Boundary Value Problems and Index Theory

Lashi Bandara

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Preface

This book was written as a consequence in developing a graduate level course in the topic during the Summer term of 2021 at the University of Potsdam, Germany. I am thankful for the feedback from participants which helped improve the quality of this document.

I am very grateful to Peter Grabs for helping me write these notes. Moreover, I very much appreciate Penelope Gehring and Peter Grabs for carefully looking through this text and offering feedback to improve its quality and legibility.

London, 25 January 2022

Lashi Bandara

1 Motivation

Any honest attempt to navigate and understand the world, both in the large and in the small, is met with fierce resistance by the presence of *boundary value problems*. As they lived out their lives, even the earliest humans could not avoid cliffs, shores, and other interfaces. All of these are examples of boundaries. The modern world demands that we rigorously understand natural phenomena through their description by *partial differential equations*. Almost always, the setting is a bounded region within a geometric structure.

The aims of this book is to give a modern mathematical treatment of the linear and elliptic aspects of this subject. Although relevant to the study of physical phenomena, the narration of this topic will be motivated by perspectives emerging from global analysis. The setting is elliptic differential operators acting on Hermitian vector bundles over measured manifolds. A description of the maximal domain of such an operator will be given in terms of a canonical operator built out of the boundary trace map. Rather than focus on particular boundary conditions, all possible boundary conditions described from a fundamental space on the boundary called the *Czech space*.

Although we commence with a description of general-order operators, we will focus our attention on first-order elliptic differential operators. An important objective of this book is to exposit the recent developments due to Bär-Bandara in this setting. The ability to characterise the toplogy of the Czech space via spectral projectors associated to an adapted operator on the boundary lies at the heart of the theory. Moreover, a graphical characterisation of elliptic boundary conditions will be demonstrated. As an application, a quick and conceptual proof of the relative index theorem of Gromov-Lawson will be given.

The book will ultimately focus on Dirac-type operators. It will culminate with the proof of the famed index theorem of Atiyah-Patodi-Singer. Applications of this theorem, including consequences to the study of positive scalar curvature metrics, will also be considered.

2 Preliminaries

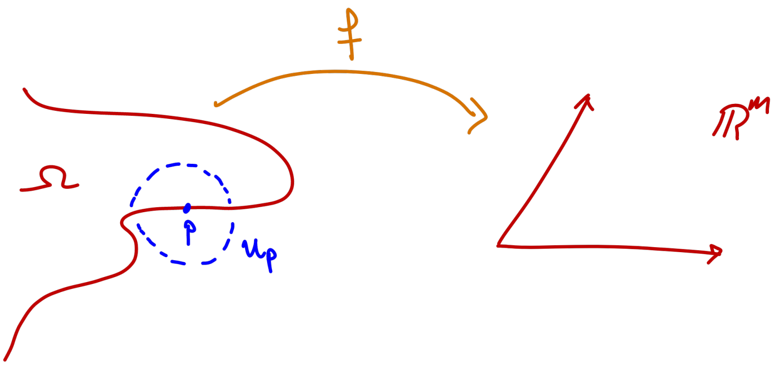
2.1 Manifolds with boundary

Manifolds with boundary are the core objects which will lie at the heart of our considerations. The purpose of this section is simply to define and recall the necessary features of these objects. For a more in depth description, see [36] by Lee or [41] by Munkres.

In time, we will need to study differential operators on manifolds with boundary. Therefore, we require their differentiable structures to be 'smooth' in an appropriate sense. This is the content of the following definition.

Definition 2.1 (Generalised smoothness). Consider an arbitrary subset $\Omega \subset \mathbb{R}^n$. Then $f : \Omega \to \mathbb{R}^n$ is *smooth* (resp. C^k) iff for all $p \in \Omega$ and every open neighbourhood $U_p \subset \mathbb{R}^n$ of p there is a smooth (resp. C^k) extension $f_{U_p} : U_p \to \mathbb{R}^n$ of $f|_{\Omega \cap U_p}$.

We will consider the half-space $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_n \geq 0\}$.



Definition 2.2. Let M be a second-countable Hausdorff space.

We consider pairs (U, ψ) where $U \subset M$ is an open set and $\psi : U \to \mathbb{R}^n_+$ is a homeomorphism to its image $\psi(U)$ which is open in $\mathbb{R}^n_+$.

