Geometry and the Kato square root problem

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History of the problem

In the 1960's, Kato considered the following *abstract evolution equation*

$$\partial_t u(t) + A(t)u(t) = f(t), \quad t \in [0, T].$$

on a Hilbert space $\mathscr{H}$.

This has a *unique strict solution* $u = u(t)$ if

$$\mathcal{D}(A(t)^\alpha) = \text{const}$$

for some $0 < \alpha \leq 1$ and $A(t)$ and $f(t)$ satisfy certain smoothness conditions.

Typically $A(t)$ is defined by an associated sesquilinear form

$$J_t : \mathcal{W} \times \mathcal{W} \to \mathbb{C}$$

where $\mathcal{W} \subset \mathscr{H}$.

Suppose $0 \leq \omega \leq \pi/2$. Then J_t is ω-sectorial means that

(i) $\mathcal{W} \subset \mathscr{H}$ is dense,
(ii) $J_t[u, u] \in S_{\omega+} = \{\zeta \in \mathbb{C} : |\arg \zeta| \leq \omega\} \cup \{0\}$, and
(iii) $\mathcal{W}$ is complete under the norm

$$\|u\|_{\mathcal{W}}^2 = \|u\|^2 + \mathrm{Re}\, J_t[u, u].$$

$T : \mathcal{D}(T) \to \mathscr{H}$ is called ω-accretive if

(i) T is densely-defined and closed,
(ii) $\langle Tu, u \rangle \in S_{\omega+}$ for $u \in \mathcal{D}(T)$, and
(iii) $\sigma(T) \subset S_{\omega+}$.

A 0-accretive operator is non-negative and self-adjoint.

Let $A(t) : \mathcal{D}(A(t)) \to \mathscr{H}$ be defined as the operator with largest domain such that

$$J_t[u, v] = \langle A(t)u, v \rangle \qquad u \in \mathcal{D}(A(t)),\ v \in \mathcal{W}.$$

The theorem of Lax-Milgram guarantees that

$$J_t \text{ is } \omega\text{-sectorial} \implies A(t) \text{ is } \omega\text{-accretive.}$$

In 1962, Kato showed in [Kato] that for $0 \leq \alpha < 1/2$ and $0 \leq \omega \leq \pi/2$,

$$\begin{aligned} &\mathcal{D}(A(t)^\alpha) = \mathcal{D}(A(t)^{*\alpha}) = \mathcal{D} = \text{const},\ \text{and} \\ &\|A(t)^\alpha u\| \simeq \|A(t)^{*\alpha} u\|, \quad u \in \mathcal{D}. \end{aligned} \tag{K_α}$$

Counter examples were known for $\alpha > 1/2$ and for $\alpha = 1/2$ when $\omega = \pi/2$.

Kato asked two questions. For $\omega < \pi/2$,

(K1) Does (K_α) hold for $\alpha = 1/2$?

(K2) For the case $\omega = 0$, we know $\mathcal{D}(\sqrt{A(t)}) = \mathcal{W}$ and (K1) is true, but is

$$\|\partial_t \sqrt{A(t)} u\| \lesssim \|u\|$$

for $u \in \mathcal{W}$?

In 1972, McIntosh provided a counter example in [Mc72] demonstrating that (K1) is false in such generality.

In 1982, McIntosh showed that (K2) also did not hold in general in [Mc82].

The *Kato square root problem* then became the following. Set

$$J[u,v] = \langle A\nabla u, \nabla v\rangle \quad u,v \in \mathrm{W}^{1,2}(\mathbb{R}^n),$$

where $A \in \mathrm{L}^\infty$ is a pointwise matrix multiplication operator satisfying the following ellipticity condition:

$$\mathrm{Re}\, J[u,u] \geq \kappa \|\nabla u\|, \quad \text{for some } \kappa > 0.$$

Under these conditions, is it true that

$$\begin{aligned} &\mathcal{D}(\sqrt{\mathrm{div}\, A\nabla}) = \mathrm{W}^{1,2}(\mathbb{R}^n) \\ &\|\sqrt{\mathrm{div}\, A\nabla} u\| \simeq \|\nabla u\|. \end{aligned} \tag{K1}$$

This was answered in the positive in 2002 by Pascal Auscher, Steve Hofmann, Michael Lacey, Alan McIntosh and Phillipe Tchamitchian in [AHLMcT].

