Graphical decompositions for general-order boundary value problems

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GEOMETRY AT INFINITY

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Obtain: $B^\perp = W_-^* \oplus \left\{ u - g^* u : u \in V_+^* \cap \mathrm{H}^{\frac{1}{2}}(\Sigma; \mathcal{E}) \right\}$.

Proof ($\Longrightarrow$)

$$W_-^* := \mathcal{P}_-^* \mathrm{L}^2(\Sigma;\mathcal{E}) \cap B^\perp \quad V_-^* := \mathcal{P}_-^* \mathrm{L}^2(\Sigma;\mathcal{E}) \cap (W_-^*)^\perp$$

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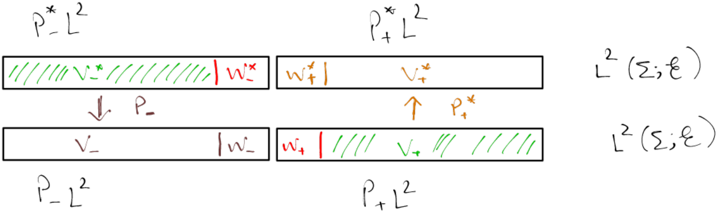

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(b) $B^\perp \subset \mathrm{H}^{\frac{1}{2}}(\Sigma;\mathcal{E})$.

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$$V_\pm = [\overline{V_\pm}^{\mathrm{H}^{-\frac{1}{2}}}, V_\pm \cap \mathrm{H}^{\frac{1}{2}}(\Sigma; \mathcal{E})]_{\theta = \frac{1}{2}}.$$

General-order

- $\check{\mathrm{H}}(\mathrm{D}) = \mathcal{P}_+ \mathbb{H}^{-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m) \oplus \mathcal{P}_- \mathbb{H}^{m-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m)$.

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- B elliptically regular $\iff$ $B \subset \mathbb{H}^{m-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m)$ and
  $B^\perp \in \mathbb{H}^{\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m_{\mathrm{op}})$.

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$$B = \{v + gv : v \in V_-\} \oplus W_+.$$

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- $W_-^* \subset \mathcal{P}_-^* \mathbb{H}^{\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m_{\mathrm{op}})$ so consider $(W_-^*)^{\perp, \mathbb{H}^{-\frac{1}{2}}(\Sigma;\mathcal{E}\otimes\mathbb{C}^m)}$ in $\mathbb{H}^{-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m)$.
- Calculation:

$$\mathbb{H}^{-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m) = \oplus_{j=0}^{m-1} \mathrm{H}^{-\frac{1}{2}-j}(\Sigma; \mathcal{E})$$

$$\supset \oplus_{j=0}^{m-1} \mathrm{H}^{\frac{1}{2}-m+j}(\Sigma; \mathcal{E}) = \mathbb{H}^{\frac{1}{2}-m}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m_{\mathrm{op}})$$

$$W_-^* := \mathcal{P}_-^* \mathbb{H}^{\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m_{\mathrm{op}}) \cap B^{\perp, \mathbb{H}^{\frac{1}{2}-m}(\Sigma;\mathcal{E}\otimes\mathbb{C}^m_{\mathrm{op}})}$$

$$W_+ := \mathcal{P}_+ \mathbb{H}^{m-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m) \cap B$$

$$V_-^* := \mathcal{P}_-^* \mathbb{H}^{\frac{1}{2}-m}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m_{\mathrm{op}}) \cap (W_-^*)^{\perp, \mathbb{H}^{-\frac{1}{2}}(\Sigma;\mathcal{E}\otimes\mathbb{C}^m)}$$

$$V_+ := \mathcal{P}_+ \mathbb{H}^{m-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m) \cap W_+^{\perp, \mathbb{H}^{\frac{1}{2}-m}(\Sigma;\mathcal{E}\otimes\mathbb{C}^m_{\mathrm{op}})}$$

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$$\begin{aligned}\mathbb{H}^{-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m) &= \oplus_{j=0}^{m-1} \mathrm{H}^{-\frac{1}{2}-j}(\Sigma; \mathcal{E}) \\ &\supset \oplus_{j=0}^{m-1} \mathrm{H}^{\frac{1}{2}-m+j}(\Sigma; \mathcal{E}) = \mathbb{H}^{\frac{1}{2}-m}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m_{\mathrm{op}})\end{aligned}$$

$\implies V_-^*$ is well-defined.