A collection $\mathscr{A}$ of such pairs is called an *atlas* if they are *compatible* and cover M, i.e. if for any $(U, \psi_U), (V, \psi_V) \in \mathscr{A}$ the *transition map*

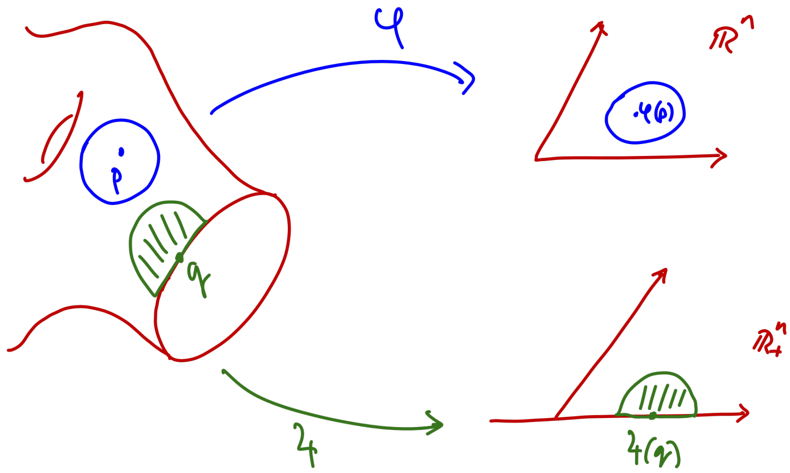
$$\psi_U \circ \psi_V^{-1} : \psi_V(U \cap V) \to \psi_U(U \cap V)$$

is smooth (in the generalised sense as above), and $\bigcup_{(U,\psi)\in\mathscr{A}} U = M$. An atlas is called *maximal* if it is maximal with these properties.

The choice of a maximal atlas $\mathscr{A}_{\max}$ makes M a *smooth n-manifold with boundary*. Any $(U,\psi) \in \mathscr{A}_{\max}$ is then called a *chart of M*, namely an *interior chart* if $\psi(U)$ is open in $\mathbb{R}^n$ or a *boundary chart* if $\psi(U)$ is open in $\mathbb{R}^n_+$ and not in $\mathbb{R}^n$.

The set $\mathring{M}$ of all points $p \in M$ for which there is an interior chart (U,ψ) with $p \in U$ is called the *interior of M*. The set $\partial M := M \setminus \mathring{M}$ is called the *boundary of M*.

Recall that a manifold in the ordinary sense is a space that is modelled on $\mathbb{R}^n$. This is captured by the fact that there, charts $\psi : U \to \mathbb{R}^n$ are asked to be homeomorphisms to their image that is open in $\mathbb{R}^n$. Taking that point of view here, we can see that a manifold with boundary is modelled on $\mathbb{R}^n_+$. With that said, we note that equivalently, we could have instead asked for our maps $\psi : U \to \mathbb{R}^n$ to be homeomorphisms to their image but $\psi(U)$ open in either $\mathbb{R}^n$ or $\mathbb{R}^n_+$. This is more convenient to work with given that then for an open subset $\Omega \subset \mathbb{R}^n$ which is a manifold with smooth boundary, the map (Ω, id) is a chart.



The following are some important facts about manifolds with boundary, which are readily verified.

Proposition 2.3. *For an n-manifold with boundary M, the following hold:*

- $\mathring{M}$ *is an n-manifold without boundary,*
- ∂M *is an $(n-1)$-manifold without boundary,*
- $M = \mathring{M} \sqcup \partial M$ *(disjoint union).*

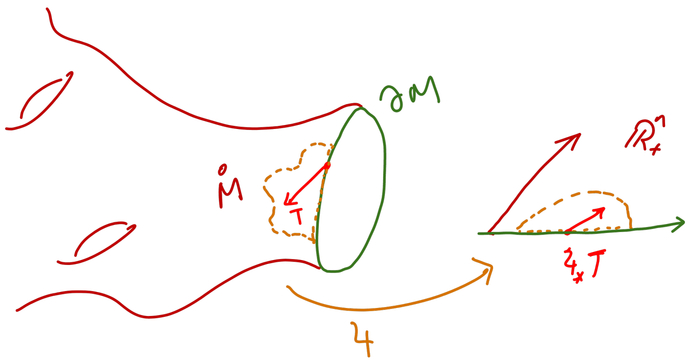
Example 2.4. 1. A manifold without boundary M is a manifold with boundary $\partial M = \varnothing$.

