The Kato square root problem on vector bundles with generalised bounded geometry

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History of the problem

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This has a *unique strict solution* $u = u(t)$ if

$$\mathcal{D}(A(t)^\alpha) = \text{const}$$

for some $0 < \alpha \leq 1$ and $A(t)$ and $f(t)$ satisfy certain smoothness conditions.

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(iii) $\mathcal{W}$ is complete under the norm

$$\|u\|_{\mathcal{W}}^2 = \|u\| + \operatorname{Re} J[u,u].$$

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A 0-accretive operator is non-negative and self-adjoint.

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$$\begin{aligned} &\mathcal{D}(A(t)^{\alpha}) = \mathcal{D}(A(t)^{*\alpha}) = \mathcal{D} = \text{const},\ \text{and} \\ &\|A(t)^{\alpha}u\| \simeq \|A(t)^{*\alpha}u\|, \quad u \in \mathcal{D}. \end{aligned} \tag{K_α}$$

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Counter examples were known for $\alpha > 1/2$ and for $\alpha = 1/2$ when $\omega = \pi/2$.

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In 1982, McIntosh showed that (K2) also did not hold in general in [Mc82].

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Under these conditions, is it true that

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This was answered in the positive in 2002 by Auscher, Hofmann, Lacey, McIntosh and Tchamitchian in [AHLMcT].

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Consider the following *uniformly elliptic* second order differential operator $\mathrm{L}_A : \mathcal{D}(\mathrm{L}_A) \subset \mathrm{L}^2(\mathcal{M}) \to \mathrm{L}^2(\mathcal{M})$ defined by

$$\mathrm{L}_A u = aS^* ASu = -a\,\mathrm{div}(A_{11}\nabla u) - a\,\mathrm{div}(A_{10}u) + aA_{01}\nabla u + aA_{00}u.$$

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That is, we assume a and $A = (A_{ij})$ are L^∞ multiplication operators and that there exist $\kappa_1, \kappa_2 > 0$ such that

$$\mathrm{Re}\,\langle av, v\rangle \geq \kappa_1 \|v\|^2,\ v \in \mathrm{L}^2$$
$$\mathrm{Re}\,\langle ASu, Su\rangle \geq \kappa_2(\|u\|^2 + \|\nabla u\|^2),\ u \in \mathrm{H}^1$$

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$$\begin{cases} \mathcal{D}(\sqrt{\mathrm{L}_A}) = \mathrm{H}^1(\mathcal{M}) \\ \left\|\sqrt{\mathrm{L}_A}u\right\| \simeq \|\nabla u\| + \|u\| = \|u\|_{\mathrm{H}^1}, \ u \in \mathrm{H}^1(\mathcal{M}) \end{cases}$$

The main theorem

Theorem (B.-Mc)

Let $\mathcal{M}$ *be a smooth, complete Riemannian manifold with* $|\mathrm{Ric}| \leq C$ *and* $\mathrm{inj}(M) \geq \kappa > 0$. *Suppose there exist* $\kappa_1, \kappa_2 > 0$ *such that*

$$\mathrm{Re}\,\langle av, v\rangle \geq \kappa_1 \|v\|^2$$
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for $v \in \mathrm{L}^2(\mathcal{M})$ *and* $u \in \mathrm{H}^1(\mathcal{M})$. *Then,* $\mathcal{D}(\sqrt{\mathrm{L}_A}) = \mathcal{D}(\nabla) = \mathrm{H}^1(\mathcal{M})$ *and* $\|\sqrt{\mathrm{L}_A} u\| \simeq \|\nabla u\| + \|u\| = \|u\|_{\mathrm{H}^1}$ *for all* $u \in \mathrm{H}^1(\mathcal{M})$.

Stability

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for $v \in \mathrm{L}^2(\mathcal{M})$ *and* $u \in \mathrm{H}^1(\mathcal{M})$. *Then for every* $\eta_i < \kappa_i$, *whenever* $\|\tilde{a}\|_\infty \leq \eta_1$, $\|\tilde{A}\|_\infty \leq \eta_2$, *the estimate*

$$\left\| \sqrt{\mathrm{L}_A}\, u - \sqrt{\mathrm{L}_{A+\tilde{A}}}\, u \right\| \lesssim (\|\tilde{a}\|_\infty + \|\tilde{A}\|_\infty)\, \|u\|_{\mathrm{H}^1}$$

holds for all $u \in \mathrm{H}^1(\mathcal{M})$. *The implicit constant depends in particular on* A, a *and* η_i.

A more general problem

We can consider the *Kato square root problem on vector bundles* by replacing $\mathcal{M}$ with $\mathcal{V}$, a smooth, complex vector bundle of rank N over $\mathcal{M}$ with metric h and connection ∇.

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We use the adaptation of the *first order systems* approach in [AKMc], which captures the Kato problem (and some other results of harmonic analysis) in terms of perturbations of *Dirac type operators*.

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Let $\Gamma_B^* = B_1 \Gamma^* B_2$, $\Pi_B = \Gamma + \Gamma_B^*$ and $\Pi = \Gamma + \Gamma^*$.

