Graphical decompositions for general-order boundary value problems

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The setting

- $\mathcal{M}$ compact smooth manifold with smooth boundary $\Sigma := \partial\mathcal{M}$.
- Smooth measure μ.
- Let $\vec{T}$ inward pointing vectorfield, and τ associated inward pointing co-vectorfield.
- $(\mathcal{E}, \mathrm{h}^{\mathcal{E}}) \to \mathcal{M}$ and $(\mathcal{F}, \mathrm{h}^{\mathcal{F}}) \to \mathcal{M}$ Hermitian bundles.
- $\mathrm{D} : \mathrm{C}^\infty(\mathcal{E}) \to \mathrm{C}^\infty(\mathcal{F})$ order $m \geq 1$ differential operator.
- D *elliptic* $\iff$ $\sigma_{\mathrm{D}}(x, \xi) : \mathcal{E}_x \to \mathcal{F}_x$ invertible for $0 \neq \xi \in \mathrm{T}^*_x\mathcal{M}$.

Unique formal adjoint $\mathrm{D}^\dagger : \mathrm{C}^\infty(\mathcal{F}) \to \mathrm{C}^\infty(\mathcal{E})$, i.e.,

$$\langle \mathrm{D}u, v\rangle_{\mathrm{L}^2(\mathcal{F};\mathrm{h}^{\mathcal{F}},\mu)} = \langle u, \mathrm{D}^\dagger v\rangle_{\mathrm{L}^2(\mathcal{E};\mathrm{h}^{\mathcal{E}},\mu)}$$

$\forall u \in \mathrm{C}^\infty_{\mathrm{cc}}(\mathcal{M};\mathcal{E}),\ v \in \mathrm{C}^\infty_{\mathrm{cc}}(\mathcal{M};\mathcal{F})$.

Define:

$$\mathrm{D}_{\max} := (\mathrm{D}^\dagger)^* \qquad \text{and} \qquad \mathrm{D}_{\min} := \overline{\mathrm{D}|_{\mathrm{C}^\infty_{\mathrm{cc}}(\mathcal{M};\mathcal{E})}}.$$

I.e.

$$\mathrm{dom}(\mathrm{D}_{\max}) := \Big\{ u \in \mathrm{L}^2(\mathcal{E};\mathrm{h}^{\mathcal{E}},\mu) : \\ \exists C_u \quad |\langle u, \mathrm{D}^\dagger v\rangle| \leq C_u \|v\|_{\mathrm{L}^2(\mathcal{F};\mathrm{h}^{\mathcal{F}},\mu)} \quad \forall v \in \mathrm{C}^\infty_{\mathrm{cc}}(\mathcal{M};\mathcal{E}) \Big\}.$$

Let $\gamma : \mathrm{C}^\infty_c(\overline{\mathcal{M}}; \mathcal{E}) \to \bigoplus_{j=0}^{m-1} \mathrm{C}^\infty_c(\Sigma; \mathcal{E})$

$$\gamma(u) = \Big(\, u|_\Sigma, \ (\partial_{\vec{T}} u)|_\Sigma, \ \dots \ , \ (\partial_{\vec{T}}^{m-1} u)|_\Sigma \Big).$$

Classic result (Seeley '66, Lions-Magenes '63 (Eng '72)):
$\gamma : \mathrm{C}^\infty(\mathcal{M}; \mathcal{E}) \to \bigoplus_{j=0}^{m-1} \mathrm{C}^\infty(\Sigma; \mathcal{E})$ extends to a bounded mapping

$$\gamma : \mathrm{dom}(\mathrm{D}_{\max}) \to \bigoplus_{j=0}^{m-1} \mathrm{H}^{-\frac{1}{2}-j}(\Sigma; \mathcal{E})$$

- $\check{\mathrm{H}}(\mathrm{D}) := \mathrm{ran}\, \gamma$ dense in $\bigoplus_{j=0}^{m-1} \mathrm{H}^{-\frac{1}{2}-j}(\Sigma; \mathcal{E})$,
- $\ker \gamma = \mathrm{H}^m_0(\mathcal{M}; \mathcal{E}) = \mathrm{dom}(\mathrm{D}_{\min})$.

Topologise $\check{\mathrm{H}}(\mathrm{D})$ such that $\gamma : {}^{\mathrm{dom}(\mathrm{D}_{\max})} \big/ {}_{\mathrm{dom}(\mathrm{D}_{\min})} \rightarrowtail\!\!\!\!\twoheadrightarrow \check{\mathrm{H}}(\mathrm{D})$.