2. The unit ball $D := \overline{B(0,1)} = \{x \in \mathbb{R}^n \mid |x|_{\mathbb{R}^n} \leq 1\}$ is a manifold with boundary $\partial D = \mathcal{S}^{n-1} = \{x \in \mathbb{R}^n \mid |x|_{\mathbb{R}^n} = 1\}$ and interior $\mathring{D} = B(0,1)$.

3. If N is a manifold without boundary, then $M := [0,\infty) \times N$ is a manifold with boundary $\partial M = \{0\} \times M$ and $\mathring{M} = (0,\infty) \times N$.

4. Non-example: If $M := \overline{Q}$ where Q is the unit cube, endowed with the differentiable structure induced from its embedding into $\mathbb{R}^n$, then the topological boundary ∂Q is not smooth. Therefore, it is not a smooth manifold with boundary.

In order to consider derivatives in the direction towards the boundary from the interior, we need know there is a special class of vectorfields on the boundary in the relevant direction. The following notion captures this.



Definition 2.5 (**T transversal/inward/outward pointing).** A vector field along the boundary $T \in C^\infty\Big(\partial M, TM|_{\partial M}\Big)$ is called *transversal* if it does not take values in $T\partial M$, i.e. if for every chart (U,ψ) and every $x \in U \cap \partial M$ we have $((\psi_* T)(x))_n \neq 0$, i.e. $(d\psi \circ T)(x) \notin \partial\mathbb{R}^n_+$. A transversal T is *inward* or *outward* pointing if

$$((\psi_* T)(x))_n > 0 \qquad \text{or} \qquad ((\psi_* T)(x))_n < 0\,,$$

respectively.

Proposition 2.6. *On a manifold with boundary, there always exists a smooth inward or outward pointing vectorfield.*

2.2 Vector bundles

In the analysis of differential operators in Euclidean settings, it is often to consider systems, i.e., functions $f : \mathbb{R}^n \to \mathbb{C}^N$, where N can be different from n. For instance, the reduction of higher order differential equations to first order typically requires a factorisation, and that entails that the dimension N will typically be much larger than n. Vector bundles are, in a phrase, 'systems in the presence of geometry'. In this section, we recall their description and salient features required in later parts.

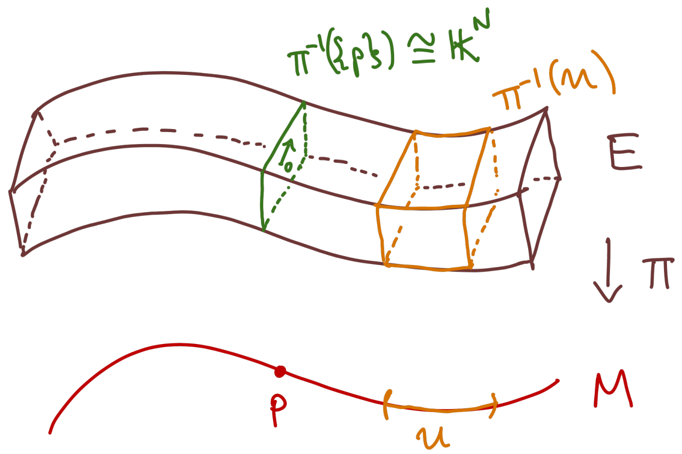
Let the field $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, although in practice, we almost always exclusively work with $\mathbb{K} = \mathbb{C}$.

Definition 2.7. E is a vector bundle over M (write $E \to M$) of rank N iff

- E is a topological space,
- there is a continuous surjection $\pi : E \to M$,
- for every $x \in M$ the fibre $E_x := \pi^{-1}(x)$ is an N-dimensional $\mathbb{K}$-vector space,
- for every $x \in M$ there is an open neighbourhood $U_x \subset M$ of x and a diffeomorphism $\Psi_x : \pi^{-1}(U_x) \to U_x \times \mathbb{K}^N$ s.t. for every $y \in U_x$ we have
  - (I) $\Psi_x(E_y) = \{y\} \times \mathbb{K}^N$, and
  - (II) $\Psi_x|_{E_y} : E_y \to \{y\} \times \mathbb{K}^N$ is a vector space isomorphism.

