When Functional Calculus, Harmonic Analysis, and Geometry party together...

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21 September 2018

SPP 2026
GEOMETRY AT INFINITY

Universität Potsdam

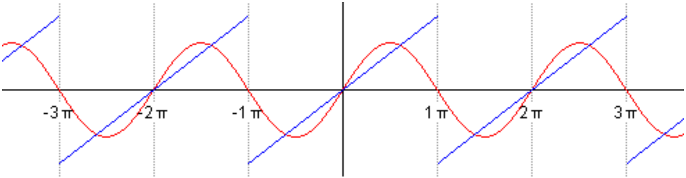
Geometry is the study of the shape of space.

Functional calculus is the ability to take functions of operators and manipulate them as if they were functions.

Harmonic analysis is the art in which a mathematical object is perceived as a signal and whose goal is to decompose this object, often in some scale-invariant way, to simpler parts which are mathematically more tractable.

Fourier series

$u \in \mathrm{L}^2(\mathbb{S}^1) \implies u(\theta) = \sum_{n=-\infty}^{\infty} a_n \mathrm{e}^{\imath n\theta},\ \exists a_n \in \mathbb{C}$.



Associated operator: $\Delta_{\mathbb{S}^1}$, Laplacian on $\mathbb{S}^1$ or equivalently $-\frac{d^2}{d\theta^2}$ with periodic boundary conditions.

Eigenvalues: $\left\{\lambda_n = n^2\right\}_{n=0}^{\infty}$.
Eigenfunctions: $\left\{\mathrm{e}^{\imath n\theta}, \mathrm{e}^{-\imath n\theta}\right\}_{n=0}^{\infty}$.

$\Delta_{\mathbb{S}^1} u(\theta) = \sum_{n=-\infty}^{\infty} a_n n^2 \mathrm{e}^{\imath n\theta}$.

Symbol - appropriate function f:

$$f(\Delta_{\mathbb{S}_1})u(\theta) := \sum_{n=-\infty}^{\infty} f(n^2)a_n \mathrm{e}^{\imath n\theta}.$$

Why?

$$\partial_t u(t,\theta) = \Delta_{\mathbb{S}^1} u(t,\theta)$$
$$\lim_{t\to 0} u(t,\theta) = u_0(\theta)$$

Unique solution: $u(t,\theta) = \mathrm{e}^{-t\Delta_{\mathbb{S}^1}} u_0 = \exp(-t\Delta_{\mathbb{S}^1})u_0$.

Functional calculus

$\mathscr{H}$ Hilbert space, D non-negative self-adjoint operator on $\mathscr{H}$, discrete spectrum $\mathrm{spec}(\mathrm{D}) = \{\lambda_i \geq 0\}$.

Eigenfunctions $\{u_i\}$. So $u \in \mathscr{H} \implies u = \sum_n a_n u_n$.

Quintessential example: $(\mathcal{M}, \mathrm{g})$ smooth compact Riemannian manifold (without boundary), $\mathscr{H} = \mathrm{L}^2(\mathcal{M}, \mathrm{g})$, $\mathrm{D} = \Delta_{\mathrm{g}} = \nabla^{*,g}\nabla$.

$$\left.\begin{array}{l} \mathrm{id} : \mathrm{H}^1(\mathcal{M}, \mathrm{g}) \hookrightarrow \mathrm{L}^2(\mathcal{M}, \mathrm{g}) \text{ compact} \\ \mathrm{dom}(\Delta_{\mathrm{g}}) = \mathrm{H}^2(\mathcal{M}, \mathrm{g}) \subset \mathrm{H}^1(\mathcal{M}, \mathrm{g}) \end{array}\right\} \implies \Delta_{\mathrm{g}} \text{ has discrete spectrum}$$

Functional calculus for D on $\mathscr{H}$:

