The Kato square root problem on vector bundles with generalised bounded geometry

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History of the problem

In the 1960's, Kato considered the following *abstract evolution equation*

$$\frac{du}{dt} + A(t)u = f(t), \quad t \in [0, T].$$

on a Hilbert space $\mathscr{H}$.

This has a *unique strict solution* $u = u(t)$ if

$$\mathcal{D}(A(t)^\alpha) = \text{const}$$

for some $0 < \alpha \leq 1$ and $A(t)$ and $f(t)$ satisfy certain smoothness conditions.

Typically $A(t)$ is defined by an associated sesquilinear form

$$J_t : \mathcal{W} \times \mathcal{W} \to \mathbb{C}$$

where $\mathcal{W} \subset \mathscr{H}$.

Suppose $0 \leq \omega \leq \pi/2$. Then J_t is ω-sectorial means that

(i) $\mathcal{W} \subset \mathscr{H}$ is dense,
(ii) $J_t[u,u] \in S_{\omega+} = \{\zeta \in \mathbb{C} : |\arg \zeta| \leq \omega\} \cup \{0\}$, and
(iii) $\mathcal{W}$ is complete under the norm

$$\|u\|_{\mathcal{W}}^2 = \|u\| + \operatorname{Re} J[u,u].$$

$T : \mathcal{D}(T) \to \mathscr{H}$ is called ω-accretive if

(i) T is densely-defined and closed,
(ii) $\langle Tu, u \rangle \in S_{\omega+}$ for $u \in \mathcal{D}(T)$, and
(iii) $\sigma(T) \subset S_{\omega+}$.

A 0-accretive operator is non-negative and self-adjoint.

Let $A(t) : \mathcal{D}(A(t)) \to \mathscr{H}$ be defined as the operator with largest domain such that

$$J_t[u,v] = \langle A(t)u, v\rangle \qquad u \in \mathcal{D}(A(t)),\ v \in \mathcal{W}.$$

The theorem of Lax-Milgram guarantees that

$$J_t \text{ is } \omega\text{-sectorial} \implies A(t) \text{ is } \omega\text{-accretive.}$$

In 1962, Kato showed in [Kato] that for $0 \leq \alpha < 1/2$ and $0 \leq \omega \leq \pi/2$,

$$\begin{aligned} &\mathcal{D}(A(t)^{\alpha}) = \mathcal{D}(A(t)^{*\alpha}) = \mathcal{D} = \text{const},\ \text{and} \\ &\|A(t)^{\alpha}u\| \simeq \|A(t)^{*\alpha}u\|, \quad u \in \mathcal{D}. \end{aligned} \tag{K_α}$$

Counter examples were known for $\alpha > 1/2$ and for $\alpha = 1/2$ when $\omega = \pi/2$.

Kato asked two questions. For $\omega < \pi/2$,

(K1) Does (K_α) hold for $\alpha = 1/2$?
(K2) For the case $\omega = 0$, we know $\mathcal{D}(\sqrt{A(t)}) = \mathcal{W}$ and (K1) is true, but is

$$\left\| \frac{d}{dt} \sqrt{A(t)} u \right\| \lesssim \|u\|$$

for $u \in \mathcal{W}$?

In 1972, McIntosh provided a counter example in [Mc72] demonstrating that (K1) is false in such generality.

In 1982, McIntosh showed that (K2) also did not hold in general in [Mc82].

The *Kato square root problem* then became the following. Set

$$J[u,v] = \langle A\nabla u, \nabla v\rangle \quad u, v \in \mathrm{H}^1(\mathbb{R}^n),$$

where $A \in \mathrm{L}^\infty$ is a pointwise matrix multiplication operator satisfying the following ellipticity condition:

$$\operatorname{Re} J[u,u] \geq \kappa \|\nabla u\|, \quad \text{for some } \kappa > 0.$$

Under these conditions, is it true that

$$\mathcal{D}(\sqrt{\operatorname{div} A\nabla}) = \mathrm{H}^1(\mathbb{R}^n)$$
$$\left\|\sqrt{\operatorname{div} A\nabla}u\right\| \simeq \|\nabla u\| \tag{K1}$$

This was answered in the positive in 2002 by Auscher, Hofmann, Lacey, McIntosh and Tchamitchian in [AHLMcT].

