An Overview of Functional Calculi for Noncommuting Operators

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Abstract

When $\mathcal{X}$ is a Banach space, the Riesz-Dunford functional calculus provides a mechanism for defining $f(T)$ when $T \in \mathcal{L}(\mathcal{X})$. We consider the situation when $A = (A_1, \dots, A_n) \in \prod_{i=1}^n \mathcal{L}(\mathcal{X})$. In particular, we do not assume that the operators A_i commute. We give a brief survey of functional calculi over suitable classes of functions containing the polynomials. Initially, we consider the *Symmetric operator calculus* defined over the over the polynomials, before examining the *Weyl functional calculus* over smooth functions when A satisfies an appropriate growth condition. Finally we employ Clifford algebras as a means to obtain the *Monogenic functional calculus* which is consistent with both the Weyl and Symmetric operator calculus.

1 Symmetric operator calculus

Let $\mathcal{X}$ be a Banach space and $A = (A_1, \dots, A_n) \in \prod_{i=1}^n \mathcal{L}(\mathcal{X})$. At the heart of the Riesz-Dunford functional calculus sits the polynomials which are the simplest functions for which a functional calculus can be defined. In analogy, using simply the boundedness of A_i, we define a functional calculus on the space of polynomials $\mathscr{P}(\mathbb{R}^n)$.

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ be a multi-index with $|\alpha| = \sum_{i=1}^n \alpha_i$ and $P^\alpha(x_1, \dots, x_n) = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. It is not difficult to see that $\mathscr{P}(\mathbb{R}^n) = \operatorname{span}\left\{P^\alpha : \alpha \in \mathbb{Z}_+^n\right\}$.

The principal difficulty in forming a functional calculus is how the calculus should behave when A_j are noncommuting. This is most easily seen in the simplest case when $n = 1$ and $P(x_1, x_2) = x_1 x_2$. Then, the choices we have for $P(x_1, x_2)$ include A_1A_2, A_2A_1, $\frac{1}{2}(A_1A_2 + A_2A_1)$.

Let us first consider the commuting case, so assume that the A_i are commuting. Then, we have a unique way of defining the functional calculus. Define:

$$P^\alpha(A_1, \dots, A_n) = \prod_{i=1}^n A^{\alpha_i} = A_1^{\alpha_1} \cdots A_n^{\alpha_n}.$$

To consider the non-commuting case, we first consider decomposing a polynomial P^α in a symmetric sense. For a multi-index $\alpha \in \mathbb{Z}_+^n$, define

$$S_\alpha = \left\{\pi : \{1, \dots, |\alpha|\} \to \{1, \dots, n\} \text{ such that } \pi^{-1}(\{j\}) = \alpha_j,\ j \in \{1, \dots, n\}\right\}.$$

That is, every map $\pi \in S_\alpha$ attains the value j exactly α_j times. In this terminology, a calculation then shows that

$$P^\alpha(x_1, \dots, x_n) = \frac{\alpha_1! \cdots \alpha_n!}{|\alpha|!} \sum_{\pi \in S_\alpha} x_{\pi(1)} \dots x_{\pi(|\alpha|)}.$$

With this in mind, we relax the commuting condition on A. This leads to the following definition.

Definition 1.1 (Symmetric operator calculus)**.** *Define the Symmetric operator calculus:*

$$P^\alpha(A_1, \dots, A_n) = \frac{\alpha_1! \cdots \alpha_n!}{|\alpha|!} \sum_{\pi \in S_\alpha} A_{\pi(1)} \cdots A_{\pi(|\alpha|)}.$$

It is also immediate that this generalises the case of commuting A. That is, for A commuting, the following calculus agrees with the calculus we defined above for commuting operators.

As an example, let $n = 2$ and $A = (A_1, A_2)$. Then,

$$P^\alpha(A_1, A_2) = \frac{2!1!}{3!}(A_1^2 A_2 + A_1 A_2 A_1 + A_2 A_1^2) = \frac{1}{3}(A_1^2 A_2 + A_1 A_2 A_1 + A_2 A_1^2).$$

It is not hard to see that if A_1, A_2 commute, then $P^\alpha(A_1, A_2) = A_1^2 A_2$.

2 Weyl Functional Calculus

Let $T \in \mathcal{L}(\mathcal{X})$. To define the Riesz-Dunford functional calculus, we use the Cauchy integral formula

$$f(\zeta) = \frac{1}{2\pi\imath} \oint_\gamma f(\xi)(\xi - \zeta)^{-1}\, d\xi.$$

as a prototype where we can consider ζ to be a 1-matrix so that $(\xi - \zeta)^{-1}$ is the resolvent of ζ. In order to construct $f(T)$, we replace the resolvent of ζ with the resolvent of T given by $\mathsf{R}_T(\xi) = (\xi I - T)^{-1}$.

Now, let $A = (A_1, \dots, A_n) \in \prod_{i=1}^n \mathcal{L}(\mathcal{X})$. Let $\mathscr{S}(\mathbb{R}^n)$ be the space of rapidly decreasing functions. Whenever $f \in \mathscr{S}(\mathbb{R}^n)$, we have the *Fourier inversion formula* [Yos95, p146, Defn 2]:

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\imath\langle \xi, x\rangle} \hat{f}(\xi)\, d\xi$$

where $\hat{f}$ represents the *Fourier transform* of f. This is the prototypical formula we use to define the Weyl functional calculus. Formally, we substitute the variable $x \in \mathbb{R}^n$ by $A \in \prod_{i=1} \mathcal{L}(\mathcal{X})$. A few tools are required to make this rigorous. As in Fourier analysis, we will require some ideas from distribution theory as we obtain the Weyl functional calculus as a Banach valued distribution.