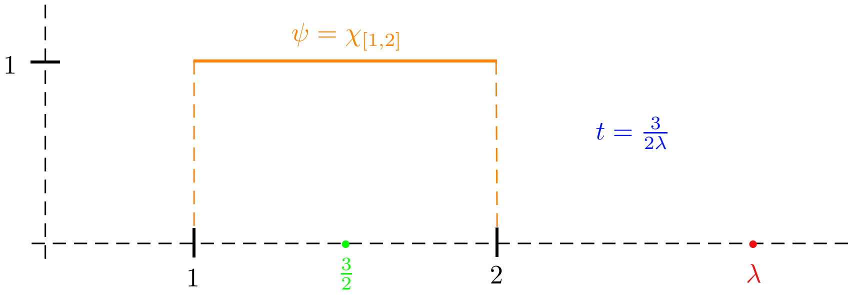
$$f(\mathrm{D})u = \sum_n f(\lambda_n) a_n u_n.$$

Picture: $u \in \mathscr{H}$ is a "signal".
"Reconstruction" of signal: $u = \sum_n a_n u_n$.

$\psi : \mathbb{R} \to \mathbb{R}$ measurable and $\psi \neq 0$ a.e. such that $\exists \alpha > 0, \exists C > 0$ satisfying

$$|\psi(x)| \leq C \left\{ |x|^{\alpha}, |x|^{-\alpha} \right\} \quad x\text{-a.e.}$$

$\psi(t\mathrm{D})u$ "band-pass filter" on u localised about frequency $\sim \frac{1}{t}$.



Assume $\ker(\mathrm{D}) = 0$ (i.e., $\lambda_1 > 0$) and

$$\begin{aligned}\|\chi_{[1,2]}(t\mathrm{D})u\|^2 &= \sum_n a_n^2 \|\chi_{[1,2]}(t\mathrm{D})u_n\|^2 \\ &= \sum_n a_n^2 \|\chi_{[1,2]}(t\lambda_n)u_n\|^2 \\ &= \sum_n a_n^2 |\chi_{[1,2]}(t\lambda_n)|^2 \|u_n\|^2.\end{aligned}$$

$$\int_0^\infty \|\chi_{[1,2]}(t\mathrm{D})u_n\|^2 \ \frac{dt}{t} = \|u_n\|^2 \int_0^\infty |\chi_{[1,2]}(t\lambda_n)|^2 \ \frac{dt}{t}.$$

$$\begin{aligned}\int_0^\infty |\chi_{[1,2]}(t\lambda_n)|^2 \ \frac{dt}{t} &= \int_0^\infty |\chi_{[1,2]}(s)| \frac{ds}{\lambda_n s} \cdot \frac{\lambda_n}{s} \\ &= \int_0^\infty |\chi_{[1,2](s)}|^2 \ \frac{ds}{s} \\ &= \int_1^2 \frac{1}{s} \ ds = \log(2).\end{aligned}$$

Then,

$$\int_0^\infty \|\chi_{[1,2]}(t\mathrm{D})u\|^2 \, \frac{dt}{t} = \sum_n a_n \log(2)\|u_n\|^2 = \log(2)\|u\|^2.$$

Similarly, for $\psi \neq 0$ a.e. with $|\psi(x)| \leq C \{|x|^\alpha, |x|^{-\alpha}\}$, x-a.e.:

$$\int_0^\infty \|\psi(t\mathrm{D})u_n\|^2 \, \frac{dt}{t} = \|u_n\|^2 \int_0^\infty |\psi(t\lambda_n)|^2 \, \frac{dt}{t} \simeq \|u_n\|^2,$$

and

$$\int_0^\infty \|\psi(t\mathrm{D})u\|^2 \, \frac{dt}{t} \simeq \|u\|^2.$$

Reconstruction of signal in norm, up to a constant independent of the signal.