Setup

Let $\mathcal{M}$ be a smooth, complete Riemannian manifold with metric g, Levi-Civita connection ∇, and volume measure $d\mu$.

Write $\mathrm{div} = -\nabla^*$ in L^2 and let $S = (I, \nabla)$.

Consider the following *uniformly elliptic* second order differential operator $\mathrm{L}_A : \mathcal{D}(\mathrm{L}_A) \subset \mathrm{L}^2(\mathcal{M}) \to \mathrm{L}^2(\mathcal{M})$ defined by

$$\mathrm{L}_A u = aS^*ASu = -a\,\mathrm{div}(A_{11}\nabla u) - a\,\mathrm{div}(A_{10}u) + aA_{01}\nabla u + aA_{00}u.$$

That is, we assume a and $A = (A_{ij})$ are L^∞ multiplication operators and that there exist $\kappa_1, \kappa_2 > 0$ such that

$$\mathrm{Re}\,\langle av, v\rangle \geq \kappa_1 \|v\|^2,\ v \in \mathrm{L}^2$$
$$\mathrm{Re}\,\langle ASu, Su\rangle \geq \kappa_2(\|u\|^2 + \|\nabla u\|^2),\ u \in \mathrm{H}^1$$

The problem

The *Kato square root problem on manifolds* is to determine when the following holds:

$$\begin{cases} \mathcal{D}(\sqrt{\mathrm{L}_A}) = \mathrm{H}^1(\mathcal{M}) \\ \left\|\sqrt{\mathrm{L}_A} u\right\| \simeq \|\nabla u\| + \|u\| = \|u\|_{\mathrm{H}^1}, \ u \in \mathrm{H}^1(\mathcal{M}) \end{cases}$$

The main theorem

Theorem (B.-Mc)

Let $\mathcal{M}$ *be a smooth, complete Riemannian manifold with* $|\mathrm{Ric}| \leq C$ *and* $\mathrm{inj}(M) \geq \kappa > 0$. *Suppose there exist* $\kappa_1, \kappa_2 > 0$ *such that*

$$\mathrm{Re}\,\langle av, v\rangle \geq \kappa_1 \|v\|^2$$
$$\mathrm{Re}\,\langle ASu, Su\rangle \geq \kappa_2 \|u\|_{\mathrm{H}^1}^2$$

for $v \in \mathrm{L}^2(\mathcal{M})$ *and* $u \in \mathrm{H}^1(\mathcal{M})$. *Then,* $\mathcal{D}(\sqrt{\mathrm{L}_A}) = \mathcal{D}(\nabla) = \mathrm{H}^1(\mathcal{M})$ *and* $\|\sqrt{\mathrm{L}_A} u\| \simeq \|\nabla u\| + \|u\| = \|u\|_{\mathrm{H}^1}$ *for all* $u \in \mathrm{H}^1(\mathcal{M})$.

Stability

Theorem (B.-Mc)

Let $\mathcal{M}$ be a smooth, complete Riemannian manifold with $|\mathrm{Ric}| \leq C$ and $\mathrm{inj}(M) \geq \kappa > 0$. Suppose that there exist $\kappa_1, \kappa_2 > 0$ such that

$$\mathrm{Re}\,\langle av, v\rangle \geq \kappa_1 \|v\|^2$$
$$\mathrm{Re}\,\langle ASu, Su\rangle \geq \kappa_2 \|u\|_{\mathrm{H}^1}^2$$

for $v \in \mathrm{L}^2(\mathcal{M})$ and $u \in \mathrm{H}^1(\mathcal{M})$. Then for every $\eta_i < \kappa_i$, whenever $\|\tilde{a}\|_\infty \leq \eta_1$, $\|\tilde{A}\|_\infty \leq \eta_2$, the estimate

$$\left\| \sqrt{\mathrm{L}_A}\, u - \sqrt{\mathrm{L}_{A+\tilde{A}}}\, u \right\| \lesssim (\|\tilde{a}\|_\infty + \|\tilde{A}\|_\infty)\, \|u\|_{\mathrm{H}^1}$$

holds for all $u \in \mathrm{H}^1(\mathcal{M})$. The implicit constant depends in particular on A, a and η_i.