Let $\Omega \subset \mathbb{R}^n$ be an open set, $f \in C^\infty(\Omega)$ and $K \Subset \Omega$ with $m < \infty$. Define $\rho_{K,m}(f) = \sup_{|\alpha| \leq m, x \in K} |D^\alpha f(x)|$, where α is a multi-index. Then, the space $\mathscr{F}(\Omega)$ denotes $C^\infty(\Omega)$ as an abstract set with the locally convex topology generated by $\{\rho_{K,m}\}$. This is, in fact, a metric space [Yos95, p27, Prop 6]. In distribution theory, we are mainly concerned with $\Omega = \mathbb{R}^n$.

The locally convex topology on $\mathscr{F}(\mathbb{R}^n)$ relates closely to the topology of its dual $\mathscr{F}(\mathbb{R}^n)'$. Therefore, given $T \in \mathscr{F}(\mathbb{R}^n)'$, we define the support of T denoted by spt T as the smallest closed set $C \subset \mathbb{R}^n$ such that for every $f \in \mathscr{F}(\mathbb{R}^n)$ with spt $f \subset \mathbb{R}^n \setminus C$ we have $T(f) = 0$ [Yos95, p62]. Then $\mathscr{F}(\mathbb{R}^n)'$ is the space of *compactly supported distributions* [Yos95, p64, Thm 2] and $\mathscr{S}(\mathbb{R}^n)'$ is the space of *tempered distributions*. Also, $\mathscr{F}(\mathbb{R}^n)' \subset \mathscr{S}(\mathbb{R}^n)'$ [Yos95, p149]. Similarly, for any Banach space $\mathcal{B}$, we write $\mathcal{L}(\mathscr{F}(\mathbb{R}^n), \mathcal{B})$ and $\mathcal{L}(\mathscr{S}(\mathbb{R}^n), \mathcal{B})$ respectively to be the space of Banach valued compactly supported distributions and tempered distributions. The support of a Banach valued distribution is defined in the same way. We are particularly interested in the case when $\mathcal{B} = \mathcal{L}(\mathcal{X})$.

The classical Paley-Wiener theorem for functions gives a characterising condition for when a function is compactly supported in terms of a decay condition on its Fourier-Laplace transform. The Paley-Wiener theorem for tempered distributions then gives a characterising condition to determine when a tempered distribution is compactly supported. That is, when a tempered distribution satisfies the characterising condition, we are able to uniquely extend its domain from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{F}(\mathbb{R}^n)$. We give the statement of Paley-Wiener theorem for Banach valued distributions as in [Jef04, Prop 2.1].

Theorem 2.1 (Banach valued Paley-Wiener)**.** *Let $\mathcal{B}$ be a Banach space and let $T \in \mathcal{L}(\mathscr{S}(\mathbb{R}^n), \mathcal{B}))$ be a tempered distribution. Then, the following statements are equivalent:*

(i) There exists an $r \geq 0$ such that spt $T \subset B_r(0)$.

(ii) There exists an $f : \mathbb{C}^n \to \mathcal{B}$ such that $\hat{T} = f$ and there exist constants $C \geq 0, s \geq 0$ such that

$$\|f(\zeta)\|_{\mathcal{B}} \leq C(1 + |\zeta|)^s e^{r|\text{Im } \zeta|}$$

for all $\zeta \in \mathbb{C}^n$.

Note here that Im $\zeta = (\text{Im } \zeta_1, \ldots, \text{Im } \zeta_n)$ and $\hat{T}$ is actually the *Fourier-Laplace* transform [Yos95, p161].

The idea is to define the functional calculus as a tempered distribution (so that we have a calculus of Schwartz functions) and then apply Theorem 2.1 to get an extension to $\mathscr{F}(\mathbb{R}^n)$. The following growth estimate is essential.

Definition 2.2 (Paley-Wiener type (s, r) and type s)**.** *Let $A = (A_1, \ldots, A_n) \in \prod_{i=1}^n \mathcal{L}(\mathcal{X})$. We say A is of* Paley-Wiener type (s, r) *if*

$$\|e^{-\imath\langle \xi, A\rangle}\|_{\mathcal{L}(\mathcal{X})} \leq C(1 + |\xi|)^s e^{r|\text{Im } \xi|}$$

for all $\xi \in \mathbb{C}^n$. We say that A is of Paley-Wiener type s *when* Im $\xi = 0$ *or equivalently $\xi \in \mathbb{R}^n$.*

Here, $\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \prod_{i=1}^n \mathcal{L}(\mathcal{X}) \to \mathcal{L}(\mathcal{X})$ is defined as $\langle \xi, A \rangle = \sum_{j=1}^n \xi_j A_j$ whenever $\xi \in \mathbb{C}^n$. We define $e^{-\imath \sum_{j=1}^n \xi_j A_j}$ via the Riesz-Dunford functional calculus. Combining these two facts, we can give meaning to the operator $e^{-\imath\langle \xi, A\rangle}$ appearing in the definition.

