_y\|$$

where $v_x = T_x^{-1} u_x$ and $v_y = T_y^{-1} u_y$. To prove L^2 continuity, it suffices to show that

$$\|T_x^{-1} v_x - T_y^{-1} v_y\| \lesssim \|A_x - A_y\|_\infty \|v_x\| + \|v_x - v_y\|.$$

Also,

$$\begin{aligned}\|T_x^{-1}v_x - T_y^{-1}v_y\| &\leq \|(T_x^{-1} - T_y^{-1})v_x\| + \|T_y^{-1}(v_x - v_y)\| \\ &\leq \|(T_x^{-1} - T_y^{-1})v_x\| + \|(T_y^{-1} - T_x^{-1})(v_x - v_y)\| \\ &\qquad + \|T_x^{-1}(v_x - v_y)\|.\end{aligned}$$

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So, for $u \in \mathrm{L}^2(\mathcal{M})$ with $\int_{\mathcal{M}} u \, d\mu_{\mathrm{g}} = 0$,

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Thus, it suffices to show

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Such an estimate follows from *holomorphic dependency of the functional calculus* if we are able to prove a *homogeneous Kato square root estimate*.

Axelsson (Rosén)-Keith-McIntosh framework

(H1) The operator $\Gamma : \mathcal{D}(\Gamma) \subset \mathscr{H} \to \mathscr{H}$ is a closed, densely-defined and nilpotent operator, by which we mean $\mathcal{R}(\Gamma) \subset \mathcal{N}(\Gamma)$,

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$$\mathrm{Re}\,\langle B_1 u, u\rangle \geq \kappa_1 \|u\|^2 \text{ and } \mathrm{Re}\,\langle B_2 v, v\rangle \geq \kappa_2 \|v\|^2,$$

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Let us now define $\Pi_B = \Gamma + B_1 \Gamma^* B_2$ with domain $\mathcal{D}(\Pi_B) = \mathcal{D}(\Gamma) \cap \mathcal{D}(B_1 \Gamma^* B_2)$.

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Π_B is an ω-bisectorial operator with $\omega \in [0, \pi/2)$.

Quadratic estimates

To say that Π_B satisfies *quadratic estimates* means that

$$\int_0^\infty \|t\Pi_B(\mathrm{I} + t^2\Pi_B^2)^{-1}u\|^2 \ \frac{dt}{t} \simeq \|u\|^2, \tag{Q}$$

for all $u \in \overline{\mathcal{R}(\Pi_B)}$.

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This implies that

$$\mathcal{D}(\sqrt{\Pi_B^2}) = \mathcal{D}(\Pi_B) = \mathcal{D}(\Gamma) \cap \mathcal{D}(\Gamma^* B_2)$$
$$\|\sqrt{\Pi_B^2}u\| \simeq \|\Pi_B u\| \simeq \|\Gamma u\| + \|\Gamma^* B_2 u\|$$

More importantly, for coefficients $A_1, A_2 \in \mathcal{L}(\mathscr{H})$ satisfying

(i) $\|A_i\|_\infty \leq \eta_i < \kappa_i$,

(ii) $A_1 A_2 \mathcal{R}(\Gamma), B_1 A_2 \mathcal{R}(\Gamma), A_1 B_2 \mathcal{R}(\Gamma) \subset \mathcal{N}(\Gamma)$, and

(iii) $A_2 A_1 \mathcal{R}(\Gamma^*), B_2 A_1 \mathcal{R}(\Gamma^*), A_2 B_1 \mathcal{R}(\Gamma^*) \subset \mathcal{N}(\Gamma^*)$,

we obtain that for an appropriately chosen $\mu < \pi/2$, and for all bounded holomorphic functions f in an open bisector containing the closed ω-bisector,

$$\|f(\Pi_B) - f(\Pi_{B+A})\| \lesssim (\|A_1\|_\infty + \|A_2\|_\infty)\|f\|_\infty. \tag{Hol}$$

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This framework and connections to the Kato square root problem can be found in their paper [AKMc].

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This framework and connections to the Kato square root problem can be found in their paper [AKMc]. This is a first-order reformulation Kato square root problem resolved by Auscher, Hofmann, Lacey, McIntosh, and Tchamitchian in [AHLMcT].

The Kato square root problem on manifolds

Let $\mathscr{H} = \mathrm{L}^2(\mathcal{M}) \oplus \mathrm{L}^2(\mathcal{M}) \oplus \mathrm{L}^2(\mathrm{T}^*\mathcal{M})$, and set $S = (\mathrm{I}, \nabla)$

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for $a \in \mathrm{L}^\infty(\mathcal{M})$ and $A \in \mathrm{L}^\infty((\mathcal{M} \times \mathbb{C}) \oplus \mathrm{T}^*\mathcal{M})$.