A more general problem

We can consider the *Kato square root problem on vector bundles* by replacing $\mathcal{M}$ with $\mathcal{V}$, a smooth, complex vector bundle of rank N over $\mathcal{M}$ with metric h and connection ∇.

These theorems are obtained as special cases of corresponding theorems on vector bundles.

We use the adaptation of the *first order systems* approach in [AKMc], which captures the Kato problem (and some other results of harmonic analysis) in terms of perturbations of *Dirac type operators*.

Axelsson (Rosén)-Keith-McIntosh framework

(H1) Let Γ be a densely-defined, closed, nilpotent operator on a Hilbert space $\mathscr{H}$,

(H2) Suppose that $B_1, B_2 \in \mathcal{L}(\mathscr{H})$ such that here exist $\kappa_1, \kappa_2 > 0$ satisfying

$$\operatorname{Re} \langle B_1 u, u \rangle \geq \kappa_1 \|u\|^2 \quad \text{and} \quad \operatorname{Re} \langle B_2 v, v \rangle \geq \kappa_2 \|v\|^2$$

for $u \in \mathcal{R}(\Gamma^*)$ and $v \in \mathcal{R}(\Gamma)$,

(H3) The operators B_1, B_2 satisfy $B_1 B_2 \mathcal{R}(\Gamma) \subset \mathcal{N}(\Gamma)$ and $B_2 B_1 \mathcal{R}(\Gamma^*) \subset \mathcal{N}(\Gamma^*)$.

Let $\Gamma_B^* = B_1 \Gamma^* B_2$, $\Pi_B = \Gamma + \Gamma_B^*$ and $\Pi = \Gamma + \Gamma^*$.

Growth restrictions

We say $\mathcal{M}$ has *exponential volume growth* if there exists $c \geq 1,\ \kappa, \lambda \geq 0$ such that

$$0 < \mu(B(x, tr)) \leq ct^{\kappa} \mathrm{e}^{\lambda tr} \mu(B(x, r)) < \infty \qquad (\mathsf{E}_{\mathsf{loc}})$$

for all $t \geq 1,\ r > 0$ and $x \in \mathcal{M}$.

For instance, if $\mathrm{Ric} \geq \eta \mathrm{g}$, for $\eta \in \mathbb{R}$, then $(\mathsf{E}_{\mathsf{loc}})$ is satisfied.

Generalised bounded geometry

We want to set $\mathscr{H} = \mathrm{L}^2(\mathcal{V})$, but we need to assume more structure in $\mathcal{V}$.

Definition (Generalised Bounded Geometry)

Suppose there exists $\rho > 0$, $C \geq 1$ such that for each $x \in \mathcal{M}$, there exists a trivialisation $\psi : B(x, \rho) \times \mathbb{C}^N \to \pi_{\mathcal{V}}^{-1}(B(x, \rho))$ satisfying

$$C^{-1} I \leq \mathrm{h} \leq C I$$

in the basis $\left\{ e^i = \psi(x, \hat{e}^i) \right\}$, where $\left\{ \hat{e}^i \right\}$ is the standard basis for $\mathbb{C}^N$. Then, we say that $\mathcal{V}$ has *generalised bounded geometry* or *GBG*. We call ρ the *GBG radius*.

Further assumptions

(H4) The bundle $\mathcal{V}$ has GBG, $\mathscr{H} = \mathrm{L}^2(\mathcal{V})$, and $\mathcal{M}$ grows at most exponentially.

(H5) The operators B_1, B_2 are matrix valued pointwise multiplication operators. That is, $B_i \in \mathrm{L}^\infty(\mathcal{M}, \mathcal{L}(\mathcal{V}))$ by which we mean that $B_i(x) \in \mathcal{L}(\pi_{\mathcal{V}}^{-1}(x))$ for every $x \in \mathcal{M}$ and there is a $C_{B_i} > 0$ so that $\|B_i(x)\|_\infty \leq C$ for almost every $x \in \mathcal{M}$.