There are important tuples of operators that are of Paley-Wiener type (s, r). Suppose that $\mathcal{X} = \mathcal{H}$ a Hilbert space and that each A_j is self adjoint. Then $\langle \xi, A\rangle$ is self adjoint for all $\xi \in \mathbb{R}^n$. Setting $r = (\sum_{j=1}^n \|A_j\|^2)^{\frac{1}{2}}$, we find that

$$\|e^{\imath\langle \xi, A\rangle}\| \leq e^{r|\text{Im } \xi|}.$$

Note that this in particular implies that each A_j is of Paley-Wiener type (s, r). However, it is not enough to assume that each A_j is of Paley-Wiener type (s, r) to conclude that A is also of Paley-Wiener type (s', r') (for $s', r' \geq 0$). This and other examples can be found in [Jef04, p18].

The lack of an order structure in a general Banach space means that we do not have the luxury of a supremum to define a Banach valued integral. Fortunately, we can use sequences. Let $(\Sigma, \mathscr{M}, \mu)$ be a measure space. Then, we say that a function $f : \Sigma \to \mathcal{X}$ is *Bochner μ-integrable* if there exists a sequence $s_n : \Sigma \to \mathcal{X}$, of μ-integrable simple functions such that $s_n \to f$ μ-a.e. and $\int_\Sigma \|s_n - s_m\| \, d\mu \to 0$ as $m, n \to \infty$. Then, we define the *Bochner integral* of f to be

$$\int_\Sigma f \, d\mu = \lim_{n\to\infty} \int_\Sigma s_n \, d\mu.$$

The limit is independent of the sequence s_n. A detailed treatment of the Bochner integral can be found in [Yos95, V, §5].

We now have sufficient tools to develop the Weyl functional calculus. The main theorem of this section is taken from [Jef04, p19]. Further properties of the Weyl functional calculus can be found in [And69].

Theorem 2.3 (Weyl functional calculus). *Let $\mathcal{X}$ be a Banach space and let $A = (A_1, \ldots, A_n) \in \prod_{i=1}^n \mathcal{L}(\mathcal{X})$. Suppose there exists $r, s \geq 0$ such that A is of Paley-Wiener type (s, r). Then there exists a unique compactly supported distribution $\mathcal{W}_A \in \mathcal{L}(\mathscr{F}(\mathbb{R}^n), \mathcal{L}(\mathcal{X}))$ which agrees with the Symmetric operator calculus for polynomials. The distribution is given by*

$$\mathcal{W}_A(f) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\imath\langle \xi, A\rangle} \hat{f}(\xi)\ d\xi$$

for every $f \in \mathscr{F}(\mathbb{R}^n)$. This integral converges as a Bochner integral in $\mathcal{L}(\mathcal{X})$, and $\operatorname{spt} \mathcal{W}_A \subset B_r(0)$. *Also, A is of Paley-Wiener type $(s, \sup|K|)$.*

In particular, we can apply the Weyl functional calculus in the case that $\mathcal{X} = \mathcal{H}$ a Hilbert space and each operator A_j is self adjoint [Jef04, Ex 2.3].

We should expect a notion of spectrum for A. In fact, there are many ways to define the *joint spectrum* of A. However, we use the following definition.

Definition 2.4 (Joint spectrum and radius). *Define the joint spectrum $\gamma(A) = \operatorname{spt} \mathcal{W}_A$. The joint spectral radius is then $r(A) = \sup|\gamma(A)|$.*

This is indeed a sensible definition of the joint spectrum. Suppose that $n = 1$. Then, we find that $\gamma(A) = \sigma(A)$. Furthermore, we have a spectral mapping theorem, $\sigma(\mathcal{W}_A(f)) = f(\sigma(A))$.

We have that whenever $T \in \mathcal{L}(\mathcal{X})$, the spectrum $\sigma(T) \subset B_{\|T\|}$. A generalisation of this result when A is Paley-Wiener type s is captured more precisely in the following theorem [Jef04, Thm 2.7].

Theorem 2.5. *Suppose A is of Paley-Wiener type s. Then A is of type (s, r) with $r = (\sum_{j=1}^n \|A_j\|^2)^{\frac{1}{2}}$, and*

$$\gamma(A) \subset \prod_{j=1}^n [-\|A_j\|, \|A_j\|] \subset \mathbb{R}^n.$$

3 Clifford Analysis

To look beyond the Weyl functional calculus to one that resembles the Riesz-Dunford calculus, we need a suitable algebra which generalises the Complex numbers, the notion of holomorphy, and Cauchy's integral theorem.

We let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$. We construct a *Clifford Algebra* over $\mathbb{F}$ as follows. Let $\{e_0, e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{R} \times \mathbb{R}^n$. We define multiplication of these basis elements in the following way:

$$\begin{aligned} e_0 &= 1 & & \\ e_j^2 &= -e_0 & & 1 \leq j \leq n \\ e_i e_j &= -e_j e_i & & 1 \leq i < j \leq n \end{aligned}$$

Whenever $1 \leq j_1 < j_2 < \cdots < j_k \leq n$, write $S = \{j_1, j_2, \ldots, j_k\}$ and define $e_S = e_{j_1} e_{j_2} \ldots e_{j_k}$. For $S = \varnothing$, define $e_\varnothing = e_0$. The Clifford algebra $\mathbb{F}_{(n)}$ is defined as

$$\mathbb{F}_{(n)} = \mathbb{F}\text{-}\operatorname{span}\{e_S : S \subset \{1, \ldots, n\}\}$$

which makes it a 2^n-dimensional algebra. If $u, v \in \mathbb{F}_{(n)}$, then $u = \sum_S u_S e_S$ and $v = \sum_R u_R e_R$ with $u_S, v_R \in \mathbb{F}$ and $uv = \sum_{S,R} u_S v_R e_S e_R$. The term u_0 is the *scalar* part of u.