Beyond self-adjointness - nonlinear perturbation problems

Let $x \mapsto A(x) \in \mathrm{L}^\infty(\mathrm{SymMat}(n))$, real self-adjoint at almost-every x and $a : \mathbb{R} \to [0, \infty]$ measurable.

$$\mathrm{L}_{A,a} u := -a \operatorname{div} A \nabla.$$

Question:

$$\|\mathrm{e}^{-tL_{A_1,a_1}} - \mathrm{e}^{-tL_{A_2,a_2}}\|_{\mathrm{L}^2 \to \mathrm{L}^2} \lesssim \|A_1 - A_2\|_{\mathrm{L}^\infty} + \|a_1 - a_2\|_{\mathrm{L}^\infty}?$$

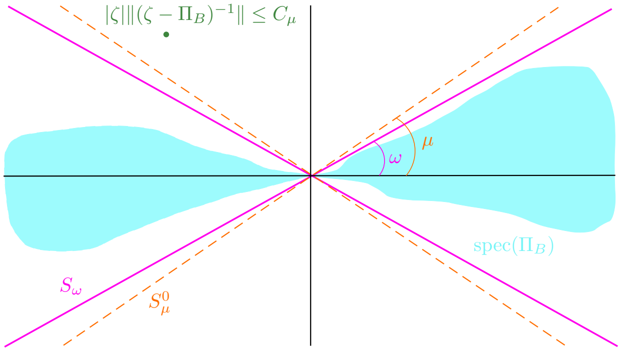
Consider:

$$\Pi_{B,b} := \begin{pmatrix} 0 & -b \operatorname{div} B \\ \nabla & 0 \end{pmatrix},$$

$x \mapsto B(x),\ x \mapsto b(x)$ $\mathbb{C}$-valued, need not be symmetric.

$$\Pi_{B,b}^2 := \begin{pmatrix} -b \operatorname{div} B \nabla & 0 \\ 0 & -\nabla b \operatorname{div} B \end{pmatrix}.$$

The operator $\Pi_{B,b}$ is ω-*bisectorial*, $\omega \in [0, \pi/2)$.



(i) $\Pi_{B,b}$ is closed,

(ii) $\text{spec}(\Pi_{B,b}) \subset S_\omega$,

(iii) $\forall \mu \in (\omega, \pi/2)$, $\exists C_\mu > 0$ s.t. $\forall \zeta \in \mathbb{C} \setminus S^o_\mu$:

$$|\zeta| \|(\zeta - \Pi_{B,b})^{-1}\| \leq C_\mu.$$

For all holomorphic ψ on S_μ^o satisfying: $\exists \alpha > 0, \exists C > 0$
$|\psi(\zeta)| \leq C \max\{|\zeta|^\alpha, |\zeta|^{-\alpha}\}$,

$$\psi(\Pi_{B,b})u := \frac{1}{2\pi\imath} \oint_\gamma \psi(\zeta)(\zeta - \Pi_{B,b})^{-1} u \, d\zeta.$$

Assume: $\exists C > 0$ such that $\|\psi(\Pi_{B,b})\|_{\mathrm{L}^2 \to \mathrm{L}^2} \leq C\|\psi\|_\infty$.
$f(\Pi_{B,b})$ defined for $f : S_\mu \to \mathbb{C}$ bounded, $f|_{S_\mu^o}$ holomorphic.

$(B', b') \mapsto f(\Pi_{B',b'})$ holomorphic (in a small neighbourhood of (B, b))
and
$$\|f(\Pi_{B,b}) - f(\Pi_{B',b'})\| \lesssim \|B - B'\|_{\mathrm{L}^\infty} + \|b - b'\|_{\mathrm{L}^\infty},$$

for (B', b') sufficiently L^∞ close to (B, b).

$$\|(\mathrm{e}^{-tL_{A_1,a_1}} - \mathrm{e}^{-tL_{A_2,a_2}})u\| = \left\| \left(\mathrm{e}^{-t\Pi^2_{B_1,b_1}} - \mathrm{e}^{-t\Pi^2_{B_2,b_2}} \right) \begin{pmatrix} u \\ 0 \end{pmatrix} \right\|$$
$$\lesssim \|A_1 - A_2\|_\infty + \|a_1 - a_2\|_\infty.$$

First-order factorisation of the *Kato square root problem* by Axelsson (Rosén)-Keith-McIntosh.