(H6) The operator Γ is a first order differential operator. That is, there exists a $C_\Gamma > 0$ such that whenever $\eta \in \mathrm{C}_\mathrm{c}^\infty(\mathcal{M})$, we have that $\eta \mathcal{D}(\Gamma) \subset \mathcal{D}(\Gamma)$ and $\mathrm{M}_\eta u(x) = [\Gamma, \eta(x)]\, u(x)$ is a multiplication operator satisfying

$$|\mathrm{M}_\eta u(x)| \leq C_\Gamma \, |\nabla \eta|_{\mathrm{T}^* M} \, |u(x)|$$

for all $u \in \mathcal{D}(\Gamma)$ and almost all $x \in \mathcal{M}$.

Dyadic cubes

Since we assume exponential growth of $\mathcal{M}$, the work of Christ in [Christ] and subsequently of Morris in [Morris] allows us to perform a *dyadic decomposition* of the manifold below a fixed "scale."

In particular, we are able to choose arbitrarily large $\mathrm{J} \in \mathbb{N}$ so that for every $j \geq \mathrm{J}$, $\mathscr{Q}^j$ is an almost everywhere decomposition of $\mathcal{M}$ by open sets.

If $l > j$ then for every *cube* $Q \in \mathscr{Q}^l$, there is a unique cube $\widehat{Q} \in \mathscr{Q}^j$ such that $Q \subset \widehat{Q}$.

Each such cube has a centre x_Q.

Each cube $Q \in \mathscr{Q}^j$ also has a diameter of at most $C_1 \delta^j$, where $C_1 > 0$ and $\delta \in (0, 1)$ are fixed, uniform quantities.

GBG coordinates

Since we assume the bundle satisfies GBG, we recall the uniform $\rho > 0$ from this criterion.

Choose $\mathrm{J} \in \mathbb{N}$ such that $C_1 \delta^{\mathrm{J}} \leq \frac{\rho}{5}$. We call $\mathrm{t_S} = \delta^{\mathrm{J}}$ the *scale*.

Call the system of trivialisations $\mathscr{C} = \left\{\psi : B(x_Q, \rho) \times \mathbb{C}^N \to \pi_{\mathcal{V}}^{-1}(B(x_Q, \rho)) \text{ s.t. } Q \in \mathscr{Q}^{\mathrm{J}}\right\}$ the *GBG coordinates*.

GBG coordinates (cont.)

Call the set of a.e. trivialisations $\mathscr{C}_{\mathrm{J}} = \left\{ \tilde{\varphi}_Q = \psi|_Q : Q \times \mathbb{C}^N \to \pi_{\mathcal{V}}^{-1}(Q) \text{ s.t. } Q \in \mathscr{Q}^{\mathrm{J}} \right\}$ the *dyadic GBG coordinates*.

For any cube Q, the unique cube $\widehat{Q} \in \mathscr{Q}^{\mathrm{J}}$ satisfying $Q \subset \widehat{Q}$ we call the *GBG cube of* Q.

The *GBG coordinate system* of Q is then $\psi : B(x_{\widehat{Q}}, \rho) \times \mathbb{C}^N \to \pi_{\mathcal{V}}^{-1}(B(x_{\widehat{Q}}, \rho))$.

Machinery for the harmonic analysis

- For $t \leq \mathrm{t_S}$, we write $\mathscr{Q}_t = \mathscr{Q}^j$ whenever $\delta^{j+1} < t \leq \delta^j$.
- For $j > \mathrm{J}$ and $Q \in \mathscr{Q}^j$ and $u = u_i e^i \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{V})$ in the GBG coordinates associated to $\widehat{Q}$. Define, the *cube integral*

$$\int_Q u = \left(\int_Q u_i \right) e^i$$

  inside $B(x_{\widehat{Q}}, \rho)$.
- The *cube average* is then defined as $u_Q(y) = \fint_Q u$, for $y \in B(x_{\widehat{Q}}, \rho)$ and 0 otherwise.
- Let $\mathcal{A}_t u(x) = u_Q(x)$ whenever $x \in Q \in \mathscr{Q}_t$.
- For each $w \in \mathbb{C}^N$, let $\gamma_t(x)w = (\Theta_t^B \omega)(x)$ where $\omega(x) = w$ in the GBG coordinates of each Q.