Kato square root problem for functions and forms

Let $\mathcal{M}$ be a smooth, complete Riemannian manifold with metric g, Levi-Civita connection ∇, and volume measure μ_{g}.

Write $\mathrm{div}_{\mathrm{g}} = -\nabla^*$ in L^2.

Let $\mathbf{\Omega}(\mathcal{M})$ denote the algebra of differential forms over $\mathcal{M}$.

Let d be the exterior derivative as an operator on $\mathrm{L}^2(\mathbf{\Omega}(\mathcal{M}))$ and d^* its adjoint, both of which are *nilpotent* operators.

The Hodge-Dirac operator is then the self-adjoint operator $\mathrm{D} = \mathrm{d} + \mathrm{d}^*$. The Hodge-Laplacian is then $\mathrm{D}^2 = \mathrm{d}\,\mathrm{d}^* + \mathrm{d}^*\,\mathrm{d}$.

Let $S = (\mathrm{I}, \nabla)$.

Assume $a \in \mathrm{L}^\infty(\mathcal{M})$ and
$A = (A_{ij}) \in \mathrm{L}^\infty(\mathcal{M}, \mathcal{L}(\mathrm{L}^2(\mathcal{M}) \oplus \mathrm{L}^2(\mathrm{T}^*\mathcal{M}))$.

Consider the following second order differential operator
$\mathrm{L}_A : \mathcal{D}(\mathrm{L}_A) \subset \mathrm{L}^2(\mathcal{M}) \to \mathrm{L}^2(\mathcal{M})$ defined by:

$$\mathrm{L}_A u = aS^*ASu = -a\,\mathrm{div}(A_{11}\nabla u) - a\,\mathrm{div}(A_{10}u) + aA_{01}\nabla u + aA_{00}u.$$

The Kato square root problem for functions is then to determine:

$\mathcal{D}(\sqrt{\mathrm{L}_A}) = \mathrm{W}^{1,2}(\mathcal{M})$ and $\|\sqrt{\mathrm{L}_A}u\| \simeq \|\nabla u\| + \|u\| = \|u\|_{\mathrm{W}^{1,2}}$ for all $u \in \mathrm{W}^{1,2}(\mathcal{M})$.

For an invertible $A \in \mathrm{L}^{\infty}(\mathcal{L}(\mathbf{\Omega}(\mathcal{M})))$, we consider perturbing D to obtain the operator $\mathrm{D}_A = \mathrm{d} + A^{-1}\mathrm{d}^* A$.

The Kato square root problem for forms is then to determine the following whenever $0 \neq \beta \in \mathbb{C}$:

$\mathcal{D}(\sqrt{\mathrm{D}_A^2 + |\beta|^2}) = \mathcal{D}(\mathrm{D}_A) = \mathcal{D}(\mathrm{d}) \cap \mathcal{D}(\mathrm{d}^* A)$ and
$\|\sqrt{\mathrm{D}_A^2 + |\beta|^2} u\| \simeq \|\mathrm{D}_A u\| + \|u\|$.

Axelsson (Rosén)-Keith-McIntosh framework

(H1) The operator $\Gamma : \mathcal{D}(\Gamma) \subset \mathscr{H} \to \mathscr{H}$ is a closed, densely-defined and nilpotent operator, by which we mean $\mathcal{R}(\Gamma) \subset \mathcal{N}(\Gamma)$,

(H2) $B_1, B_2 \in \mathcal{L}(\mathscr{H})$ and there exist $\kappa_1, \kappa_2 > 0$ satisfying the accretivity conditions

$$\mathrm{Re}\,\langle B_1 u, u\rangle \geq \kappa_1 \|u\|^2 \text{ and } \mathrm{Re}\,\langle B_2 v, v\rangle \geq \kappa_2 \|v\|^2,$$

for $u \in \mathcal{R}(\Gamma^*)$ and $v \in \mathcal{R}(\Gamma)$, and

(H3) $B_1 B_2 \mathcal{R}(\Gamma) \subset \mathcal{N}(\Gamma)$ and $B_2 B_1 \mathcal{R}(\Gamma^*) \subset \mathcal{N}(\Gamma^*)$.

