Boundary value problems for general first-order elliptic operators

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Setup

- M smooth manifold with smooth compact boundary $\Sigma = \partial M$;
- τ interior co-vectorfield along ∂M;
- μ smooth volume measure on M and ν induced smooth volume measure on Σ;
- (E, h^E), $(F, h^F) \to M$ Hermitian vector bundles over M;
- D first-order elliptic differential operator from E to F;
- D and D^* complete - i.e., $\mathrm{C}^\infty_c(E;F)$ and $\mathrm{C}^\infty_c(F;E)$ dense in $\mathrm{dom}(D_{\max})$ and $\mathrm{dom}(D^*_{\max})$ respectively.

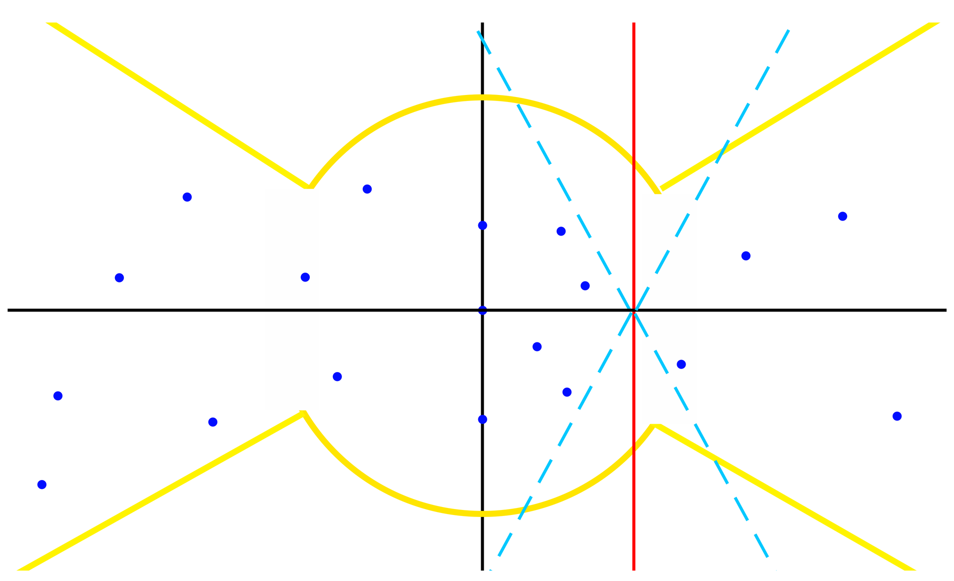
Adapted boundary operator

Principal symbol for D and D^*: $\sigma_D(x,\xi)$ and $\sigma_{D^*}(x,\xi)$, define $\sigma_0(x) := \sigma_D(x,\tau(x))$.

A and $\tilde{A}$ are *adapted boundary operators* (to D or D^* respectively) on $E_\Sigma := E|_\Sigma$ and $F_\Sigma := F|_\Sigma$ respectively if their principal symbols are given by:

$$\sigma_A(x,\xi) = \sigma_D(x,\tau(x))^{-1} \circ \sigma_D(x,\xi) \quad \text{and} \quad \sigma_{\tilde{A}}(x,\xi) = \sigma_{D^*}(x,\tau(x))^{-1} \circ \sigma_{D^*}(x,\xi).$$

- Exists and are elliptic differential operators of order 1.
- Unique up to an operator of order zero.
- Discrete spectrum, generally non-orthogonal eigenspaces.
- **No additional assumptions on A (i.e., self-adjointness) apart from ellipticity of D:**



Admissible cut $r \in \mathbb{R}$: the line $l_r := \{\zeta \in \mathbb{C} : \mathrm{Re}\,\zeta = r\}$ is not in the spectrum of A (yields $A_r := A - r$ invertible bi-sectorial).

An admissible cut always exists.

$\chi^\pm(A_r) : \mathrm{L}^2(E_\Sigma) \to \mathrm{L}^2(E_\Sigma)$ spectral projectors to the left and right of l_r - pseudos of order zero.