Boundary conditions

- *Boundary condition*: $B \subset \check{\mathrm{H}}(\mathrm{D})$ *closed* subspace.
  $\rightsquigarrow \quad \mathrm{D}_B$ *closed* operator.
- $\mathrm{D}_{\min} \subset \mathrm{D}_{\mathrm{ext}} \subset \mathrm{D}_{\max}$ closed extension $\iff$ $B_{\mathrm{ext}} := \{\gamma u : u \in \mathrm{dom}(\mathrm{D}_{\mathrm{ext}})\}$ boundary condition with $\mathrm{D}_{B_{\mathrm{ext}}} = \mathrm{D}_{\mathrm{ext}}$.
- *Elliptically regular boundary condition*: $\mathrm{D}_B^* = \mathrm{D}^\dagger_{B^\dagger}$ and

$$\mathrm{dom}(\mathrm{D}_B) \subset \mathrm{H}^m(\mathcal{M};\mathcal{E}) \quad \text{and} \quad \mathrm{dom}(\mathrm{D}_B^*) \subset \mathrm{H}^m(\mathcal{M};\mathcal{F}).$$

Equivalently,

$$B \subset \bigoplus_{j=0}^{m-1} \mathrm{H}^{m-\frac{1}{2}-j}(\Sigma;\mathcal{E}) \quad \text{and} \quad B^\dagger \subset \bigoplus_{j=0}^{m-1} \mathrm{H}^{m-\frac{1}{2}-j}(\Sigma;\mathcal{E}).$$

Define:

$$\mathbb{H}^s(\Sigma;\mathcal{E}\otimes\mathbb{C}^m) := \bigoplus_{j=0}^{m-1} \mathrm{H}^{s-j}(\Sigma;\mathcal{E})$$

$$\mathbb{H}^s(\Sigma;\mathcal{E}\otimes\mathbb{C}^m_{\mathrm{op}}) := \bigoplus_{j=0}^{m-1} \mathrm{H}^{s+j}(\Sigma;\mathcal{E}).$$

Boundary decomposing projector $\mathcal{P}_+$:

(i) $\mathcal{P}_+ : \mathbb{H}^\alpha(\Sigma;\mathcal{E}\otimes\mathbb{C}^m) \to \mathbb{H}^\alpha(\Sigma;\mathcal{E}\otimes\mathbb{C}^m)$ projection for $\alpha \in \left\{-\frac{1}{2}, m-\frac{1}{2}\right\}$,

(ii) $\mathcal{P}_+ : \check{\mathrm{H}}(D) \to \check{\mathrm{H}}(D)$ and $\mathcal{P}_- := (I - \mathcal{P}_+) : \check{\mathrm{H}}(D) \to \mathbb{H}^{m-\frac{1}{2}}(\Sigma;\mathcal{E}\otimes\mathbb{C}^m)$,

(iii) $\|u\|_{\check{\mathrm{H}}(\mathrm{D})} \simeq \|\mathcal{P}_- u\|_{\mathbb{H}^{m-\frac{1}{2}}(\Sigma;\mathcal{E})} + \|\mathcal{P}_+ u\|_{\mathbb{H}^{-\frac{1}{2}}(\Sigma;\mathcal{E})}$.

Examples

(i) First-order (Bär-Bandara '20):
$\mathbb{H}^s(\Sigma;\mathcal{E}\otimes\mathbb{C}^m) = \mathbb{H}^s(\Sigma;\mathcal{E}\otimes\mathbb{C}^m_{\mathrm{op}}) = \mathrm{H}^s(\Sigma;\mathcal{E})$.

Let $\mathrm{A} : \mathrm{L}^2(\Sigma;\mathcal{E}) \to \mathrm{L}^2(\Sigma;\mathcal{E})$ *boundary adapted operator*, i.e.,

$$\sigma_{\mathrm{A}}(x,\xi) = \sigma_D(x,\tau(x))^{-1} \circ \sigma_D(x,\xi).$$

Can be chosen invertible bisectorial and $\mathcal{P}_+ = \chi^+(\mathrm{A})$

$$\check{\mathrm{H}}(\mathrm{D}) = \chi^-(\mathrm{A})\mathrm{H}^{\frac{1}{2}}(\Sigma;\mathcal{E}) \oplus \chi^+(\mathrm{A})\mathrm{H}^{-\frac{1}{2}}(\Sigma;\mathcal{E}).$$

(ii) General-order (Seeley '66): $\mathcal{C}_{\mathrm{D}} := \gamma\ker(\mathrm{D}_{\max})$.