Theorem (B., McIntosh ([BMc], 2012))

Let $(\mathcal{M}, \mathrm{g})$ *be a smooth, complete Riemannian manifold with* $|\mathrm{Ric}| \leq C$ *and* $\mathrm{inj}(M) \geq \kappa > 0$*. Suppose that the following ellipticity condition holds: there exist* $\kappa_1, \kappa_2 > 0$ *such that* $\mathrm{Re}\,\langle au, u\rangle \geq \kappa_1 \|u\|^2$ *and*

$$\mathrm{Re}(\langle A_{11}\nabla v, \nabla v\rangle + \langle A_{10} v, \nabla v\rangle + \langle A_{01}\nabla v, v\rangle + \langle A_{00} v, v\rangle) \geq \kappa_2 \|v\|^2_{\mathrm{W}^{1,2}}$$

for all $u \in \mathrm{L}^2(\mathcal{M})$ *and* $v \in \mathrm{W}^{1,2}(\mathcal{M})$*. Let* $D_A u = -a \,\mathrm{div}\, A_{11}\nabla u - a\,\mathrm{div}\, A_{10} u + aA_{01}\nabla u + aA_{00}u$*. Then, the quadratic estimates* (Q) *are satisfied,* $\mathcal{D}(\sqrt{D_A}) = \mathcal{D}(\nabla) = \mathrm{W}^{1,2}(\mathcal{M})$ *with* $\|\sqrt{D_A} u\| \simeq \|\nabla u\| + \|u\| = \|u\|_{\mathrm{W}^{1,2}}$ *for all* $u \in \mathrm{W}^{1,2}(\mathcal{M})$*, and*

$$\|\sqrt{D_A} u - \sqrt{D_B u}\| \lesssim \|A - B\|_\infty \|u\|_{\mathrm{W}^{1,2}},$$

whenever b, B *are coefficients that satisfy accretivity assumptions with* $\eta_i < \kappa_i$

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$$\|u\| \lesssim \|\Pi u\|,$$

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  for $u \in \mathcal{D}(\Pi) \cap \mathcal{R}(\Pi)$. This is almost *trivial* for the inhomogeneous problem.

The homogeneous Kato square root problem on compact manifolds

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By self-adjointness for Π and, if the coefficients satisfy (H1)-(H3) by bi-sectoriality,

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Thus, we have that $\mathrm{L}^2(\mathcal{M}) = \mathcal{N}(\nabla) \oplus^{\perp} \overline{\mathcal{R}(\mathrm{div})}$ and $\mathrm{L}^2(\mathrm{T}^*\mathcal{M}) = \mathcal{N}(\mathrm{div}) \oplus^{\perp} \overline{\mathcal{R}(\nabla)}$.

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Moreover,

$$\overline{\mathcal{R}(\mathrm{div})} = \left\{ u \in \mathrm{L}^2(\mathcal{M}) : \int_{\mathcal{M}} u \, d\mu_{\mathrm{g}} = 0 \right\} = \mathcal{R}.$$

Now, let $u \in \mathcal{R}(\Pi) \cap \mathcal{D}(\Pi)$. So $u = (u_1, u_2)$, and

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Now, $u_2 = \nabla v_2$, for some $v_2 \in \mathcal{D}(\Pi)$. So,

$$\begin{aligned}\|\operatorname{div} u_2\| = \|\Delta v_2\| = \|\sqrt{\Delta}\sqrt{\Delta} v_2\| \geq C\|\sqrt{\Delta} v_2\| \\ = C\|\nabla v_2\| = C\|u_2\|.\end{aligned}$$

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That is,

$$\|\Pi u\| \geq C\|u\|.$$

Theorem (B., ([B], 2015))

On a compact manifold $\mathcal{M}$ with a smooth metric g, the operator Π_B admits a bounded functional calculus. In particular, $\mathcal{D}(\sqrt{b \operatorname{div} B\nabla}) = \mathrm{W}^{1,2}(\mathcal{M})$ and $\|\sqrt{b \operatorname{div} B\nabla} u\| \simeq \|\nabla u\|$. Moreover, whenever $\|\tilde{b}\|_\infty < \eta_1$ and $\|\tilde{B}\|_\infty < \eta_2$, where $\eta_i < \kappa_i$, we have the following Lipschitz estimate

$$\|\sqrt{b \operatorname{div} B\nabla} u - \sqrt{(b+\tilde{b}) \operatorname{div}(B+\tilde{B})\nabla} u\| \lesssim (\|\tilde{b}\|_\infty + \|\tilde{B}\|_\infty)\|\nabla u\|$$

whenever $u \in \mathrm{W}^{1,2}(\mathcal{M})$. The implicit constant depends on b, B and η_i.

Corollary

Fix $x \in \mathcal{M}$ and $u \in \mathrm{W}^{1,2}(\mathcal{M})$. If $\|A_x - A_y\| \leq \zeta < \kappa_x$, then for $u \in \mathrm{W}^{1,2}(\mathcal{M})$,

$$\|\sqrt{\mathrm{L}_x} u - \sqrt{\mathrm{L}_y} u\| \lesssim \|A_x - A_y\|_\infty \|\nabla u\|.$$

The implicit constant depends on ζ and A_x.

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Fix $x \in \mathcal{M}$ *and* $u \in \mathrm{W}^{1,2}(\mathcal{M})$. *If* $\|A_x - A_y\| \leq \zeta < \kappa_x$, *then for* $u \in \mathrm{W}^{1,2}(\mathcal{M})$,

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The implicit constant depends on ζ *and* A_x.

Corollary

Fix $x \in \mathcal{M}$ *and suppose that* $\|A_x - A_y\| \leq \zeta < \kappa_x$. *Then,*

$$\|\mathrm{L}_x^{-1}\eta_x - \mathrm{L}_y^{-1}\eta_y\| \lesssim \|A_x - A_y\|_\infty \|\eta_x\| + \|\eta_x - \eta_y\|,$$

whenever $\eta_x, \eta_y \in \mathrm{L}^2(\mathcal{M})$ *satisfies* $\int_{\mathcal{M}} \eta_x \, d\mu_{\mathrm{g}} = \int_{\mathcal{M}} \eta_y \, d\mu_{\mathrm{g}} = 0$. *The implicit constant depends on* ζ, κ_x, *and* A_x.

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