We equip the algebra with an involution. The *Clifford conjugate* for a basis element e_S is the element $\overline{e}_S$ satisfying $\overline{e}_S e_S = 1 = e_S \overline{e}_S$. Observe then that $\overline{e}_S = \pm e_S$, with the sign chosen appropriately. Then, the Clifford conjugate of a general $u \in \mathbb{F}_{(n)}$ is defined as $\overline{u} = \sum_S \overline{u}_S \overline{e}_S$. A calculation reveals that $\overline{uv} = \overline{v}\,\overline{u}$. Also, note that $u\overline{v} = \sum_S u_S \overline{v}_S + \sum_{S \neq R} u_S \overline{v}_R e_S \overline{e}_R$. From this observation we define an inner product $\langle u, v \rangle = (u\overline{v})_0 = \sum_S u_S \overline{v}_S$. We write $|\cdot| = |\cdot|_2$ for the associated norm.

We can embed $\mathbb{R}^{n+1}$ in $\mathbb{F}_{(n)}$ by identifying it with the subspace $\mathbb{R}\text{-}\operatorname{span}\{e_0, \ldots, e_n\}$ via the map $(x_0, \ldots, x_n) \mapsto \sum_{j=0}^n x_j e_j$. Then, whenever $m \leq n$, we can consider $\mathbb{R}^m \subset \mathbb{R}^{n+1}$ by identifying $\mathbb{R}^m$ with the subspace $\operatorname{span}\{e_1, \ldots, e_m\}$ and so via transitivity, we can embed $\mathbb{R}^m$ in $\mathbb{F}_{(n)}$.

Not every element of $\mathbb{F}_{(n)}$ is invertible [MP87]. However, if $x \in \mathbb{R}^{n+1}$, then it does have an inverse. An easy calculation shows that $\overline{e}_j = -e_j$ whenever $1 \leq j \leq n$. Given an element $x \in \mathbb{R}^{n+1}$ the conjugate $\overline{x} = x_0 - \sum_{j=1}^n x_j e_j$. The *Kelvin inverse* of x is then given by

$$x^{-1} = \frac{\overline{x}}{|x|^2} = \frac{x_0 - \sum_{j=1}^n x_j e_j}{\sum_{j=0}^n x_j^2}.$$

The Clifford algebra $\mathbb{R}_{(1)}$ can be identified with the Complex numbers. This can be seen by identifying $1 \mapsto 1$ and $\imath \mapsto e_1$. Similarly, the algebra $\mathbb{R}_{(2)}$ can be identified with the *Quaternions*.

A *Banach module* over $\mathbb{F}_{(n)}$ is a Banach space $\mathcal{X}$ over $\mathbb{F}$ with an operation of multiplication by elements of $\mathbb{F}_{(n)}$ with a $\kappa \geq 1$ such that

$$\|xu\| \leq \kappa |u| \|x\| \qquad \text{and} \qquad \|ux\| \leq \kappa |u| \|x\|$$

for all $x \in \mathcal{X}$ and $u \in \mathbb{F}_{(n)}$. In some sense, a Banach module is a generalisation of the concept of a space over a field, since in the module we are able to multiply by elements of $\mathbb{F}_{(n)}$.

Suppose $\mathcal{X}$ and $\mathcal{Y}$ are Banach modules over $\mathbb{F}_{(n)}$. We say that $A : \mathcal{X} \to \mathcal{Y}$ is a *right module homomorphism* if $(Ax)u = A(xu)$ for all $x \in \mathcal{X}$ and $u \in \mathbb{F}_{(n)}$. Certainly, we also have a notion of *left module homomorphisms.* Namely, it is a map $B : \mathcal{X} \to \mathcal{Y}$ satisfying $(ux)B = u(xB)$. That this situation arises can be seen by considering the operator $A = \sum_{j=0}^n A_j e_j$, with $A_j \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Then, in general, we do not expect $xA = Ax$. The space of continuous right homomorphisms is denoted $\mathcal{L}_{(n)}(\mathcal{X}, \mathcal{Y})$ and it is considered as a Banach space with the uniform operator topology.

Let $\mathcal{X}_{(n)} = \mathcal{X} \otimes \mathbb{F}_{(n)}$. Every $\xi \in \mathcal{X}_{(n)}$ arises in the form $\xi = \sum_S x_S \otimes e_S$, where $x_S \in \mathcal{X}$. For convenience, we omit the $\otimes$ and simply write $\xi = \sum_S x_S e_S$. Multiplication by $u \in \mathbb{F}_{(n)}$ is defined by $\xi u = \sum_{S,R} u_R x_S e_S e_R$ and $u\xi = \sum_{S,R} u_R x_S e_R e_S$. We equip $\xi \in \mathcal{X}_{(n)}$ with the norm $\|\xi\| = (\sum_S \|x_S\|^2)^{\frac{1}{2}}$. It follows then that $\mathcal{X}_{(n)}$ is a Banach module with $\kappa = 1$. We highlight a trivial fact. Since $\mathbb{F}$ is a Banach space over $\mathbb{F}$, the module $\mathbb{F} \otimes \mathbb{F}_{(n)} \cong \mathbb{F}_{(n)}$ is in fact a Banach module over $\mathbb{F}_{(n)}$. This observation shows our choice of notation for a Clifford algebra is consistent with that of a Banach module. Furthermore, the space $(\mathcal{L}(\mathcal{X}, \mathcal{Y}))_{(n)}$ can be identified with $\mathcal{L}_{(n)}(\mathcal{X}_{(n)}, \mathcal{Y}_{(n)})$ [Jef04, §3.2].