Exists classical pseudo-differential projector $\mathcal{P}_{\mathcal{C}_{\mathrm{D}}}$ of order zero such that

$$\mathcal{C}_{\mathrm{D}} = \mathcal{P}_{\mathcal{C}_{\mathrm{D}}}\mathbb{H}^{m,-\frac{1}{2}}(\Sigma;\mathcal{E}\otimes\mathbb{C}) \quad \text{and}$$

$$\check{\mathrm{H}}(\mathrm{D}) = (1-\mathcal{P}_{\mathcal{C}_{\mathrm{D}}})\mathbb{H}^{m-\frac{1}{2}}(\Sigma;\mathcal{E}\otimes\mathbb{C}) \oplus \mathcal{P}_{\mathcal{C}_{\mathrm{D}}}\mathbb{H}^{-\frac{1}{2}}(\Sigma;\mathcal{E}\otimes\mathbb{C}).$$

First-order case

- $\hat{\mathrm{H}}(\mathrm{D}, \mathcal{P}_+) := \mathcal{P}_-^* \mathrm{H}^{-\frac{1}{2}}(\Sigma; \mathcal{E}) \oplus \mathcal{P}_+^* \mathrm{H}^{\frac{1}{2}}(\Sigma; \mathcal{E})$.
- L^2-induced perfect pairing : $\langle \cdot, \cdot \rangle : \check{\mathrm{H}}(\mathrm{D}) \times \hat{\mathrm{H}}(\mathrm{D}, \mathcal{P}_+) \to \mathbb{C}$.
- $\exists \sigma_0 \in \mathrm{C}^\infty(\Sigma; \mathcal{E} \otimes \mathcal{F}^*)$ invertible $\sigma_0^* \check{\mathrm{H}}(\mathrm{D}^\dagger) = \hat{\mathrm{H}}(\mathrm{D}, \mathcal{P}_+)$.
- $B^\dagger \subset \check{\mathrm{H}}(\mathrm{D}^\dagger)$ closed $\iff$
  $B^\perp := \left\{ v \in \hat{\mathrm{H}}(\mathrm{D}, \mathcal{P}_+) : \langle u, v \rangle = 0 \ \forall u \in B \right\} = \sigma_0^* B^\dagger$.
- $\forall u \in \mathrm{dom}(\mathrm{D}_{\max}) \ \forall v \in \mathrm{dom}(\mathrm{D}^\dagger_{\max})$:

$$\langle \mathrm{D}_{\max} u, v \rangle_{\mathrm{L}^2(\mathcal{M};\mathcal{F})} - \langle u, \mathrm{D}^\dagger_{\max} v \rangle_{\mathrm{L}^2(\mathcal{M};\mathcal{E})} = -\langle u|_\Sigma, \sigma_0^* \, v|_\Sigma \rangle_{\check{\mathrm{H}}(\mathrm{D}) \times \hat{\mathrm{H}}(\mathrm{D}, \mathcal{P}_+)}.$$

- B elliptically regular $\iff$ $B \subset \mathrm{H}^{\frac{1}{2}}(\Sigma; \mathcal{E})$ and $B^\perp \in \mathrm{H}^{\frac{1}{2}}(\Sigma; \mathcal{E})$.

B elliptically regular if and only if *graphical decomposition* (w.r.t. B and boundary decomposing $\mathcal{P}_+$) holds:

(F1) $\exists\ W_\pm, V_\pm$ mutually complementary subspaces such that

$$V_\pm \oplus W_\pm = \mathcal{P}_\pm \mathrm{L}^2(\Sigma; \mathcal{E}),$$

(follows that $V_\pm^* \oplus W_\pm^* = \mathcal{P}_\pm^* \mathrm{L}^2(\Sigma; \mathcal{E})$)

(F2) $W_\pm$ are finite dimensional with $W_\pm, W_\pm^* \subset \mathrm{H}^{\frac{1}{2}}(\Sigma; \mathcal{E})$, and