Theorem (McIntosh 1986)

Let T be an ω-bisectorial operator, $\omega \in [0, \pi/2)$, on a Hilbert space $\mathscr{H}$. Then, $\mathscr{H} = \ker(T) \oplus \overline{\operatorname{ran}(T)}$ and for $\mu \in (\omega, \pi/2)$, the following are equivalent:

- $\exists C > 0$ *such that* $\|\psi(T)\|_{\mathscr{H} \to \mathscr{H}} \leq C\|\psi\|_\infty$, $\forall \psi \in \Psi(S^o_\mu)$.
- $\exists \psi \in \Psi(S^o_\mu)$, *not identically zero on either sector, such that*

$$\|u\|^2_{\psi,T} := \int_0^\infty \|\psi(tT)u\|^2 \, \frac{dt}{t} \simeq \|u\|^2, \quad \forall u \in \overline{\operatorname{ran}(T)}. \qquad \text{(Qest)}$$

- $\forall \psi \in \Psi(S^o_\mu)$ *not identically zero on either sector,* (Qest) *holds.*

Nomenclature: say T has a H^∞ functional calculus.


(McIntosh)
SA Ingram
1946
MATTHEW
Malcolm Ingram
Sandy Ingram
RB 8/1946
SAlly In
1946

Picture: $\psi(tT)$ band-pass filter for the spectrum with real part $\frac{1}{t}$.
Geometric interpretation: $\|u\|_{\psi,T}$ *new* but comparable norm (shape) on $\overline{\mathrm{ran}(T)}$.

Quadratic estimates (Qest) in PDE have connections to the so-called *square function estimates*.

Can be computed via real-variable harmonic analysis methods, à la Calderón-Zygmund.

McIntosh convergence lemma: Given $f : S_\mu \to \mathbb{C}$ bounded and $f|_{S^o_\mu}$ holomorphic, there exists $f_n \in \Psi(S^o_\mu)$ such that $\{f_n(T)\}$ is Cauchy.

H^∞ functional calculus:

$$f(T)u := f(0)\,\mathbf{P}_{\ker(T),\overline{\mathrm{ran}(T)}}\,u + \lim_{n\to\infty} f_n(T)u.$$

Back to $\|\psi(\Pi_{B,b})\|_{L^2 \to L^2} \lesssim \|\psi\|_\infty$.

Proved by Axelsson (Rosén)-Keith-McIntosh (for coefficients also bounded below) in 2005 by showing

$$\int_0^\infty \left\| \frac{t\Pi_{B,b}}{1+t^2\Pi_{B,b}^2} u \right\|^2 \frac{dt}{t} \simeq \|u\|^2, \ \forall u \in \overline{\mathrm{ran}(\Pi_{B,b})}.$$

Corresponds to $\psi \in \Psi(S_\mu^o)$ given by:

$$\psi(\zeta) = \frac{\zeta}{1+\zeta^2}.$$

Proof involves dyadic structures, local $T(b)$ theorems, etc.

Original Kato square root problem were similar in their estimates, proved by Auscher-Hofmann-Lacey-McIntosh-Tchamitchian in 2002.

Geometry!

$(\mathcal{M}, \mathrm{g})$ smooth, complete Riemannian manifold.

Laplacian with respect to g:

$$\Delta_{\mathrm{g}} := -\operatorname{div}_{\mathrm{g}} \nabla, \quad \operatorname{div}_{\mathrm{g}} := -\nabla^{*,\mathrm{g}}$$

Let h be another metric with $C \geq 1$ satisfying:

$$C^{-1}|u|_{\mathrm{g}(x)} \leq |u|_{\mathrm{h}(x)} \leq C|u|_{\mathrm{g}(x)}, \quad \forall u \in \mathrm{T}_x\mathcal{M}.$$

$\exists A \in \mathrm{C}^\infty \cap \mathrm{L}^\infty(\mathrm{Sym}(\mathrm{T}^*\mathcal{M} \otimes \mathrm{T}\mathcal{M}))$ such that

$$\mathrm{h}_x(u, v) = \mathrm{g}_x(A(x)u, v), \ \forall u, v \in \mathrm{T}_x\mathcal{M}$$
$$d\mu_{\mathrm{h}}(x) = \theta(x)\ d\mu_{\mathrm{g}}(x), \quad \theta(x) := \sqrt{\det A(x)}.$$

Then, $\Delta_{\mathrm{h}} = -\theta^{-1} \operatorname{div}_{\mathrm{g}} A\theta\nabla$.