Let us now define $\Pi_B = \Gamma + B_1 \Gamma^* B_2$ with domain $\mathcal{D}(\Pi_B) = \mathcal{D}(\Gamma) \cap \mathcal{D}(B_1 \Gamma^* B_2)$.

Quadratic estimates

To say that Π_B satisfies *quadratic estimates* means that

$$\int_0^\infty \|t\Pi_B(\mathrm{I} + t^2\Pi_B^2)^{-1}u\|^2 \ \frac{dt}{t} \simeq \|u\|^2, \tag{Q}$$

for all $u \in \overline{\mathcal{R}(\Pi_B)}$.

This implies that

$$\mathcal{D}(\sqrt{\Pi_B^2}) = \mathcal{D}(\Pi_B) = \mathcal{D}(\Gamma) \cap \mathcal{D}(\Gamma^* B_2)$$
$$\|\sqrt{\Pi_B^2}u\| \simeq \|\Pi_B u\| \simeq \|\Gamma u\| + \|\Gamma^* B_2 u\|$$

The main theorem on manifolds

Theorem (B.-Mc, 2012)

Let $\mathcal{M}$ be a smooth, complete Riemannian manifold with $|\mathrm{Ric}| \leq C$ *and* $\mathrm{inj}(M) \geq \kappa > 0$. *Suppose the following ellipticity condition holds: there exists* $\kappa_1, \kappa_2 > 0$ *such that*

$$\mathrm{Re}\,\langle av, v\rangle \geq \kappa_1 \|v\|^2$$
$$\mathrm{Re}\,\langle ASu, Su\rangle \geq \kappa_2 \|u\|^2_{\mathrm{W}^{1,2}}$$

for $v \in \mathrm{L}^2(\mathcal{M})$ *and* $u \in \mathrm{W}^{1,2}(\mathcal{M})$. *Then,*
$\mathcal{D}(\sqrt{\mathrm{L}_A}) = \mathcal{D}(\nabla) = \mathrm{W}^{1,2}(\mathcal{M})$ *and*
$\|\sqrt{\mathrm{L}_A}u\| \simeq \|\nabla u\| + \|u\| = \|u\|_{\mathrm{W}^{1,2}}$ *for all* $u \in \mathrm{W}^{1,2}(\mathcal{M})$.

Lipschitz estimates

Since we allow the coefficients a and A to be *complex*, we obtain the following stability result as a consequence:

Theorem (B.-Mc, 2012)

Let $\mathcal{M}$ be a smooth, complete Riemannian manifold with $|\mathrm{Ric}| \leq C$ and $\mathrm{inj}(\mathcal{M}) \geq \kappa > 0$. Suppose that there exist $\kappa_1, \kappa_2 > 0$ such that $\mathrm{Re}\,\langle av, v\rangle \geq \kappa_1 \|v\|^2$ and $\mathrm{Re}\,\langle ASu, Su\rangle \geq \kappa_2 \|u\|^2_{\mathrm{W}^{1,2}}$ for $v \in \mathrm{L}^2(\mathcal{M})$ and $u \in \mathrm{W}^{1,2}(\mathcal{M})$. Then for every $\eta_i < \kappa_i$, whenever $\|\tilde{a}\|_\infty \leq \eta_1$, $\|\tilde{A}\|_\infty \leq \eta_2$, the estimate

$$\|\sqrt{\mathrm{L}_A}\, u - \sqrt{\mathrm{L}_{A+\tilde{A}}}\, u\| \lesssim (\|\tilde{a}\|_\infty + \|\tilde{A}\|_\infty)\|u\|_{\mathrm{W}^{1,2}}$$

holds for all $u \in \mathrm{W}^{1,2}(\mathcal{M})$. The implicit constant depends in particular on A, a and η_i.