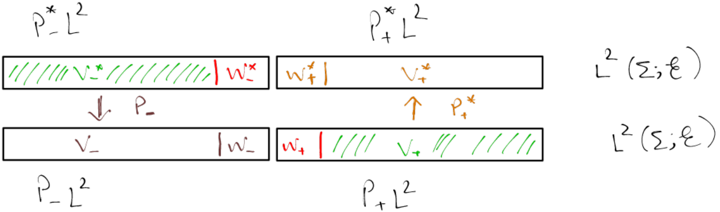
(F3) $\exists g : V_- \to V_+$ bounded linear map with $g(V_- \cap \mathrm{H}^{\frac{1}{2}}(\Sigma; \mathcal{E})) \subset V_+ \cap \mathrm{H}^{\frac{1}{2}}(\Sigma; \mathcal{E})$ and $g^*(V_+^* \cap \mathrm{H}^{\frac{1}{2}}(\Sigma; \mathcal{E})) \subset V_-^* \cap \mathrm{H}^{\frac{1}{2}}(\Sigma; \mathcal{E})$ such that

$$B = W_+ \oplus \left\{ v + gv : v \in V_- \cap \mathrm{H}^{\frac{1}{2}}(\Sigma; \mathcal{E}) \right\}.$$

Obtain: $B^\perp = W_-^* \oplus \left\{ u - g^* u : u \in V_+^* \cap \mathrm{H}^{\frac{1}{2}}(\Sigma; \mathcal{E}) \right\}$.

Proof ($\Longrightarrow$)

$$W_-^* := \mathcal{P}_-^* \mathrm{L}^2(\Sigma;\mathcal{E}) \cap B^\perp \quad V_-^* := \mathcal{P}_-^* \mathrm{L}^2(\Sigma;\mathcal{E}) \cap (W_-^*)^\perp$$
$$W_+ := \mathcal{P}_+ \mathrm{L}^2(\Sigma;\mathcal{E}) \cap B \quad V_+ := \mathcal{P}_+ \mathrm{L}^2(\Sigma;\mathcal{E}) \cap W_+^\perp$$



$$W_- := \mathcal{P}_- W_-^* \quad V_- := \mathcal{P}_- V_-^*$$
$$W_+^* := \mathcal{P}_+^* W_+ \quad V_+^* := \mathcal{P}_+^* V_+.$$

Key points of proof

- $\mathcal{P}_- B$ and $\mathcal{P}_+^* B^\perp$ closed subspaces, $W_+ = \ker(\mathcal{P}_-|_B)$ and $W_-^* = \ker(\mathcal{P}_+^*|_{B^\perp})$.

  (a) $\|u\|_{\check{\mathrm{H}}(\mathrm{D})} \simeq \|u\|_{\mathrm{H}^{\frac{1}{2}}}$ for $u \in B$ since B closed in $\check{\mathrm{H}}(\mathrm{D})$ *and* $\mathrm{H}^{\frac{1}{2}}(\Sigma; \mathcal{E})$.

  (b) $\|u\|_{\mathrm{H}^{\frac{1}{2}}} \simeq \|\mathcal{P}_- u\|_{\mathrm{H}^{\frac{1}{2}}} + \|\mathcal{P}_+ u\|_{\mathrm{H}^{-\frac{1}{2}}}$.

  (c) $\mathcal{P}_+ B \subset \mathrm{H}^{\frac{1}{2}}(\Sigma; \mathcal{E})$ since it is boundary decomposing and $\mathrm{H}^{\frac{1}{2}}(\Sigma; \mathcal{E}) \hookrightarrow \mathrm{H}^{-\frac{1}{2}}(\Sigma; \mathcal{E})$ compact embedding.

  (d) Implies $\mathcal{P}_-|_B$ has closed range and finite dimensional kernel.

- $\mathcal{P}_- B = V_- \cap \mathrm{H}^{\frac{1}{2}}(\Sigma; \mathcal{E})$ and $\mathcal{P}_+^* B^\perp = V_+^* \cap \mathrm{H}^{\frac{1}{2}}(\Sigma; \mathcal{E})$

  Key point: obtain $W_-^* = \mathcal{P}_-^* \mathrm{L}^2(\Sigma; \mathcal{E}) \cap B^\perp$ as

  $$W_-^* = \mathcal{P}_-^* \mathrm{H}^{-\frac{1}{2}}(\Sigma; \mathcal{E}) \cap B^{\perp, \mathrm{H}^{-\frac{1}{2}}} = (\mathcal{P}_- B)^{\perp, \mathcal{P}_-^* \mathrm{H}^{-\frac{1}{2}}}.$$

  Requires:

  (a) $u \in \mathcal{P}_-^* \mathrm{H}^{-\frac{1}{2}}(\Sigma; \mathcal{E}) \cap B^{\perp, \mathrm{H}^{-\frac{1}{2}}} \implies u \in \hat{\mathrm{H}}(\mathrm{D}, \mathcal{P}_+) \implies u \in B^\perp$.
  (b) $B^\perp \subset \mathrm{H}^{\frac{1}{2}}(\Sigma; \mathcal{E})$.