Cancellation assumption

(H7) There exists $c > 0$ such that for all $t \leq \mathrm{t_S}$ and $Q \in \mathscr{Q}_t$,

$$\left| \int_Q \Gamma u \ d\mu \right| \leq c\mu(Q)^{\frac{1}{2}} \|u\| \quad \text{and} \quad \left| \int_Q \Gamma^* v \ d\mu \right| \leq c\mu(Q)^{\frac{1}{2}} \|v\|$$

for all $u \in \mathcal{D}(\Gamma),\ v \in \mathcal{D}(\Gamma^*)$ satisfying $\mathrm{spt}\ u,\ \mathrm{spt}\ v \subset Q$.

Dyadic Poincaré assumption

(H8) There exists $C_P,\ C_C,\ c,\ \tilde{c} > 0$ and an operator $\Xi : \mathcal{D}(\Xi) \subset \mathrm{L}^2(\mathcal{V}) \to \mathrm{L}^2(\mathscr{N})$, where $\mathscr{N}$ is a normed bundle over $\mathcal{M}$ with norm $|\cdot|_{\mathscr{N}}$ and $\mathcal{D}(\Pi) \cap \mathcal{R}(\Pi) \subset \mathcal{D}(\Xi)$ satisfying for all $u \in \mathcal{D}(\Pi) \cap \mathcal{R}(\Pi)$,

-1 (Dyadic Poincaré)

$$\int_B |u - u_Q|^2 \ d\mu \leq C_P(1 + r^\kappa e^{\lambda c r t})(rt)^2 \int_{\tilde{c}B} \left(|\Xi u|^2_{\mathscr{N}} + |u|^2 \right) \ d\mu$$

for all balls $B = B(x_Q, rt)$ with $r \geq C_1/\delta$ where $Q \in \mathscr{Q}_t$ with $t \leq \mathrm{t_S}$, and

-2 (Coercivity)

$$\|\Xi u\|^2_{\mathrm{L}^2(\mathscr{N})} + \|u\|^2_{\mathrm{L}^2(\mathcal{V})} \leq C_C \|\Pi u\|^2_{\mathrm{L}^2(\mathcal{V})}.$$

Kato square root type estimate

Proposition

Suppose $\mathcal{M}$ is a smooth, complete Riemannian manifold and $\mathcal{V}$ is a smooth vector bundle over $\mathcal{M}$. If (H1)-(H8) are satisfied, then

(i) $\mathcal{D}(\Gamma) \cap \mathcal{D}(\Gamma_B^*) = \mathcal{D}(\Pi_B) = \mathcal{D}(\sqrt{\Pi_B^2})$, *and*

(ii) $\|\Gamma u\| + \|\Gamma_B u\| \simeq \|\Pi_B u\| \simeq \|\sqrt{\Pi_B^2} u\|$, *for all* $u \in \mathcal{D}(\Pi_B)$.

Kato square root problem on vector bundles

Theorem (B.-Mc.)

Suppose $\mathcal{M}$ grows at most exponentially and satisfies a local Poincaré inequality on functions. Further, suppose that both $\mathcal{V}$ and $\mathrm{T}^*\mathcal{M}$ have GBG, and

(i) *the metric h and ∇ are compatible,*

(ii) *there exists $C > 0$ such that in each GBG chart we have that $\left|\nabla e^j\right|, \left|\nabla dx^i\right|, \left|\partial_k \mathrm{h}^{ij}\right|, \left|\partial_k \mathrm{g}^{ij}\right| \leq C$ a.e.,*

(iii) *there exist $\kappa_1, \kappa_2 > 0$ such that $\mathrm{Re}\,\langle au, u\rangle \geq \kappa_1 \|u\|^2$ and $\mathrm{Re}\,\langle ASv, Sv\rangle \geq \kappa_2 \|v\|_{\mathrm{H}^1}^2$ for all $u \in \mathrm{L}^2(\mathcal{V})$ and $v \in \mathrm{H}^1(\mathcal{V})$, and*

(iv) *we have that $\mathcal{D}(\Delta) \subset \mathrm{H}^2(\mathcal{V})$, and there exist $C' > 0$ such that $\|\nabla^2 u\| \leq C' \|(I + \Delta)u\|$ whenever $u \in \mathcal{D}(\Delta)$.*

Then, $\mathcal{D}(\sqrt{\mathrm{L}_A}) = \mathcal{D}(\nabla) = \mathrm{H}^1(\mathcal{V})$ with $\|\sqrt{\mathrm{L}_A} u\| \simeq \|\nabla u\| + \|u\| = \|u\|_{\mathrm{H}^1}$ for all $u \in \mathrm{H}^1(\mathcal{V})$.