Now we can begin to consider some analytic properties of Clifford valued functions. Our approach is taken from [MP87] and [Jef04]. A more measure theoretic approach can be found in [Mit94, §1.2].

Given $\Omega \subset \mathbb{R}^{n+1}$ an open set, any function $f : \Omega \to \mathbb{F}_{(n)}$ can be written as $f = \sum_S f_S e_S$. Often, we regard f as a function $f : \Omega \subset \mathbb{F}_{(n)} \to \mathbb{F}_{(n)}$ via the canonical embedding. We say $f \in C^\infty(\Omega, \mathbb{F}_{(n)})$ if $f_S \in C^\infty(\Omega)$ for every S. For such a function, letting ∂_j denote the partial derivative in the direction j, we define

$$D = \partial_0 e_0 + \sum_{j=1}^{n} \partial_j e_j.$$

Then, $Df = \sum_S (\partial_0 f_S e_S + \sum_{j=1}^n \partial_j f_S e_j e_S)$ and $fD = \sum_S (\partial_0 f_S e_S + \sum_{j=1}^n \partial_j f_S e_S e_j)$. Such a function is said to be *left monogenic* if $Df = 0$ and *right monogenic* if $fD = 0$ on Ω. In the case of $\mathbb{R}_{(1)} \cong \mathbb{C}$, we find that $Df = fD$ and $Df = 0$ are exactly the holomorphic functions. The notion of monogenic generalises holomorphy.

In a first step towards a generalisation of Cauchy's integral theorem, we have the following important result [FB82, §8.8].

Theorem 3.1 (Existence and uniqueness of fundamental solution)**.** *There exists a unique left and right fundamental solution $E \in L^1_{loc}(\mathbb{R}^{n+1}, \mathbb{F}_{(n)})$ in the sense of distributions $\mathscr{S}(\mathbb{R}^{n+1})_{(n)}$ to the equation $DE = ED = \delta_0 e_0$. When $x \neq 0$, the fundamental solution is given by*

$$E(x) = \frac{1}{\omega_{n+1}} \frac{\overline{x}}{|x|^{n+1}}$$

where $\frac{1}{\omega_{n+1}} = \frac{1}{2}\pi^{-\frac{n+1}{2}}\Gamma(\frac{n+1}{2})$, the area of the unit sphere S^n. Furthermore, E is both left and right monogenic on $\mathbb{R}^{n+1} \setminus \{0\}$.

A detailed treatment of distributions taking values in Banach modules is found in [FB82, §2]. From the fundamental solution, we can construct a generalised *Cauchy kernel*. Analogous to the case of complex variables, we write the Cauchy kernel as $G_x(\omega) = E(\omega - x)$ for all $\omega \neq x$ and the following theorem shows that it is indeed a generalisation of the classical Cauchy kernel.

Theorem 3.2 (Monogenic Cauchy's integral theorem)**.** *Suppose that $\Omega \subset \mathbb{R}^{n+1} \subset \mathbb{F}_{(n)}$ is a bounded open set with a smooth boundary $\partial\Omega$, and $\nu : \mathbb{R}^{n+1} \subset \mathbb{F}_{(n)} \to \mathbb{R}^{n+1} \subset \mathbb{F}_{(n)}$ is the unit outer normal to $\partial\Omega$. Furthermore, let μ denote the surface measure on $\partial\Omega$. Suppose $f, g : \Omega' \to \mathbb{F}_{(n)}$ where $\overline{\Omega} \subset \Omega'$ and Ω' are open sets. If f is left monogenic, and g is right monogenic. Then the following hold:*

1. *$\int_{\partial\Omega} G_x(\omega)\nu(\omega)f(\omega)\ d\mu(\omega) = f(x)$ when $x \in \Omega$ and 0 otherwise,*
2. *$\int_{\partial\Omega} g(\omega)\nu(\omega)G_x(\omega)\ d\mu(\omega) = g(x)$ when $x \in \Omega$ and 0 otherwise,*
3. *$\int_{\partial\Omega} g(\omega)\nu(\omega)f(\omega)\ d\mu(\omega) = 0.$*

A proof of the left monogenic case involving Stokes' theorem and differential forms can be found in [FB82, §9]. The right monogenic case can be proved similarly. A slightly different proof for both cases can be found in [Mit94, §1.2].

There are also analogues of the important classical theorems in complex analysis such as Liouville's theorem, and Morera's theorem. Details can be found in [MP87] and [FB82, Ch.2].

The best example for the theory is taken when $n = 1$. Identifying $\mathbb{R}_{(1)}$ with $\mathbb{C}$, we find that $D = \partial_x + \imath\partial_y$, and the Cauchy kernel becomes $G_x(\omega) = \frac{1}{2\pi}\frac{\overline{\omega - x}}{|\omega-x|^2} = \frac{1}{2\pi}\frac{1}{\omega-x}$. A straightforward calculation then reveals that $\int_{\partial\Omega} G_x(\omega)\nu(\omega)f(\omega)\,d\mu(\omega) = \frac{1}{\imath}\int_\gamma G_x(z)f(z)\,dz$, which is the classical Cauchy integral theorem.