- Define

$$X_- := \mathcal{P}_-|_{B \cap W_+^\perp} : B \cap W_+^\perp \to \mathcal{P}_- B$$
$$X_+^* := \mathcal{P}_+^*|_{B^\perp \cap (W_-^*)^\perp} : B^\perp \cap (W_-^*)^\perp \to \mathcal{P}_+^* B^\perp.$$

- X_- and X_+ are bounded and invertible isomorphisms (in the induced topology).
- $g_0 := P_{V_+, W_- \oplus V_- \oplus W_+} \circ (X_-)^{-1}$ and $h_0 := P_{V_-^*, W_-^* \oplus V_+^* \oplus W_+^*} \circ (X_+^*)^{-1}$.
- g_0 and $-h_0$ are adjoints w.r.t. induced pairing.
- g obtained from interpolation i.e.,

$$V_\pm = [\overline{V_\pm}^{\mathrm{H}^{-\frac{1}{2}}}, V_\pm \cap \mathrm{H}^{\frac{1}{2}}(\Sigma; \mathcal{E})]_{\theta = \frac{1}{2}}.$$

General-order

- $\check{\mathrm{H}}(\mathrm{D}) = \mathcal{P}_+ \mathbb{H}^{-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m) \oplus \mathcal{P}_- \mathbb{H}^{m-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m)$.
- $\hat{\mathrm{H}}(\mathrm{D}, \mathcal{P}_+) := \mathcal{P}_-^* \mathbb{H}^{\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m_{\mathrm{op}}) \oplus \mathcal{P}_+^* \mathbb{H}^{\frac{1}{2}-m}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m_{\mathrm{op}})$.
- L^2-induced perfect pairing : $\langle \cdot, \cdot \rangle : \check{\mathrm{H}}(\mathrm{D}) \times \hat{\mathrm{H}}(\mathrm{D}, \mathcal{P}_+) \to \mathbb{C}$.
- $\exists \mathfrak{a} \in \mathrm{Diff}_{m-1}(\Sigma; \mathcal{E} \otimes \mathcal{F}^*)$ invertible $\mathfrak{a}^* \check{\mathrm{H}}(\mathrm{D}^\dagger) = \hat{\mathrm{H}}(\mathrm{D}, \mathcal{P}_+)$.
- $B^\dagger \subset \check{\mathrm{H}}(\mathrm{D}^\dagger)$ closed $\iff$
  $B^\perp := \left\{ v \in \hat{\mathrm{H}}(\mathrm{D}, \mathcal{P}_+) : \langle u, v \rangle = 0 \ \forall u \in B \right\} = \mathfrak{a}^* B^\dagger$.
- $\forall u \in \mathrm{dom}(\mathrm{D}_{\max}) \ \forall v \in \mathrm{dom}(\mathrm{D}^\dagger_{\max})$:

$$\langle \mathrm{D}_{\max} u, v \rangle_{\mathrm{L}^2(\mathcal{M};\mathcal{F})} - \langle u, \mathrm{D}^\dagger_{\max} v \rangle_{\mathrm{L}^2(\mathcal{M};\mathcal{E})} = -\langle \gamma u, \mathfrak{a}^* \ \gamma v \rangle_{\check{\mathrm{H}}(\mathrm{D}) \times \hat{\mathrm{H}}(\mathrm{D}, \mathcal{P}_+)}.$$

- B elliptically regular $\iff$ $B \subset \mathbb{H}^{m-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m)$ and
  $B^\perp \in \mathbb{H}^{\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m_{\mathrm{op}})$.