We call these (U_x, Ψ_x) *local trivialisations.*

When (U, ψ) is a chart for M and simultaneously (U, Ψ) is a local trivialisation, then we call (U, ψ) or more precisely (U, ψ, Ψ) a *trivialising chart for E* or simply a *trivialising chart* when E is clear from context.



A vector bundle $E \to M$ which is diffeomorphic to $M \times \mathbb{K}^N$ is called *trivial.*

Example 2.8. 1. Let (U, ψ) be a local trivialisation of a vector bundle $E \to M$. Then the bundle $E|_U := \pi^{-1}(U) \to U$ is trivial, so every vector bundle is *locally trivial* motivating the name 'local trivialisation'.

2. $E := M \times \mathbb{C}^N$ is trivial and of complex rank N.

3. $E := TM$ or T^*M is a vector bundle of real rank n. These are not trivial in general, e.g. $T\mathcal{S}^2$ is not trivial.

4. Given an $\mathbb{R}$-vector bundle $E \to M$, its complexification $E_{\mathbb{C}}$ with fibres $(E_{\mathbb{C}})_x = E_x \otimes_{\mathbb{R}} \mathbb{C}$ is a $\mathbb{C}$-vector bundle.

5. $E := T^{(p,q)}M$ is the (p,q)-tensor bundle.

6. $E := \Lambda M = \bigoplus_{p=0}^{n} \Lambda^p M$ is the bundle of differential forms.

7. For a vector bundle $E \to M$ of rank N, we consider the vector bundle $E|_{\partial M} \to \partial M$, which is also of rank N.

8. Given a transversal $T \in \mathrm{C}^\infty(\partial M, TM|_{\partial M})$, we obtain a splitting
$$TM|_{\partial M} = T\partial M \oplus \operatorname{span}\{T\}\,,$$
where $\operatorname{span}\{T\}$ is understood to be the bundle with fibres $\operatorname{span}\{T\}_x := \operatorname{span}\{T_x\}$. When g is a Riemannian metric, T can be obtained orthogonal to $T\partial M$ and is then called a normal vector field. In this case, we call $N\partial M := \operatorname{span}\{T\}$ the normal bundle. It is easy to see that this is a trivial line bundle.

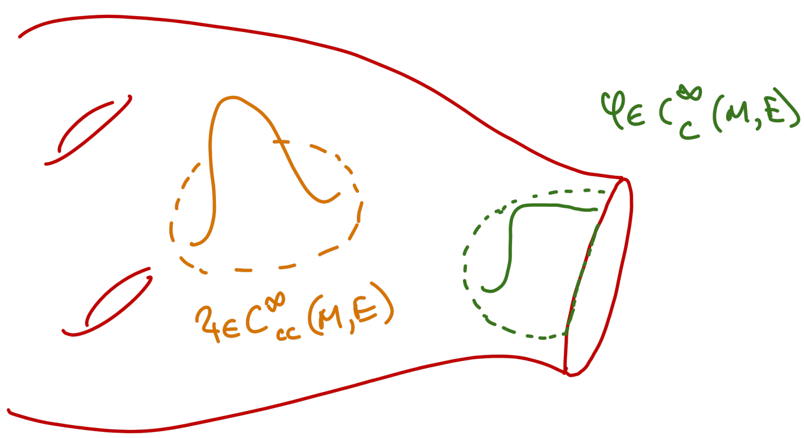
9. The fibres of the dual bundle E^* are the dual spaces of the fibres of E.

2.3 Function spaces

Let $E \to M$ and let $U \subset M$ open or $U \subset \partial M$ open. By $\mathrm{C}^k(U,E)$ we denote the set of C^k-sections of the bundle $E|_U \to U$, i.e. the C^k-functions $\varphi : U \to E$ satisfying $\pi \circ \varphi = \mathrm{id}_U$.

Then
$$\mathrm{C}^k_{\mathrm{c}}(M,E) := \left\{\varphi \in \mathrm{C}^k(U,E) \;\middle|\; \operatorname{spt}(\varphi) \text{ is compact}\right\},$$
$$\mathrm{C}^k_{\mathrm{cc}}(M,E) := \left\{\varphi \in \mathrm{C}^k_{\mathrm{c}}(U,E) \;\middle|\; \operatorname{spt}(\varphi) \subset \mathring{M}\right\}.$$

In other words, a general $\phi \in \mathrm{C}^k_{\mathrm{c}}(U,E)$ might satisfy $\operatorname{spt}\phi \cap \partial M \neq \varnothing$. In contrast, $\psi \in \mathrm{C}^k_{\mathrm{cc}}(U,E)$ always satisfies $\operatorname{spt}\psi \cap \partial M = \varnothing$.