Monogeniety, like holomorphy, has a vector valued analogue. Let $\mathcal{B}$ be a Banach space. Consider $f : \Omega \subset \mathbb{R}^{n+1} \subset \mathbb{F}_{(n)} \to \mathcal{B}_{(n)}$. Recall that *weak holomorphy* is defined via the dual $\mathcal{B}' = \mathcal{L}(\mathcal{B}, \mathbb{C})$ [Yos95, Ch.5,§3]. For a notion of *weak monogenic* we can consider $\mathcal{L}_{(n)}(\mathcal{B}_{(n)}, \mathbb{F}_{(n)}) \cong \mathcal{L}(\mathcal{B}, \mathbb{F})_{(n)} \cong \mathcal{B}'_{(n)}$. Then, the notion of *strong monogenic* corresponds to limits in the original topology. As in the holomorphic case, we have an equivalence of monogeniety in the weak and strong sense [Jef04, Prop 3.2]. In particular, Theorem 3.2 holds in the vector valued case. These facts are still valid if we replace the Banach space $\mathcal{X}$ with $\mathcal{E}$, a locally convex linear space over $\mathbb{F}$ [Jef04, §3.4]. Then, $\mathcal{E}_{(n)}$ is then called a *locally convex module*. However, of particular importance is the case when $\mathcal{X}$ is a Banach space and $\mathcal{B} = \mathcal{L}(\mathcal{X})$.

It will be useful to extend analytic functions on neighbourhoods of $\mathbb{R}^n$ to monogenic functions on an appropriate neighbourhood of $\mathbb{R}^{n+1}$. Without loss of generality (that is, by translation), assume that $\Omega \subset \mathbb{R}^n$ such that $0 \in \Omega$. Let $f : \Omega \to \mathbb{C}$ be analytic. So, we have the following Taylor expansion of f at 0:

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{l_1=1}^{n} \cdots \sum_{l_k=1}^{n} a_{l_1 \ldots l_j} x_{l_1} \cdots x_{l_j}.$$

For the function $\pi_j : \mathbb{R}^n \to \mathbb{R}$ given by $\pi_j(x) = x_j$, the *monogenic extension* is given by the function $z_j : \mathbb{R}^{n+1} \to \mathbb{F}_{(n)}$ defined by $z_j(x) = x_j e_0 - x_0 e_j$. For each multi-index $l_1 \ldots l_k$, define $V^{l_1 \ldots l_k} : \mathbb{R}^{n+1} \to \mathbb{F}_{(n)}$ by

$$V^{l_1 \ldots l_k}(x) = \frac{1}{k!} \sum_{(j_1, \ldots, j_k) \in \sigma(l_1, \ldots, l_k)} z_{j_1} \cdots z_{j_k}$$

and $\sigma(1_1, \ldots, l_k)$ represent distinguishable permutations of $l_1, \ldots, l_k$. Also, $V^{l_1 \ldots l_k} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \subset \mathbb{F}_{(n)}$ [Jef04, §3.5]. Then, there exists a neighbourhood $\Omega' \subset \mathbb{R}^{n+1}$ on which the following series is convergent and we define:

$$\tilde{f}(x) = \sum_{k=0}^{\infty} \sum_{(l_1, \ldots, l_k)} a_{l_1 \ldots l_k} V^{l_1 \ldots l_k}(x).$$

The function $\tilde{f}$ is called the monogenic extension of f. It takes values in $\mathbb{R}^{n+1} \subset \mathbb{F}_{(n)}$.

There are some natural function spaces associated with this machinery. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set. Then the space $\mathcal{M}_n(\Omega) = \mathcal{M}(\Omega, \mathbb{F}_{(n)})$ consists of left monogenic functions $f : \Omega \to \mathbb{F}_{(n)}$ and it is equipped with the compact-open topology. For a closed $C \subset \mathbb{R}^{n+1}$ the space $\mathcal{M}_n(C) = \mathcal{M}(C, \mathbb{F}_{(n)})$ is defined as the strict inductive limit of $\mathcal{M}_n(\Omega)$ such that $C \subset \Omega$ and Ω is open. More details can be found in [Jef04, §3.8] and [FB82, Ch3].

4 Monogenic Functional Calculus

The key step in defining the Monogenic functional calculus is to define the operator valued Cauchy kernel $G_\omega(A)$. This can be done via an extended Weyl functional calculus and alternatively, via a monogenic expansion. These two approaches are consistent. We proceed as in [Jef04, §4].

Recall that the Weyl functional calculus defined in §2 is a compactly supported distribution and so it acts on functions defined on all of $\mathbb{R}^n$ when A is of Paley-Wiener type (s,r). We first extend $\mathcal{W}_A$ to act on functions defined on arbitrary open sets containing the support of the distribution. Let $\Omega \subset \mathbb{R}^n$ be open such that spt $\mathcal{W}_A \subset \Omega$ and suppose $f \in \mathscr{F}(\Omega)$. Let $\tilde{f}$ be any extension of f to the whole of $\mathbb{R}^n$. Then the extension $\mathcal{W}_A(f) = \mathcal{W}_A(\tilde{f})$ is a well defined map since any compactly supported distribution vanishes outside its support [Yos95, Ch.1, §13].