- $\mathbb{H}^{m-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m) \subset \mathbb{H}^{\frac{1}{2}-m}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m_{\mathrm{op}}) \implies V_+$ well-defined.

$$W_+^* := \mathcal{P}_+^* \mathbb{H}^{\frac{1}{2}-m}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m_{\mathrm{op}}) \cap V_+^{\perp, \mathbb{H}^{\frac{1}{2}-m}(\Sigma; \mathcal{E}\otimes\mathbb{C}^m_{\mathrm{op}})}$$

$$W_- := \mathcal{P}_- \mathbb{H}^{m-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m) \cap (V_-^*)^{\perp, \mathbb{H}^{m-\frac{1}{2}}(\Sigma; \mathcal{E}\otimes\mathbb{C}^m)}$$

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Not hard to show these are ranges of the respective adjoint projectors.

Proof highlights

- $\mathcal{P}_{-}B$ and $\mathcal{P}_{+}^{*}B^{\perp}$ closed subspaces

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$$\mathbb{H}^{m-\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m) \xhookrightarrow{\text{compact}} \mathbb{H}^{-\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m)$$

$$\mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\text{op}}) \to \mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\text{op}}) \xhookrightarrow{\text{compact}} \mathbb{H}^{\frac{1}{2}-m}(\Sigma; E \otimes \mathbb{C}^m_{\text{op}})$$

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$$\begin{array}{l} \mathbb{H}^{m-\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m) \xhookrightarrow{\text{compact}} \mathbb{H}^{-\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m) \\ \mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\text{op}}) \to \mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\text{op}}) \xhookrightarrow{\text{compact}} \mathbb{H}^{\frac{1}{2}-m}(\Sigma; E \otimes \mathbb{C}^m_{\text{op}}) \end{array}.$$

- Not immediate that $V_-^* \oplus W_-^* = \mathcal{P}_-^* \mathbb{H}^{m-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m)$.

Proof highlights

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$$\begin{gathered}\mathbb{H}^{m-\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m) \xhookrightarrow{\text{compact}} \mathbb{H}^{-\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m) \\ \mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\mathrm{op}}) \to \mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\mathrm{op}}) \xhookrightarrow{\text{compact}} \mathbb{H}^{\frac{1}{2}-m}(\Sigma; E \otimes \mathbb{C}^m_{\mathrm{op}})\end{gathered}.$$

- Not immediate that $V_-^* \oplus W_-^* = \mathcal{P}_-^* \mathbb{H}^{m-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m)$.
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- Not immediate that $V_-^* \oplus W_-^* = \mathcal{P}_-^* \mathbb{H}^{m-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m)$.
  - $V_-^* \cap W_-^* = 0$ obtained from $\langle w, w \rangle = \|w\|^2_{\mathrm{L}^2}$ when $w \in \mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\text{op}}) \subset \oplus_{j=0}^{m-1} \mathrm{L}^2(\Sigma; E)$.

Proof highlights

- $\mathcal{P}_- B$ and $\mathcal{P}_+^* B^\perp$ closed subspaces, $W_+ = \ker(\mathcal{P}_-|_B)$ and $W_-^* = \ker(\mathcal{P}_+^*|_{B^\perp})$.

$$\mathbb{H}^{m-\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m) \xhookrightarrow{\text{compact}} \mathbb{H}^{-\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m)$$
$$\mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\mathrm{op}}) \to \mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\mathrm{op}}) \xhookrightarrow{\text{compact}} \mathbb{H}^{\frac{1}{2}-m}(\Sigma; E \otimes \mathbb{C}^m_{\mathrm{op}}).$$

- Not immediate that $V_-^* \oplus W_-^* = \mathcal{P}_-^* \mathbb{H}^{m-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m)$.
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  - $\mathcal{P}_-^* \mathbb{H}^{\frac{1}{2}-m}(\Sigma; E \otimes \mathbb{C}^m_{\mathrm{op}}) \subset V_-^* \oplus W_-^*$ requires projector to W_-^* along V_-^*.

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$$\mathbb{H}^{m-\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m) \xhookrightarrow{\text{compact}} \mathbb{H}^{-\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m)$$
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  - $\mathcal{P}_-^* \mathbb{H}^{\frac{1}{2}-m}(\Sigma; E \otimes \mathbb{C}^m_{\mathrm{op}}) \subset V_-^* \oplus W_-^*$ requires projector to W_-^* along V_-^*.
  - $Pu = \sum_{i=1}^{\dim W_-^*} \langle u, e_i \rangle$, where e_i is a basis with $\langle e_i, e_j \rangle = \delta_{ij}$.