Curvature endomorphism for forms

Let $\{\theta^i\}$ be an orthonormal frame at x for $\mathbf{\Omega}^1(\mathcal{M}) = \mathrm{T}^*\mathcal{M}$.

Denote the components of the curvature tensor in this frame by Rm_{ijkl}. The curvature endomorphism is then the operator

$$\mathrm{R}\,\omega = \mathrm{Rm}_{ijkl}\,\theta^i \wedge (\theta^j \llcorner (\theta^k \wedge (\theta^l \llcorner \omega)))$$

for $\omega \in \mathbf{\Omega}_x(\mathcal{M})$.

This can be seen as an extension of Ricci curvature for forms, since $\mathrm{g}(\mathrm{R}\,\omega, \eta) = \mathrm{Ric}(\omega^\flat, \eta^\flat)$ whenever $\omega, \eta \in \mathbf{\Omega}^1_x(\mathcal{M})$ and where $\flat : \mathrm{T}^*\mathcal{M} \to \mathrm{T}\mathcal{M}$ is the flat isomorphism through the metric g.

The Weitzenböck formula then asserts that $\mathrm{D}^2 = \mathrm{tr}_{12}\,\nabla^2 + \mathrm{R}\,$.

Theorem (B., 2012)

Let $\mathcal{M}$ *be a smooth, complete Riemannian manifold and let* $\beta \in \mathbb{C} \setminus \{0\}$. *Suppose there exist* $\eta, \kappa > 0$ *such that* $|\mathrm{Ric}| \leq \eta$ *and* $\mathrm{inj}(\mathcal{M}) \geq \kappa$. *Furthermore, suppose there is a* $\zeta \in \mathbb{R}$ *satisfying* $\mathrm{g}(\mathrm{R}\, u, u) \geq \zeta \, |u|^2$, *for* $u \in \mathbf{\Omega}_x(\mathcal{M})$ *and* $A \in \mathrm{L}^\infty(\mathcal{L}(\mathbf{\Omega}(\mathcal{M})))$ *and* $\kappa_1 > 0$ *satisfying*

$$\mathrm{Re}\,\langle Au, u\rangle \geq \kappa_1 \|u\|^2.$$

Then, $\mathcal{D}(\sqrt{\mathrm{D}_A^2 + |\beta|^2}) = \mathcal{D}(\mathrm{D}_A) = \mathcal{D}(\mathrm{d}) \cap \mathcal{D}(\mathrm{d}^* A)$ *and* $\|\sqrt{\mathrm{D}_A^2 + |\beta|^2} u\| \simeq \|\mathrm{D}_A\, u\| + \|u\|$.

The Kato problem for functions are captured in the AKM framework on letting $\mathscr{H} = \mathrm{L}^2(\mathcal{M}) \oplus (\mathrm{L}^2(\mathcal{M}) \oplus \mathrm{L}^2(\mathrm{T}^*\mathcal{M}))$ and letting

$$\Gamma = \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix},\ \Gamma^* = \begin{pmatrix} 0 & S^* \\ 0 & 0 \end{pmatrix},\ B_1 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix},\ B_2 = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}.$$

For the case of forms, the setup takes the form, $\mathscr{H} = \mathrm{L}^2(\boldsymbol{\Omega}(\mathcal{M})) \oplus \mathrm{L}^2(\boldsymbol{\Omega}(\mathcal{M}))$ and

$$\Gamma = \begin{pmatrix} \mathrm{d} & 0 \\ \beta & -\mathrm{d} \end{pmatrix}, \Gamma^* = \begin{pmatrix} \delta & \overline{\beta} \\ 0 & -\delta \end{pmatrix}, B_1 = \begin{pmatrix} A{-}1 & 0 \\ 0 & A^{-1} \end{pmatrix}, B_2 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Geometry and harmonic analysis

Harmonic analytic methods are used to prove quadratic estimates (Q).

The idea is to reduce the quadratic estimate (Q) to a *Carleson measure* estimate. This is achieved via a *local* $T(b)$ argument.