Growth restrictions

We say $\mathcal{M}$ has *exponential volume growth* if there exists $c \geq 1,\ \kappa, \lambda \geq 0$ such that

$$0 < \mu(B(x,tr)) \leq ct^{\kappa} \mathrm{e}^{\lambda tr} \mu(B(x,r)) < \infty \tag{E$_{\text{loc}}$}$$

for all $t \geq 1,\ r > 0$ and $x \in \mathcal{M}$.

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for all $t \geq 1,\ r > 0$ and $x \in \mathcal{M}$.

For instance, if $\mathrm{Ric} \geq \eta \mathrm{g}$, for $\eta \in \mathbb{R}$, then ($\mathsf{E}_{\mathsf{loc}}$) is satisfied.

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Definition (Generalised Bounded Geometry)

Suppose there exists $\rho > 0$, $C \geq 1$ such that for each $x \in \mathcal{M}$, there exists a trivialisation $\psi : B(x, \rho) \times \mathbb{C}^N \to \pi_{\mathcal{V}}^{-1}(B(x, \rho))$ satisfying

$$C^{-1} I \leq \mathrm{h} \leq C I$$

in the basis $\left\{ e^i = \psi(x, \hat{e}^i) \right\}$, where $\left\{ \hat{e}^i \right\}$ is the standard basis for $\mathbb{C}^N$. Then, we say that $\mathcal{V}$ has *generalised bounded geometry* or *GBG*. We call ρ the *GBG radius*.

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(H5) The operators B_1, B_2 are matrix valued pointwise multiplication operators. That is, $B_i \in \mathrm{L}^\infty(\mathcal{M}, \mathcal{L}(\mathcal{V}))$ by which we mean that $B_i(x) \in \mathcal{L}(\pi_{\mathcal{V}}^{-1}(x))$ for every $x \in \mathcal{M}$ and there is a $C_{B_i} > 0$ so that $\|B_i(x)\|_\infty \leq C$ for almost every $x \in \mathcal{M}$.

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(H6) The operator Γ is a first order differential operator. That is, there exists a $C_\Gamma > 0$ such that whenever $\eta \in \mathrm{C}_\mathrm{c}^\infty(\mathcal{M})$, we have that $\eta\mathcal{D}(\Gamma) \subset \mathcal{D}(\Gamma)$ and $\mathrm{M}_\eta u(x) = [\Gamma, \eta(x)]\, u(x)$ is a multiplication operator satisfying

$$|\mathrm{M}_\eta u(x)| \leq C_\Gamma \, |\nabla \eta|_{\mathrm{T}^* M} \, |u(x)|$$

for all $u \in \mathcal{D}(\Gamma)$ and almost all $x \in \mathcal{M}$.

Dyadic cubes

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Each cube $Q \in \mathscr{Q}^j$ also has a diameter of at most $C_1 \delta^j$, where $C_1 > 0$ and $\delta \in (0, 1)$ are fixed, uniform quantities.

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Call the system of trivialisations
$\mathscr{C} = \left\{\psi : B(x_Q, \rho) \times \mathbb{C}^N \to \pi_{\mathcal{V}}^{-1}(B(x_Q, \rho)) \text{ s.t. } Q \in \mathscr{Q}^{\mathrm{J}}\right\}$ the *GBG coordinates*.

GBG coordinates (cont.)

Call the set of a.e. trivialisations $\mathscr{C}_{\mathrm{J}} = \left\{ \tilde{\varphi}_Q = \psi|_Q : Q \times \mathbb{C}^N \to \pi_{\mathcal{V}}^{-1}(Q) \text{ s.t. } Q \in \mathscr{Q}^{\mathrm{J}} \right\}$ the *dyadic GBG coordinates*.

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The *GBG coordinate system* of Q is then $\psi : B(x_{\widehat{Q}}, \rho) \times \mathbb{C}^N \to \pi_{\mathcal{V}}^{-1}(B(x_{\widehat{Q}}, \rho))$.

Machinery for the harmonic analysis

- For $t \leq \mathrm{t_S}$, we write $\mathscr{Q}_t = \mathscr{Q}^j$ whenever $\delta^{j+1} < t \leq \delta^j$.

Machinery for the harmonic analysis

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- Let $\mathcal{A}_t u(x) = u_Q(x)$ whenever $x \in Q \in \mathscr{Q}_t$.
- For each $w \in \mathbb{C}^N$, let $\gamma_t(x)w = (\Theta^B_t \omega)(x)$ where $\omega(x) = w$ in the GBG coordinates of each Q.

Cancellation assumption

(H7) There exists $c > 0$ such that for all $t \leq \mathrm{t_S}$ and $Q \in \mathscr{Q}_t$,

$$\left| \int_Q \Gamma u \, d\mu \right| \leq c\mu(Q)^{\frac{1}{2}} \|u\| \quad \text{and} \quad \left| \int_Q \Gamma^* v \, d\mu \right| \leq c\mu(Q)^{\frac{1}{2}} \|v\|$$

for all $u \in \mathcal{D}(\Gamma)$, $v \in \mathcal{D}(\Gamma^*)$ satisfying $\mathrm{spt}\ u$, $\mathrm{spt}\ v \subset Q$.