Proof highlights

- $\mathcal{P}_- B$ and $\mathcal{P}_+^* B^\perp$ closed subspaces, $W_+ = \ker(\mathcal{P}_-|_B)$ and $W_-^* = \ker(\mathcal{P}_+^*|_{B^\perp})$.

$$\begin{aligned} &\mathbb{H}^{m-\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m) \xhookrightarrow{\text{compact}} \mathbb{H}^{-\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m) \\ &\mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\text{op}}) \to \mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\text{op}}) \xhookrightarrow{\text{compact}} \mathbb{H}^{\frac{1}{2}-m}(\Sigma; E \otimes \mathbb{C}^m_{\text{op}}) \end{aligned}.$$

- Not immediate that $V_-^* \oplus W_-^* = \mathcal{P}_-^* \mathbb{H}^{m-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m)$.
  - $V_-^* \cap W_-^* = 0$ obtained from $\langle w, w \rangle = \|w\|^2_{\mathrm{L}^2}$ when $w \in \mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\text{op}}) \subset \oplus_{j=0}^{m-1} \mathrm{L}^2(\Sigma; E)$.
  - $\mathcal{P}_-^* \mathbb{H}^{\frac{1}{2}-m}(\Sigma; E \otimes \mathbb{C}^m_{\text{op}}) \subset V_-^* \oplus W_-^*$ requires projector to W_-^* along V_-^*.
  - $Pu = \sum_{i=1}^{\dim W_-^*} \langle u, e_i \rangle$, where e_i is a basis with $\langle e_i, e_j \rangle = \delta_{ij}$. Possible only since $e_i \in \mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\text{op}}) \subset \mathbb{H}^{-\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\text{op}})$.

- $V_+^* \cap \mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\mathrm{op}}) = \mathcal{P}_+^* B^\perp$

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- $V_+^* \cap \mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\mathrm{op}}) = \mathcal{P}_+^* B^\perp$ and $V_- = \mathcal{P}_- B$.
  - Key:

$$W_+ = (\mathcal{P}_+^* B^\perp)^{\perp, \mathbb{H}^{-\frac{1}{2}}(\Sigma; E\otimes\mathbb{C}^m)} \cap \mathcal{P}_+^* \mathbb{H}^{-\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m)$$

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$$W_-^* = (\mathcal{P}_- B)^{\perp, \mathbb{H}^{\frac{1}{2}-m}(\Sigma; E \otimes \mathbb{C}^m_{\mathrm{op}})} \cap \mathcal{P}_- \mathbb{H}^{\frac{1}{2}-m}(\Sigma; E \otimes \mathbb{C}^m_{\mathrm{op}}).$$

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  - Need: $B \subset \mathbb{H}^{m-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m)$ and $B^\perp \in \mathbb{H}^{\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m_{\text{op}})$

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- $X_- := \mathcal{P}_-|_{B \cap W_+^\perp} : B \cap W_+^\perp \to \mathcal{P}_- B$.

- $V_+^* \cap \mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\rm op}) = \mathcal{P}_+^* B^\perp$ and $V_- = \mathcal{P}_- B$.
  - Key:

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  $X_+^* := \mathcal{P}_+^*|_{B^\perp \cap (W_-^*)^{\perp, \mathbb{H}^{-\frac{1}{2}}(\Sigma; E\otimes\mathbb{C}^m)}} : B^\perp \cap (W_-^*)^{\perp, \mathbb{H}^{-\frac{1}{2}}(\Sigma; E\otimes\mathbb{C}^m)} \to \mathcal{P}_+^* B^\perp$.

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- $X_- := \mathcal{P}_-|_{B \cap W_+^\perp} : B \cap W_+^\perp \to \mathcal{P}_- B$.

  $X_+^* := \mathcal{P}_+^*|_{B^\perp \cap (W_-^*)^{\perp, \mathbb{H}^{-\frac{1}{2}}(\Sigma; E\otimes\mathbb{C}^m)}} : B^\perp \cap (W_-^*)^{\perp, \mathbb{H}^{-\frac{1}{2}}(\Sigma; E\otimes\mathbb{C}^m)} \to \mathcal{P}_+^* B^\perp$.

  Both maps bounded isomorphisms to their ranges.

- $g : V_- \to V_+$, bounded in the $\mathbb{H}^{m-\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m)$ norm, defined as

$$g := P_{V_+, W_+ \oplus \mathcal{P}_- \mathbb{H}^{m-\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m)} \circ (X_-)^{-1}.$$

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- $h : V_+^* \cap \mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\mathrm{op}}) \to V_-^* \cap \mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\mathrm{op}})$

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- From $B \perp B^\perp$ in the $\check{\mathrm{H}}(\mathrm{D}) \times \hat{\mathrm{H}}(\mathrm{D}, \mathcal{P}_+)$, obtain

$$g^*(V_+^* \cap \mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\mathrm{op}})) = -h(V_+^* \cap \mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\mathrm{op}})) \subset V_-^* \cap \mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\mathrm{op}}).$$