Now, we extend $\mathcal{W}_A$ to a map $\mathscr{F}(\Omega)_{(n)} \to \mathcal{L}(\mathcal{X})_{(n)}$ by embedding $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$ where $\Omega \subset \mathbb{R}^{n+1}$ is an open set and spt $\mathcal{W}_A \subset \Omega$. So, identify $\mathbb{R}^n$ as the space $\{x \in \mathbb{R}^{n+1} : x_0 = 0\}$. Then, spt $\mathcal{W}_A \subset \Omega \cap \mathbb{R}^n$ and $\Omega \cap \mathbb{R}^n$ is open in $\mathbb{R}^n$. If $f \in \mathscr{F}(\Omega)_{(n)}$ then $f = \sum_S f_S e_S$ and we define $\mathcal{W}_A(f) = \sum_S \mathcal{W}_A(f_S|_{\Omega\cap\mathbb{R}^n})e_S$. We have the following important observation for analytic f [Jef04, §4.1].

Theorem 4.1. *Let f be an analytic function in an open set Ω in $\mathbb{R}^n$ such that* spt $\mathcal{W}_A \subset \Omega$*, and let $\tilde{f} : \tilde{\Omega} \to \mathbb{F}_{(n)}$ be the monogenic extension. Then, $\mathcal{W}_A(\tilde{f}) = \mathcal{W}_A(f)e_0$.*

As a consequence, we can consider $\mathcal{W}_A(\tilde{f}) \in \mathcal{L}(\mathcal{X})$ whenever f is analytic.

For, any $\omega \in \mathbb{R}^{n+1} \setminus$ spt $\mathcal{W}_A$, there exists an open set $\Omega_\omega \subset \mathbb{R}^{n+1}$ such that spt $\mathcal{W}_A \subset \Omega_\omega$ and $\omega \notin \Omega_\omega$ and $G_\omega \in \mathscr{F}(\Omega_\omega)_{(n)}$. Assuming that A is of type Paley-Wiener s, we define $G_\omega(A) = \mathcal{W}_A(G_\omega)$. Then, by [Jef04, Cor 4.3], the map $\omega \mapsto G_\omega(A)$ is both left and right monogenic on $\mathbb{R}^{n+1} \setminus$ spt $\mathcal{W}_A$. Since spt $\mathcal{W}_A = \gamma(A)$, the set of singularities of $\omega \to G_\omega(A)$ is exactly $\gamma(A)$. Also, we can apply the vector valued version of Theorem 3.2 to justify the following definition.

Definition 4.2 (Monogenic functional calculus for Paley-Winer type s operators)**.** *Let A be of Paley-Wiener type s, and let $G_\omega(A) = \mathcal{W}_A(G_\omega)$. Suppose $\Omega, \Omega' \subset \mathbb{R}^{n+1}$ be open sets such that $\gamma(A) \subset \overline{\Omega} \subset \Omega'$ with Ω bounded. Also, suppose $\partial\Omega$ is smooth with unit outer normal ν and surface measure μ. If $f : \Omega \to \mathbb{F}_{(n)}$ is left monogenic and $g : \Omega \to \mathbb{F}_{(n)}$ is right monogenic define*

$$f(A) = \int_{\partial\Omega} G_\omega(A)\nu(\omega)f(\omega)\ d\mu(\omega) \qquad \text{and} \qquad g(A) = \int_{\partial\Omega} g(\omega)\nu(\omega)G_\omega(A)\ d\mu(\omega)$$

We find that $\mathcal{W}_A(f) = f(A)$ and $\mathcal{W}_A(g) = g(A)$ [Jef04, Cor 4.5]. Therefore, for A that is of Paley-Wiener s, we have only rephrased the Weyl functional calculus in another language.

We now abandon the Paley-Wiener type s condition on A. For suitably large ω, we define $G_\omega(A)$ in terms of a power series which is motivated by the $n = 1$ case where we have $\mathsf{R}_{A_j}(\zeta) = \sum_{k=0}^{\infty} A_j^i \zeta^{-k-1}$ for $|\zeta| > \|A_j\|$. Let $R_A = (1+\sqrt{2})\|\sum_{j=1}^{n} A_j e_j\|$ and whenever $\omega \in \mathbb{R}^{n+1}$, let $a_{l_1\dots l_k} = (-1)^k(\partial_{l_1}\cdots\partial_{l_k}G_\omega)(0)$. Also, define

$$V^{l_1\dots l_k}(A) = \frac{1}{k!}\sum_{(j_1,\dots,j_k)\in\sigma(l_1,\dots,l_k)} A_{j_1}\cdots A_{j_k}$$

and as before, $\sigma(1_1,\dots,l_k)$ represent distinguishable permutations of $l_1,\dots,l_k$. Then, for $|\omega| > R_A$, we define the Cauchy kernel

$$G_\omega(A) = \sum_{k=0}^{\infty}\sum_{(l_1,\dots,l_k)} a_{l_1\dots l_k}V^{l_1\dots l_k}(A)$$

since this series is absolutely convergent for $|\omega| > R_A$ by [Jef04, Lem 4.7]. This definition is consistent by the following theorem [Jef04, §4.2].

Theorem 4.3. *Suppose A is of Paley-Wiener type s and $|\omega| > R_A$. Then $G_\omega(A) = \mathcal{W}_A(G_\omega)$.*

Let $\mathcal{A}_{R_A} = \{\omega \in \mathbb{R}^{n+1} : |\omega| > R_A\}$. Now, again assume that A is arbitrary, and let $\Omega \subset \mathbb{R}^{n+1}$ be the largest open set which is the domain of a function $\omega \mapsto \tilde{G}_\omega(A) : \Omega \to \mathcal{L}(\mathcal{X})_{(n)}$ such that $[\omega \mapsto G_\omega(A)] = [\omega \mapsto \tilde{G}_\omega(A)|_{\mathcal{A}_{R_A}}]$. That this is sensible is a consequence of the following theorem [Jef04, Thm 4.8].