If $E := M \times \mathbb{R}$ is the trivial line bundle, we write $\mathrm{C}^k(U) := \mathrm{C}^k(U, E)$ and similarly for the other function spaces.

2.4 Riemannian and Hermitian structures

For an $\mathbb{R}$-vector bundle $E \to M$, a section $h \in \mathrm{C}^\infty(M, E^* \otimes E^*)$ which for all $x \in M$ satisfies $h(x)[u, v] = h(x)[v, u]$ for all $u, v \in E_x$ and $h(x)[u, u] > 0$ for all $u \in E_x \setminus \{0\}$ is called a *Riemannian structure/metric* on E.

Similarly, for a $\mathbb{C}$-vector bundle $E \to M$, a section $h \in \mathrm{C}^\infty(M, E^* \otimes_{\mathbb{R}} E^*)$ which for all $x \in M$ satisfies $h(x)[u, v] = \overline{h(x)[v, u]}$ for all $u, v \in E_x$ and $h(x)[u, u] > 0$ for all $u \in E_x \setminus \{0\}$ is called *Hermitian structure/metric* on E.

When the field is clear from context, we say that h is a *metric* on E.

Definition 2.9. For $x \in M$ and $u \in E_x$ we set $|u|_{h(x)} := \sqrt{h(x)[u, u]}$.

Example 2.10.
1. $E := M \times \mathbb{C}^N$, $h(x)[u, v] = u \cdot \bar{v}$ or $h(x)[u, v] = \xi(x) u \cdot \bar{v}$ for $\xi > 0$.
2. A Riemannian metric g on TM induces Riemannian structures on T^*M, $T^{(p,q)}M$, ΛM, etc., and $g|_{N\partial M}$ is a metric on $N\partial M$.

2.5 Measures and integration

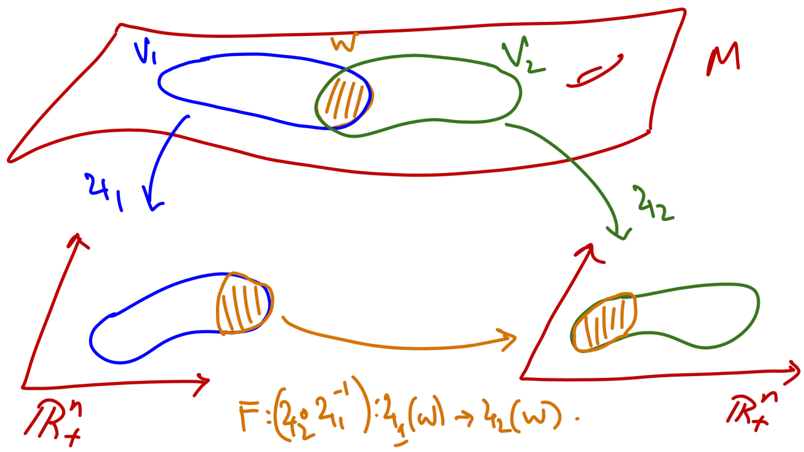
The locally Euclidean nature of a manifold not only affords it with a differentiable structure, it also carries a canonical measure structure. By this, we mean that notions of objects being *measurable* as well as the notion of sets of zero measure could be understood without alluding to a fixed reference measure. As we shall see later, these notions agree with the induced notions when we have an induced measure from a sufficiently wide class of geometric structures on a manifold. The approach we take here is to consider measures induced from sections of a certain vector bundle called the *density bundle*. This is a more convenient object in the context of manifolds with boundary since a density on the manifold induces a density on the boundary on choosing an inward or outward pointing vectorfield.

Definition 2.11.
- Let (U, ψ) be a chart. Then the pullback Lebesgue measure $d\psi^* \mathscr{L}$ of the Lebesgue measure $d\mathscr{L}$ on U is defined by requiring
$$\int_U f \, d\psi^* \mathscr{L} := \int_{\psi(U)} \left(f \circ \psi^{-1}\right) \, d\mathscr{L}.$$
for every integrable $f : U \to \mathbb{R}$.
- $A \subset M$ is called *measurable* if $A \cap U$ is $d\psi^* \mathscr{L}$-measurable in all charts (U, ψ).