Geometry enters the picture precisely in the harmonic analysis. We need to perform harmonic analysis on vector fields, not just functions.

One can show that this is *not* artificial - the Kato problem on functions immediately provides a solution to the dual problem on vector fields.

Elements of the proofs

Similar in structure to the proof of [AKMc] which is inspired from the proof in [AHLMcT].

- A dyadic decomposition of the space
- A notion of averaging (in an integral sense)
- Poincaré inequality - on both functions and vector fields
- Control of ∇^2 in terms of Δ.

Rough metrics

Definition (Rough metric)

Let g be a $(2,0)$ symmetric tensor field with measurable coefficients and that for each $x \in \mathcal{M}$, there is some chart (U, ψ) near x and a constant $C \geq 1$ such that

$$C^{-1} \left|u\right|_{\psi^* \delta(y)} \leq \left|u\right|_{\mathrm{g}(y)} \leq C \left|u\right|_{\psi^* \delta(y)},$$

for almost-every $y \in U$ and where δ is the Euclidean metric in $\psi(U)$. Then we say that g is a rough metric, and such a chart (U, ψ) is said to satisfy the *local comparability condition*.

Metric perturbations

Definition

We say that two rough metrics g and $\tilde{\mathrm{g}}$ are C-close if

$$C^{-1} \left|u\right|_{\tilde{\mathrm{g}}(x)} \leq \left|u\right|_{\mathrm{g}(x)} \leq C \left|u\right|_{\tilde{\mathrm{g}}(x)}$$

for almost-every $x \in \mathcal{M}$ where $C \geq 1$. Two such metrics are said to be C-close everywhere if this inequality holds for every $x \in \mathcal{M}$.

We also say that g and $\tilde{\mathrm{g}}$ are close if there exists some $C \geq 1$ for which they are C-close.

For two continuous metrics, C-close and C-close everywhere coincide.

Proposition

Let g *and* $\tilde{\mathrm{g}}$ *be two rough metrics that are* C*-close. Then, there exists* $\mathrm{B} \in \mathbf{\Gamma}(\mathrm{T}^*\mathcal{M} \otimes \mathrm{T}\mathcal{M})$ *such that it is symmetric, almost-everywhere positive and invertible, and*

$$\tilde{\mathrm{g}}_x(\mathrm{B}(x)u, v) = \mathrm{g}_x(u, v)$$

for almost-every $x \in \mathcal{M}$*. Furthermore, for almost-every* $x \in \mathcal{M}$*,*

$$C^{-2} |u|_{\tilde{\mathrm{g}}(x)} \leq |\mathrm{B}(x)u|_{\tilde{\mathrm{g}}(x)} \leq C^2 |u|_{\tilde{\mathrm{g}}(x)},$$

and the same inequality with $\tilde{\mathrm{g}}$ *and* g *interchanged. If* $\tilde{\mathrm{g}} \in \mathrm{C}^k$ *and* $\mathrm{g} \in \mathrm{C}^l$ *(with* $k, l \geq 0$*), then the properties of* B *are valid for all* $x \in \mathcal{M}$ *and* $\mathrm{B} \in \mathrm{C}^{\min\{k,l\}}(\mathrm{T}^*\mathcal{M} \otimes \mathrm{T}\mathcal{M})$*.*

The measure $\mu_{\mathrm{g}}(x) = \theta(x)\ d\mu_{\tilde{\mathrm{g}}}(x)$, where $\theta(x) = \sqrt{\det B(x)}$.
Consequently,

(i) whenever $p \in [1, \infty)$, $\mathrm{L}^p(\mathcal{T}^{(r,s)}\mathcal{M}, \mathrm{g}) = \mathrm{L}^p(\mathcal{T}^{(r,s)}\mathcal{M}, \tilde{\mathrm{g}})$ with

$$C^{-\left(r+s+\frac{n}{2p}\right)} \|u\|_{p,\tilde{\mathrm{g}}} \leq \|u\|_{p,\mathrm{g}} \leq C^{r+s+\frac{n}{2p}} \|u\|_{p,\tilde{\mathrm{g}}},$$