B elliptically regular if and only if the *general graphical decomposition* holds:

(G1) there exist mutually complementary subspaces $W_\pm$ and $V_\pm$ of $\mathbb{H}^{m-\frac{1}{2}}(\Sigma;\mathcal{E}\otimes\mathbb{C}^m)$ satisfying:

$$W_\pm \oplus V_\pm = \mathcal{P}_\pm \mathbb{H}^{m-\frac{1}{2}}(\Sigma;\mathcal{E}\otimes\mathbb{C}^m),$$

(G2) it holds that ${W_-}^* \subset \mathbb{H}^{\frac{1}{2}}(\Sigma;\mathcal{E}\otimes\mathbb{C}^m_{\mathrm{op}})$, and ${W_\pm}^* \subset \mathbb{H}^{\frac{1}{2}-m}(\Sigma;E\otimes\mathbb{C}^m_{\mathrm{op}})$ and the subspaces $W_\pm \subset \mathbb{H}^{m-\frac{1}{2}}(\Sigma;\mathcal{E}\otimes\mathbb{C}^m)$ are finite dimensional,

(G3) there exists a continuous map $g : V_- \to V_+$ such that

$$g^*(V_+^* \cap \mathbb{H}^{\frac{1}{2}}(\Sigma;\mathcal{E}\otimes\mathbb{C}^m_{\mathrm{op}})) \subset V_-^* \cap \mathbb{H}^{\frac{1}{2}}(\Sigma;\mathcal{E}\otimes\mathbb{C}^m_{\mathrm{op}}),$$

where g^* denotes the adjoint in the induced L^2-pairing between $\mathbb{H}^{m-\frac{1}{2}}(\Sigma;\mathcal{E}\otimes\mathbb{C}^m)$ and $\mathbb{H}^{\frac{1}{2}-m}(\Sigma;\mathcal{E}\otimes\mathbb{C}^m_{\mathrm{op}})$ and

$$B = \{v + gv : v \in V_-\} \oplus W_+.$$

$$W_-^* := \mathcal{P}_-^* \mathbb{H}^{\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m_{\text{op}}) \cap B^{\perp, \mathbb{H}^{\frac{1}{2}-m}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m_{\text{op}})}$$

$$W_+ := \mathcal{P}_+ \mathbb{H}^{m-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m) \cap B$$

$$V_-^* := \mathcal{P}_-^* \mathbb{H}^{\frac{1}{2}-m}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m_{\text{op}}) \cap (W_-^*)^{\perp, \mathbb{H}^{-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m)}$$

$$V_+ := \mathcal{P}_+ \mathbb{H}^{m-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m) \cap W_+^{\perp, \mathbb{H}^{\frac{1}{2}-m}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m_{\text{op}})}$$

- $W_-^* \subset \mathcal{P}_-^* \mathbb{H}^{\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m_{\text{op}})$ so consider $(W_-^*)^{\perp, \mathbb{H}^{-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m)}$ in $\mathbb{H}^{-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m)$.
- Calculation:

$$\mathbb{H}^{-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m) = \oplus_{j=0}^{m-1} \mathrm{H}^{-\frac{1}{2}-j}(\Sigma; \mathcal{E})$$
$$\supset \oplus_{j=0}^{m-1} \mathrm{H}^{\frac{1}{2}-m+j}(\Sigma; \mathcal{E}) = \mathbb{H}^{\frac{1}{2}-m}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m_{\text{op}})$$

$\implies V_-^*$ is well-defined.

- $\mathbb{H}^{m-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m) \subset \mathbb{H}^{\frac{1}{2}-m}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m_{\text{op}}) \implies V_+$ well-defined.

$$W_+^* := \mathcal{P}_+^* \mathbb{H}^{\frac{1}{2}-m}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m_{\mathrm{op}}) \cap V_+^{\perp, \mathbb{H}^{\frac{1}{2}-m}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m_{\mathrm{op}})}$$

$$W_- := \mathcal{P}_- \mathbb{H}^{m-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m) \cap (V_-^*)^{\perp, \mathbb{H}^{m-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m)}$$

$$V_+^* := \mathcal{P}_+^* \mathbb{H}^{\frac{1}{2}-m}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m_{\mathrm{op}}) \cap W_+^{\perp, \mathbb{H}^{\frac{1}{2}-m}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m_{\mathrm{op}})}$$

$$V_- := \mathcal{P}_- \mathbb{H}^{m-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m) \cap (W_-^*)^{\perp, \mathbb{H}^{m-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m)}$$

Not hard to show these are ranges of the respective adjoint projectors.