- Space: $\check{H}(A) := \chi^-(A_r)\mathrm{H}^{\frac12}(E_\Sigma) \oplus \chi^+(A_r)\mathrm{H}^{-\frac12}(E_\Sigma)$.
- Norm: $\|u\|^2_{\check{H}(A)} := \|\chi^-(A_r)u\|^2_{\mathrm{H}^{\frac12}} + \|\chi^+(A_r)u\|^2_{\mathrm{H}^{-\frac12}}$.
- *Norms corresponding to two different spectral cuts are comparable.*

Theorem 1: Maximal domains and $\check{H}(A)$, $\check{H}(\tilde{A})$ spaces

- $\mathrm{C}^\infty_c(E)$ is dense in $\mathrm{dom}(D_{\max})$ and $\mathrm{dom}((D^*)_{\max})$ with respect to corresponding graph norms.
- The trace maps $\mathrm{C}^\infty_c(E) \to \mathrm{C}^\infty_c(E_\Sigma)$ and $\mathrm{C}^\infty_c(F) \to \mathrm{C}^\infty_c(F_\Sigma)$ given by $u \mapsto u|_\Sigma$ extend uniquely to surjective bounded linear maps $\mathrm{dom}(D_{\max}) \to \check{H}(A)$ and $\mathrm{dom}((D^*)_{\max}) \to \check{H}(\tilde{A})$.
- The spaces
$$\mathrm{dom}(D_{\max}) \cap \mathrm{H}^1_{\mathrm{loc}}(E_\Sigma) = \left\{u \in \mathrm{dom}(D_{\max}) : u|_\Sigma \in \mathrm{H}^{\frac12}(E_\Sigma)\right\}$$
$$\mathrm{dom}((D^*)_{\max}) \cap \mathrm{H}^1_{\mathrm{loc}}(F_\Sigma) = \left\{u \in \mathrm{dom}((D^*)_{\max}) : u|_\Sigma \in \mathrm{H}^{\frac12}(F_\Sigma)\right\}.$$
- For all $u \in \mathrm{dom}(D_{\max})$ and $v \in \mathrm{dom}((D^*)_{\max})$,
$$\langle D_{\max}u, v\rangle_{\mathrm{L}^2(F)} - \langle u, (D^*)_{\max}v\rangle_{\mathrm{L}^2(E)} = -\left\langle \sigma_0 u|_\Sigma, v|_\Sigma\right\rangle_{\mathrm{L}^2(F_\Sigma)}.$$

Theorem 2: Higher regularity

$$\mathrm{dom}(D_{\max}) \cap \mathrm{H}^{k+1}_{\mathrm{loc}}(E) = \left\{u \in \mathrm{dom}(D_{\max}) : Du \in \mathrm{H}^k_{\mathrm{loc}}(F) \text{ and } \chi^+(A_r)(u|_\Sigma) \in \mathrm{H}^{k+\frac12}(E_\Sigma)\right\}.$$

Proof ingredients of Theorems 1 and 2:

- Identification of $\mathrm{dom}(A_r) = \mathrm{dom}(A_r^*)$ by elliptic pseudo-differential operator theory.
- H^∞ functional calculus for the invertible sectorial operator $|A_r| := A_r \,\mathrm{sgn}(A_r)$.
- Semigroup theory and Kato square root problem methods: ellipticity via equivalent norm for which $|A_r|$ is maximal-accretive.
- Maximal regularity (via H^∞ functional calculus) for higher regularity.

Boundary conditions and the associated operator

A *closed* linear subspace $B \subset \check{H}(A)$ is called a *boundary condition* for D. Associated operator domains:

$$\mathrm{dom}(D_{B,\max}) = \left\{u \in \mathrm{dom}(D_{\max}) : u|_\Sigma \in B\right\}$$
$$\mathrm{dom}(D_B) = \left\{u \in \mathrm{dom}(D_{\max}) \cap \mathrm{H}^1_{\mathrm{loc}}(E_\Sigma) : u|_\Sigma \in B\right\},$$

and similarly for the formal adjoint D^* with A replaced by $\tilde{A}$.