(ii) for $p = \infty$, $\mathrm{L}^\infty(\mathcal{T}^{(r,s)}\mathcal{M}, \mathrm{g}) = \mathrm{L}^\infty(\mathcal{T}^{(r,s)}\mathcal{M}, \tilde{\mathrm{g}})$ with

$$C^{-(r+s)} \|u\|_{\infty,\tilde{\mathrm{g}}} \leq \|u\|_{\infty,\mathrm{g}} \leq C^{r+s} \|u\|_{\infty,\tilde{\mathrm{g}}},$$

(iii) the Sobolev spaces $\mathrm{W}^{1,p}(\mathcal{M}, \mathrm{g}) = \mathrm{W}^{1,p}(\mathcal{M}, \tilde{\mathrm{g}})$ and $\mathrm{W}^{1,p}_0(\mathcal{M}, \mathrm{g}) = \mathrm{W}^{1,p}_0(\mathcal{M}, \tilde{\mathrm{g}})$ with

$$C^{-\left(1+\frac{n}{2p}\right)} \|u\|_{\mathrm{W}^{1,p},\tilde{\mathrm{g}}} \leq \|u\|_{\mathrm{W}^{1,p},\mathrm{g}} \leq C^{1+\frac{n}{2p}} \|u\|_{\mathrm{W}^{1,p},\tilde{\mathrm{g}}},$$

(iv) the Sobolev spaces $\mathrm{W}^{\mathrm{d},p}(\mathcal{M}, \mathrm{g}) = \mathrm{W}^{\mathrm{d},p}(\mathcal{M}, \tilde{\mathrm{g}})$ and $\mathrm{W}^{\mathrm{d},p}_0(\mathcal{M}, \mathrm{g}) = \mathrm{W}^{\mathrm{d},p}_0(\mathcal{M}, \tilde{\mathrm{g}})$ with

$$C^{-\left(n+\frac{n}{2p}\right)} \|u\|_{\mathrm{W}^{\mathrm{d},p},\tilde{\mathrm{g}}} \leq \|u\|_{\mathrm{W}^{\mathrm{d},p},\mathrm{g}} \leq C^{n+\frac{n}{2p}} \|u\|_{\mathrm{W}^{\mathrm{d},p},\tilde{\mathrm{g}}},$$

(v) the divergence operators satisfy $\mathrm{div}_{D,\mathrm{g}} = \theta^{-1} \mathrm{div}_{D,\tilde{\mathrm{g}}} \theta \mathrm{B}$ and $\mathrm{div}_{N,\mathrm{g}} = \theta^{-1} \mathrm{div}_{N,\tilde{\mathrm{g}}} \theta \mathrm{B}$.

Case of functions

Theorem (B, 2014)

Let $\tilde{\mathrm{g}}$ *be a smooth, complete metric and suppose that there exists* $\kappa > 0$ *and* $\eta > 0$ *such that*

(i) $\mathrm{inj}(\mathcal{M}, \tilde{\mathrm{g}}) \geq \kappa$ *and,*

(ii) $|\mathrm{Ric}(\tilde{\mathrm{g}})| \leq \eta$*.*

Then, for any rough metric g *that is close, the Kato square root problem for functions has a solution on* $(\mathcal{M}, \mathrm{g})$*.*

Case of forms

Theorem (B, 2014)

Let g *be a rough metric close to* $\tilde{\mathrm{g}}$*, a smooth, complete metric, and suppose that:*

(i) *there exists* $\kappa > 0$ *such that* $\mathrm{inj}(\mathcal{M}, \tilde{\mathrm{g}}) \geq \kappa$*,*

(ii) *there exists* $\eta > 0$ *such that* $|\mathrm{Ric}(\tilde{\mathrm{g}})| \leq \eta$*, and*

(iii) *there exists* $\zeta \in \mathbb{R}$ *such that* $\tilde{\mathrm{g}}(\mathrm{R}\,\omega, \omega) \geq \zeta\, |\omega|_{\tilde{\mathrm{g}}}^2$ *.*

Then, the Kato square root problem for forms has a solution on $(\mathcal{M}, \mathrm{g})$*.*

Compact manifolds with rough metrics

Theorem (B, 2014)

Let $\mathcal{M}$ be a smooth, compact manifold and g a rough metric. Then, the Kato square root problem (on functions and forms, respectively) has a solution.