Dyadic Poincaré assumption

(H8) There exists $C_P,\ C_C,\ c,\ \tilde{c} > 0$ and an operator $\Xi : \mathcal{D}(\Xi) \subset \mathrm{L}^2(\mathcal{V}) \to \mathrm{L}^2(\mathscr{N})$, where $\mathscr{N}$ is a normed bundle over $\mathcal{M}$ with norm $|\cdot|_{\mathscr{N}}$ and $\mathcal{D}(\Pi) \cap \mathcal{R}(\Pi) \subset \mathcal{D}(\Xi)$ satisfying for all $u \in \mathcal{D}(\Pi) \cap \mathcal{R}(\Pi)$,

-1 (Dyadic Poincaré)

$$\int_B |u - u_Q|^2 \ d\mu \leq C_P(1 + r^\kappa e^{\lambda c r t})(rt)^2 \int_{\tilde{c}B} \left(|\Xi u|^2_{\mathscr{N}} + |u|^2 \right) \ d\mu$$

for all balls $B = B(x_Q, rt)$ with $r \geq C_1/\delta$ where $Q \in \mathscr{Q}_t$ with $t \leq \mathrm{t_S}$, and

-2 (Coercivity)

$$\|\Xi u\|^2_{\mathrm{L}^2(\mathscr{N})} + \|u\|^2_{\mathrm{L}^2(\mathcal{V})} \leq C_C \|\Pi u\|^2_{\mathrm{L}^2(\mathcal{V})} .$$

Kato square root type estimate

Proposition

Suppose $\mathcal{M}$ is a smooth, complete Riemannian manifold and $\mathcal{V}$ is a smooth vector bundle over $\mathcal{M}$. If (H1)-(H8) are satisfied, then

(i) $\mathcal{D}(\Gamma) \cap \mathcal{D}(\Gamma_B^*) = \mathcal{D}(\Pi_B) = \mathcal{D}(\sqrt{\Pi_B^2})$, *and*

(ii) $\|\Gamma u\| + \|\Gamma_B u\| \simeq \|\Pi_B u\| \simeq \|\sqrt{\Pi_B^2} u\|$, *for all* $u \in \mathcal{D}(\Pi_B)$.

Kato square root problem on vector bundles

Theorem (B.-Mc.)

Suppose $\mathcal{M}$ grows at most exponentially and satisfies a local Poincaré inequality on functions. Further, suppose that both $\mathcal{V}$ and $\mathrm{T}^*\mathcal{M}$ have GBG, and

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(iii) *there exist $\kappa_1, \kappa_2 > 0$ such that $\mathrm{Re}\,\langle au, u\rangle \geq \kappa_1 \left\|u\right\|^2$ and $\mathrm{Re}\,\langle ASv, Sv\rangle \geq \kappa_2 \left\|v\right\|_{\mathrm{H}^1}^2$ for all $u \in \mathrm{L}^2(\mathcal{V})$ and $v \in \mathrm{H}^1(\mathcal{V})$,*

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Then, $\mathcal{D}(\sqrt{\mathrm{L}_A}) = \mathcal{D}(\nabla) = \mathrm{H}^1(\mathcal{V})$ with $\|\sqrt{\mathrm{L}_A}u\| \simeq \|\nabla u\| + \|u\| = \|u\|_{\mathrm{H}^1}$ for all $u \in \mathrm{H}^1(\mathcal{V})$.

Set $\mathscr{H} = \mathrm{L}^2(\mathcal{V}) \oplus (\mathrm{L}^2(\mathcal{V}) \oplus \mathrm{L}^2(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V}))$.

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Kato square root problem for tensors

Theorem (B.-Mc.)

Let $\mathcal{M}$ *be a smooth, complete Riemannian manifold with* $|\mathrm{Ric}| \leq C$ *and* $\mathrm{inj}(M) \geq \kappa > 0$. *Suppose that there exist* $C' > 0$ *such that* $\|\nabla^2 u\| \leq C' \|(I + \Delta)u\|$ *whenever* $u \in \mathcal{D}(\Delta) \subset \mathrm{H}^2(\mathcal{T}^{(p,q)}\mathcal{M})$. *Then,* $\mathcal{D}(\sqrt{\mathrm{L}_A}) = \mathcal{D}(\nabla) = \mathrm{H}^1(\mathcal{T}^{(p,q)}\mathcal{M})$ *and* $\|\sqrt{\mathrm{L}_A} u\| \simeq \|\nabla u\| + \|u\| = \|u\|_{\mathrm{H}^1}$ *for all* $u \in \mathrm{H}^1(\mathcal{T}^{(p,q)}\mathcal{M})$.

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See the observation following Theorem 1.2 in [Hebey].

- The proposition guarantees GBG coordinates for $\mathcal{V} = \mathcal{T}^{(p,q)}\mathcal{M}$.

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See Proposition 3.3 in [Hebey].

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