Set $\mathscr{H} = \mathrm{L}^2(\mathcal{V}) \oplus (\mathrm{L}^2(\mathcal{V}) \oplus \mathrm{L}^2(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V}))$.

Define

$$\Gamma = \begin{bmatrix} 0 & 0 \\ S & 0 \end{bmatrix}, \ B_1 = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } B_2 = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}.$$

Then,

$$\Gamma^* = \begin{bmatrix} 0 & S^* \\ 0 & 0 \end{bmatrix} \text{ and } \Pi_B^2 = \begin{bmatrix} \mathrm{L}_A & 0 \\ 0 & * \end{bmatrix}$$

Kato square root problem for tensors

Theorem (B.-Mc.)

Let $\mathcal{M}$ *be a smooth, complete Riemannian manifold with* $|\mathrm{Ric}| \leq C$ *and* $\mathrm{inj}(M) \geq \kappa > 0$. *Suppose that there exist* $C' > 0$ *such that* $\left\|\nabla^2 u\right\| \leq C' \left\|(I + \Delta)u\right\|$ *whenever* $u \in \mathcal{D}(\Delta) \subset \mathrm{H}^2(\mathcal{T}^{(p,q)}\mathcal{M})$. *Then,* $\mathcal{D}(\sqrt{\mathrm{L}_A}) = \mathcal{D}(\nabla) = \mathrm{H}^1(\mathcal{T}^{(p,q)}\mathcal{M})$ *and* $\|\sqrt{\mathrm{L}_A} u\| \simeq \|\nabla u\| + \|u\| = \|u\|_{\mathrm{H}^1}$ *for all* $u \in \mathrm{H}^1(\mathcal{T}^{(p,q)}\mathcal{M})$.

Ricci, injectivity bounds and GBG

Proposition

Suppose there is a $\kappa, \eta > 0$ *such that* $\mathrm{inj}(\mathcal{M}) \geq \kappa$ *and* $|\mathrm{Ric}| \leq \eta$. *Then for* $A > 1$ *and* $\alpha \in (0,1)$, *there exists* $r_H(n, A, \alpha, \kappa, \eta) > 0$ *such that for each* $x \in \mathcal{M}$, *there is a coordinate system corresponding to* $B(x, r_H)$ *satisfying:*

(i) $A^{-1}\delta_{ij} \leq \mathrm{g}_{ij} \leq A\delta_{ij}$ *as bilinear forms and,*

(ii) $$\sum_l r_H \sup_{y \in B(x, r_H)} |\partial_l \mathrm{g}_{ij}(y)| + \sum_l r_H^{1+\alpha} \sup_{y \neq z \in B(x, r_H)} \frac{|\partial_l \mathrm{g}_{ij}(z) - \partial_l \mathrm{g}_{ij}(y)|}{d(y,z)^\alpha} \leq A - 1.$$

See the observation following Theorem 1.2 in [Hebey].

- The proposition guarantees GBG coordinates for $\mathcal{V} = \mathcal{T}^{(p,q)}\mathcal{M}$.
- The proposition gives $\left|\nabla e^j\right|, \left|\nabla dx^i\right|, \left|\partial_k \mathrm{h}^{ij}\right|, \left|\partial_k \mathrm{g}^{ij}\right| \leq C$ in each GBG coordinate chart $\left\{e^j\right\}$ for $\mathcal{T}^{(p,q)}\mathcal{M}$.
- The Ricci bounds guarantee exponential volume growth and the local Poincaré inequality.

By invoking Theorem 5 (on vector bundles), we obtain Theorem 6 (finite rank tensors).

Theorem 1 follows from Theorem 6 since $\left\|\nabla^2 u\right\| \lesssim \|(I + \Delta)u\|$ is a consequence of the *Bochner-Lichnerowicz-Weitzenböck* identity, Ricci curvature bounds and uniform lower bounds on injectivity radius.

See Proposition 3.3 in [Hebey].

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