Cones and induced metrics

Let $\mathcal{C}^n_{r,h}$ be the n-cone of height $h > 0$ and radius $r > 0$. The cone can be realised as the image of the graph function

$$F_{r,h}(x) = \left(x, h\left(1 - \frac{|x|}{r}\right)\right).$$

Let U be an open set in $\mathbb{R}^n$ such that $B_r(0) \subset U$. Then, define $G_{r,h} : U \to \mathbb{R}^{n+1}$ as the map $F_{r,h}$ whenever $x \in B_r(0)$ and $(x, 0)$ otherwise.

Then we obtain that the map $G_{r,h}$ satisfies

$$|x - y| \leq |G_{r,h}(x) - G_{r,h}(y)| \leq \sqrt{1 + (hr^{-1})^2}\, |x - y|.$$

Proposition

Let $\gamma : I \to U$ be a smooth curve such that $\gamma(0) \not\in \{0\} \cup \partial B_r(0)$. Then,

$$\left|\gamma'(0)\right| \leq \left|(G_{r,h} \circ \gamma)'(0)\right| \leq \sqrt{1 + \frac{h^2}{r^2}} \left|\gamma'(0)\right|.$$

Moreover, for $u \in T_x U$, $x \not\in \{0\} \cup \partial B_r(0)$ (and in particular for almost-every x),

$$|u|_\delta \leq |u|_{\mathrm{g}} \leq \sqrt{1 + \frac{h^2}{r^2}} |u|_\delta ,$$

where δ is the usual inner product on U induced by $\mathbb{R}^n$.

A particular consequence is that the metrics $\mathrm{g} = G_{r,h}^* \delta_{\mathbb{R}^{n+1}}$ and $\delta_{\mathbb{R}^n}$ are $\sqrt{1 + (hr^{-1})^2}$-close on U.

Lemma

Given $\varepsilon > 0$, there exists two points x, x' and distinct minimising smooth geodesics $\gamma_{1,\varepsilon}$ and $\gamma_{2,\varepsilon}$ between x and x' of length ε. Furthermore, there are two constants $C_{1,r,h,\varepsilon}, C_{2,r,h,\varepsilon} > 0$ depending on h, r and ε such that the geodesics $\gamma_{1,\varepsilon}$ and $\gamma_{2,\varepsilon}$ are contained in $G_{r,h}(A_\varepsilon)$ where A_ε is the Euclidean annulus

$$\left\{x \in B_r(0) : C_{1,r,h,\varepsilon} < |x| < C_{2,r,h,\varepsilon}\right\}.$$

Theorem (B., 2014)

For any $C > 1$, *there exists a smooth metric* g *which is* C*-close to the Euclidean metric* δ *for which* $\mathrm{inj}(\mathbb{R}^2, \mathrm{g}) = 0$. *Furthermore, the Kato square root problem for functions can be solved for* $(\mathbb{R}^2, \mathrm{g})$ *under the.*

In higher dimensions, we obtain a similar result since the 2-dimensional cone can be realised as a totally geodesic submanifold.

Theorem (B., 2014)

Let $\mathcal{M}$ be a smooth manifold of dimension at least 2 and g a continuous metric. Given $C > 1$, and a point $x_0 \in \mathcal{M}$, there exists a rough metric h such that:

(i) *it induces a length structure and the metric $\mathrm{d_g}$ preserves the topology of $\mathcal{M}$,*

(ii) *it is smooth everywhere except x_0,*

(iii) *the geodesics through x_0 are Lipschitz,*

(iv) *it is C-close to g,*

(v) $\mathrm{inj}(\mathcal{M} \setminus \{x_0\}, \mathrm{h}) = 0.$

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