- For boundary condition B, the operator D_B closed and between D_{cc} (on $\mathrm{C}^\infty_{cc}(E)$) and $D_{\max}$.
- D_c closed extension of D_{cc}, then $B := \left\{u|_\Sigma : u \in \mathrm{dom}(D_c)\right\}$ is a boundary condition and $D_c = D_{B,\max}$.
- Boundary condition $B \subset \mathrm{H}^{\frac12}(E_\Sigma)$ if and only if $D_B = D_{B,\max}$.
- Adjoint boundary condition B^{ad} so that $D_B^{\mathrm{ad}} = D^*_{B^{\mathrm{ad}}}$:
$$B^{\mathrm{ad}} := \left\{v \in \check{H}(-\tilde{A}) : \langle \sigma_0 u, v\rangle_{\mathrm{L}^2(F_\Sigma)} = 0 \quad \forall u \in B\right\}$$

Elliptic boundary conditions

$B \subset \mathrm{H}^{\frac12}(E_\Sigma)$ boundary condition is called *elliptic* if there exists an admissible cut $r \in \mathbb{R}$ and:

- $W_\pm$, $V_\pm$ are mutually complementary subspaces such that
$$V_\pm \oplus W_\pm = \chi^\pm(A_r)\mathrm{L}^2(E_\Sigma),$$
- $W_\pm$ are finite dimensional with $W_\pm, W_\pm^* \subset \mathrm{H}^{\frac12}(E_\Sigma)$, and
- $g : V_- \to V_+$ bounded linear map with $g(V_-^{\frac12}) \subset V_+^{\frac12}$ and $g^*((V_+^*)^{\frac12}) \subset (V_-^*)^{\frac12}$ such that
$$B = W_+ \oplus \left\{v + gv : v \in V_-^{\frac12}\right\}.$$

$B \subset \mathrm{H}^{\frac12}(E_\Sigma)$ be a subspace, then the following are equivalent:

- B a boundary condition and $B^{\mathrm{ad}} \subset \mathrm{H}^{\frac12}(F_\Sigma)$,
- the definition is satisfied for any admissible spectral cut $r \in \mathbb{R}$,
- B an elliptic boundary condition.

For elliptic boundary condition B, have B^{ad} elliptic boundary condition for D^* and
$$\sigma_0^*(B^{\mathrm{ad}}) = W_-^* \oplus \left\{u - g^*u : u \in (V_+^*)^{\frac12}\right\}.$$

Pseudo-local and local boundary conditions

- For classical pseudo-differential projector P of order zero (not necessarily orthogonal), the space
$$B = P(\mathrm{H}^{\frac12}(E_\Sigma))$$
is called a *pseudo-local boundary condition*.
- Boundary condition $B \subset \mathrm{H}^{\frac12}(E_\Sigma)$ a *local boundary condition* if there exists a sub-bundle $E' \subset E_\Sigma$ such that
$$B = \mathrm{H}^{\frac12}(E').$$

Theorem 3: Characterisation of pseudo-local boundary conditions

Given a pseudo-local boundary condition $B = P(\mathrm{H}^{\frac12}(E_\Sigma))$, the following are equivalent:

- B an elliptic boundary condition,
- for admissible cut $r \in \mathbb{R}$, the operator
$$P - \chi^+(A_r) : \mathrm{L}^2(E_\Sigma) \to \mathrm{L}^2(E_\Sigma)$$
is Fredholm,
- for admissible cut $r \in \mathbb{R}$, the operator
$$P - \chi^+(A_r) : \mathrm{L}^2(E_\Sigma) \to \mathrm{L}^2(E_\Sigma)$$
is elliptic classical pseudo of order zero.

If B is a pseudo-local elliptic boundary condition and $D_B u$ is smooth, then u is smooth up to the boundary.