Proof highlights

- $\mathcal{P}_- B$ and $\mathcal{P}_+^* B^\perp$ closed subspaces, $W_+ = \ker(\mathcal{P}_-|_B)$ and $W_-^* = \ker(\mathcal{P}_+^*|_{B^\perp})$.

$$\mathbb{H}^{m-\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m) \xhookrightarrow{\text{compact}} \mathbb{H}^{-\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m)$$
$$\mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\text{op}}) \to \mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\text{op}}) \xhookrightarrow{\text{compact}} \mathbb{H}^{\frac{1}{2}-m}(\Sigma; E \otimes \mathbb{C}^m_{\text{op}}).$$

- Not immediate that $V_-^* \oplus W_-^* = \mathcal{P}_-^* \mathbb{H}^{m-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m)$.
  - $V_-^* \cap W_-^* = 0$ obtained from $\langle w, w \rangle = \|w\|^2_{\mathrm{L}^2}$ when $w \in \mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\text{op}}) \subset \oplus_{j=0}^{m-1} \mathrm{L}^2(\Sigma; E)$.
  - $\mathcal{P}_-^* \mathbb{H}^{\frac{1}{2}-m}(\Sigma; E \otimes \mathbb{C}^m_{\text{op}}) \subset V_-^* \oplus W_-^*$ requires projector to W_-^* along V_-^*.
  - $Pu = \sum_{i=1}^{\dim W_-^*} \langle u, e_i \rangle$, where e_i is a basis with $\langle e_i, e_j \rangle = \delta_{ij}$. Possible only since $e_i \in \mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\text{op}}) \subset \mathbb{H}^{-\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\text{op}})$.

- $V_+^* \cap \mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\mathrm{op}}) = \mathcal{P}_+^* B^\perp$ and $V_- = \mathcal{P}_- B$.
  - Key:

$$W_+ = (\mathcal{P}_+^* B^\perp)^{\perp, \mathbb{H}^{-\frac{1}{2}}(\Sigma; E\otimes\mathbb{C}^m)} \cap \mathcal{P}_+^* \mathbb{H}^{-\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m)$$
$$W_-^* = (\mathcal{P}_- B)^{\perp, \mathbb{H}^{\frac{1}{2}-m}(\Sigma; E\otimes\mathbb{C}^m_{\mathrm{op}})} \cap \mathcal{P}_- \mathbb{H}^{\frac{1}{2}-m}(\Sigma; E \otimes \mathbb{C}^m_{\mathrm{op}}).$$

  - Need: $B \subset \mathbb{H}^{m-\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m)$ and $B^\perp \in \mathbb{H}^{\frac{1}{2}}(\Sigma; \mathcal{E} \otimes \mathbb{C}^m_{\mathrm{op}})$
- $X_- := \mathcal{P}_-|_{B \cap W_+^\perp} : B \cap W_+^\perp \to \mathcal{P}_- B$.

  $X_+^* := \mathcal{P}_+^*|_{B^\perp \cap (W_-^*)^{\perp, \mathbb{H}^{-\frac{1}{2}}(\Sigma; E\otimes\mathbb{C}^m)}} : B^\perp \cap (W_-^*)^{\perp, \mathbb{H}^{-\frac{1}{2}}(\Sigma; E\otimes\mathbb{C}^m)} \to \mathcal{P}_+^* B^\perp$.

  Both maps bounded isomorphisms to their ranges.

- $g : V_- \to V_+$, bounded in the $\mathbb{H}^{m-\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m)$ norm, defined as

$$g := P_{V_+, W_+ \oplus \mathcal{P}_- \mathbb{H}^{m-\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m)} \circ (X_-)^{-1}.$$

- $h : V_+^* \cap \mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\mathrm{op}}) \to V_-^* \cap \mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\mathrm{op}})$bounded in the $\mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\mathrm{op}})$ norm defined as

$$h := P_-^* \circ (X_+^*)^{-1}.$$

- From $B \perp B^\perp$ in the $\check{\mathrm{H}}(\mathrm{D}) \times \hat{\mathrm{H}}(\mathrm{D}, \mathcal{P}_+)$, obtain

$$g^*(V_+^* \cap \mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\mathrm{op}})) = -h(V_+^* \cap \mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\mathrm{op}})) \subset V_-^* \cap \mathbb{H}^{\frac{1}{2}}(\Sigma; E \otimes \mathbb{C}^m_{\mathrm{op}}).$$