Π_B has H^∞ functional calculus $\implies$

$$\|e^{-t\Delta_g} - e^{-t\Delta_h}\| \lesssim \|g - h\|_{L^\infty}$$

$\forall h$ metrics sufficiently L^∞-close to g.

- Axelsson (Rosén)-Keith-McIntosh 2005: $\mathcal{M}$ compact, g smooth.
- Morris 2012: $\mathcal{M} \subset \mathbb{R}^{n+k}$, smooth Euclidean submanifold, $|\mathbf{II}| < \infty$.
- Bandara-McIntosh 2016: $(\mathcal{M}, g)$ complete, smooth, $|\mathrm{Ric}|_g < \infty$, $\mathrm{inj}(\mathcal{M}, g) \geq \kappa$, $\exists \kappa > 0$.
- Bandara 2017: $\mathcal{M}$ compact, g "rough" (non-smooth, measurable coefficient).

☠ Also: application to regularity properties of a geometric flow "tangential" in a suitably weak sense to the Ricci flow.

Bär-Bandara 2018: $\mathcal{M}$ manifold with *compact* boundary $\Sigma := \partial\mathcal{M}$. $E, F \to \mathcal{M}$ vector bundles, $D : \mathrm{C}^\infty(E) \to \mathrm{C}^\infty(F)$ first-order elliptic differential operator.

A an "adapted" operator to the boundary. Ellipticity of $D \implies A$ ω-bisectorial (up to the addition of a lower order term).

$u \mapsto u|_\Sigma$ extends to a bounded surjection $\mathrm{dom}(\mathrm{D}_{\max}) \to \check{H}(A)$,

$$\check{H}(A) := \chi^-(A)\mathrm{H}^{\frac{1}{2}}(E_\Sigma) \oplus \chi^+(A)\mathrm{H}^{-\frac{1}{2}}(E_\Sigma).$$

Boundary conditions are $B \subset \check{H}(A)$ closed subspaces which yield closed operators D_B with

$$\mathrm{dom}(D_B) = \left\{u \in \mathrm{dom}(D_{\max}) : u|_\Sigma \in B\right\}.$$

Motivations: physics: Rarita-Schwinger operator on $3/2$-spinors, index theory: Atiyah-Patodi-Singer corresponds to $B = \chi^-(A)\mathrm{H}^{\frac{1}{2}}(\not{\Delta}\mathcal{M}|_\Sigma)$.

Beyond the bi-sectorial regime

Bandara-McIntosh-Rosén 2018: $(\mathcal{M}, \mathrm{g})$ Spin-manifold, complete, $|\mathrm{Ric_g}| + |\nabla^{\mathrm{g}}\mathrm{Ric_g}| < C_{\mathrm{g}}$, $\mathrm{inj}(\mathcal{M}, \mathrm{g}) \geq \kappa$, $\exists \kappa > 0$.

Given $C > 0$ $\forall \mathrm{h}$ metrics with $\|\mathrm{h} - \mathrm{g}\|_{\mathrm{L}^\infty} < 1$and $\|\nabla^{\mathrm{g}} h\|_{\mathrm{L}^\infty} \leq C$,

$$\left\| \frac{\not{D}_{\mathrm{g}}}{\sqrt{1 + \not{D}_{\mathrm{g}}^2}} - \frac{\not{D}_{\mathrm{h}}}{\sqrt{1 + \not{D}_{\mathrm{h}}^2}} \right\|_{\mathrm{L}^2 \to \mathrm{L}^2} \lesssim \|\mathrm{g} - \mathrm{h}\|_{\mathrm{L}^\infty}.$$

Bandara-Rosén 2018: Similar results for $\mathcal{M}$ with compact boundary, similar hypothesis but with perturbation of *local* boundary conditions.

Motivations: "spectral flows", index theory, and physics

"But I always say, one's company, two's a crowd, and three's a party."

- Andy Warhol