Theorem 4.4. *If A is of Paley-Wiener type s, then* spt $\mathcal{W}_A = \gamma(A) = \mathbb{R}^{n+1} \setminus \Omega$.

However, it is not possible to use $\omega \mapsto \tilde{G}_\omega(A)$ to define the monogenic functional calculus since this extension may not be unique.

We introduce the crucial sufficient condition that we will assume from now on: suppose A satisfies $\sigma(\langle A, \xi\rangle) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^n$. Then, there exists a monogenic extension $\omega \mapsto \tilde{G}_\omega(A) : \Omega \to \mathcal{L}(\mathcal{X})_{(n)}$ of the function $\omega \mapsto G_\omega(A)$ where $\Omega = (\mathbb{R}^{n+1} \setminus \mathbb{R}^n) \cup \mathcal{A}_{R_A}$ [Jef04, Thm 4.12]. By taking a union of all monogenic extensions with open domains Ω such that $(\mathbb{R}^{n+1} \setminus \mathbb{R}^n) \cup \mathcal{A}_{R_A} \subset \Omega$, we have the following important theorem.

Theorem 4.5 (Existence of a unique maximal monogenic extension)**.** *Suppose A satisfies the condition*

$$\sigma(\langle A, \xi\rangle) \subset \mathbb{R}, \qquad \text{for all } \xi \in \mathbb{R}^n.$$

Then, there exists a largest open set Ω such that $(\mathbb{R}^{n+1} \setminus \mathbb{R}^n) \cup \mathcal{A}_{R_A} \subset \Omega$ and and a unique left and right maximal monogenic extension $\omega \mapsto G_\omega(A) : \Omega \to \mathcal{L}(\mathcal{X})_{(n)}$.

Then, the monogenic spectrum $\gamma(A)$ is defined to be the complement of the domain of the maximal monogenic extension. Since $\gamma(A) \subset \overline{B_{R_A}}$, we have the following theorem analogous to the $n = 1$ case [Jef04, Thm 4.16].

Theorem 4.6. *The monogenic spectrum $\gamma(A)$ is a nonempty compact subset of $\mathbb{R}^n$.*

Suppose A satisfies $\sigma(\langle A, \xi\rangle) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^n$. Let $\Omega \subset \Omega' \subset \mathbb{R}^{n+1}$ be open sets such that $\gamma(A) \subset \Omega$, with $\overline{\Omega} \subset \Omega'$ and Ω is bounded with smooth boundary $\partial\Omega$ equipped with a unit outer normal ν and surface measure μ. If $f : \Omega' \to \mathbb{R}^{n+1}$ is left monogenic, then

$$f(A) = \int_{\partial\Omega} G_\omega(A)\nu(\omega)f(\omega)\ d\mu(\omega).$$

As before, we can appeal to the vector valued version of Theorem 3.2 to find that integral exists and so the definition makes sense. Indeed, for any $f \in \mathcal{M}_n(\gamma(A))$, there exists Ω, Ω' as above and this leads to the following definition.

Definition 4.7 (Monogenic functional calculus)**.** *The map $f \mapsto f(A)$ whenever $f \in \mathcal{M}_n(\gamma(A))$ is called the* monogenic functional calculus.

The Riesz-Dunford functional calculus is continous, is consistent with the polynomial functional calculus, and maps the Cauchy kernel to resolvents. A corresponding statement for the monogenic functional calculus is the following, which is a combination of [Jef04, Prop 4.19], [Jef04, Prop 4.20] and [Jef04, Thm 4.22].

Theorem 4.8. *Suppose A satisfies $\sigma(\langle A, \xi\rangle) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^n$.*

1. *The Monogenic functional calculus is a continuous map $f \mapsto f(A)$ for $f \in \mathcal{M}_n(\gamma(A))$.*

2. *If $P \in \mathscr{P}(\mathbb{R}^n)$, then the Monogenic functional calculus agrees with the Symmetric operator calculus.*

3. *Suppose $\Omega \subset \mathbb{R}^{n+1}$ is an open set, $\gamma(A) \subset \Omega$, with smooth boundary $\partial\Omega$ together with a unit outer normal ν and surface measure μ. Then, whenever $\omega \in \mathbb{R}^{n+1} \setminus \overline{\Omega}$,*

$$G_\omega(A) = \int_{\partial\Omega} G_\zeta(A)\nu(\zeta)G_\omega(\zeta)\ d\mu(\zeta).$$

4. *If $\Omega \subset \mathbb{R}^n$ is an open set such that $\gamma(A) \subset \Omega$ and $f : \Omega \to \mathbb{C}$ is a real analytic function, then $f(A) \in \mathcal{L}(\mathcal{X})$.*

5. *If A is also Paley-Wiener type s then the Monogenic functional calculus agrees with the Weyl functional calculus for $f \in \mathcal{M}_n(\gamma(A))$.*

There are many analogous properties that the Monogenic functional calculus shares with both the Riesz-Dunford calculus and the Weyl functional calculus. The Monogenic functional calculus also enjoys some spectral mapping properties. Furthermore, it is possible to perform spectral decompositions analogous to the case of the Riesz-Dunford calculus. A detailed treatment of these facts can be found in [Jef04, §4.3] and [Jef04, §4.4].

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