- $A \subset M$ is said to be of *null measure* or a *null set* if $d\psi^* \mathscr{L}(A \cap U) = 0$ for every chart (U, ψ).

The measurable and null sets generate a σ-algebra. This is readily verified by fixing any smooth Riemannian metric g on M and then verifying that the $\mathrm{d}\mu_g$-measurable sets and $\mathrm{d}\mu_g$-zero measure sets, where $d\mu_g$ is the induced measure, are respectively measurable and null in this sense. In particular, this allows us to consider measurable sections of $E \to M$ without first having to fix a reference measure. We denote this set of sections by $\mathrm{MeasSect}(M, E)$. Details regarding these facts can be found in [8].

Consider the intersection $W := V_1 \cap V_2$ of two charts (V_1, ψ_1) and (V_2, ψ_2) with coordinates $x = \psi_1$ and $y = \psi_2$.



Let $F : \psi_1(W) \to \psi_2(W)$ be a diffeomorphism between open sets of $\mathbb{R}^n$ and $\xi : M \to \mathbb{R}$ with $\mathrm{spt}(\xi) \subset W$. Then

$$\int_{\psi_2(W)} (\xi \circ \psi_2^{-1})(y)\, \mathrm{d}\mathscr{L}(y) = \int_{\psi_1(W)} (\xi \circ \psi_1^{-1}) |DF(x)|\, \mathrm{d}\mathscr{L}(x)\,,$$

where $\psi_2(W) = F(\psi_1(W))$, and $|DF(x)| := \left|\det\left(\left(\frac{\partial y^i}{\partial x^j}\right)_{i,j}\right)\right|$.

Concern 1: We require a geometric gadget to account for the factor $\det(DF)$ in such coordinate transformations.

Concern 2: Given a measure $\mathrm{d}\mu$ on M, how do we get from it a measure $\mathrm{d}\nu$ on ∂M?

The goal of this section would be to construct a certain vector bundle which will allow us to simultaneously address both concerns.

Let W be a finite dimensional $\mathbb{R}$-vector space.

Definition 2.12. A map $\xi : W^n \to \mathbb{R}$ is called a *density* on W if

$$\xi(Tw_1, \dots, Tw_n) = |\det(T)|\xi(w_1, \dots, w_n)$$

for all $T : W \to W$ linear and $w_i \in W$.

Then $|\Lambda|W := \{\xi : W^n \to \mathbb{R} \mid \xi \text{ is a density on } W\}$.

Proposition 2.13. *We have the following properties.*

I) The set $|\Lambda|W$ is an $\mathbb{R}$-vector space.

II) Let $\xi_1, \xi_2 \in |\Lambda|W$ and $\xi_1(e_1, \dots, e_n) = \xi_2(e_1, \dots, e_n)$ for a basis $\{e_i\}$ of W. Then $\xi_1 = \xi_2$.

III) Let $\omega \in \Lambda^n W$ and set $|\omega|(x_1, \dots, x_n) := |\omega(x_1, \dots, x_n)|$. Then $|\omega| \in |\Lambda|W$.

IV) The vector space $|\Lambda|W$ is 1-dimensional and $|\Lambda|W = \operatorname{span}\{|\omega|\}$ for all $\omega \in \Lambda^n W \setminus \{0\}$.

Proof. Left as an exercise. □

Definition 2.14 (Density bundle). As a set, define

$$|\Lambda|M := \{(p, \xi) \mid p \in M, \xi \in |\Lambda|T_pM\}.$$

To make it a bundle we need trivialisations. In a chart U with coordinates x define

$$\Phi_U : \pi^{-1}(U) \to U \times \mathbb{R}$$

by

$$\Phi_U(c|\omega(p)|) := (p, c),$$

where $c \in \mathbb{R}$ and $\omega(p) = dx^1|_p \wedge \dots \wedge dx^n|_p$.

Then in charts U_1 and U_2 with coordinates x and y, respectively, we get

$$\left(\Phi_{U_1} \circ \Phi_{U_2}^{-1}\right)(p, 1) = \left(p, \det\left(\left.\frac{\partial y^i}{\partial x^i}\right|_p\right)\right).$$

Remark 2.15. The density bundle $|\Lambda|M$ cannot be identified with the forms $\Lambda^n M$ unless M is orientable.

Definition 2.16. A density $\mu \in C^\infty(M, |\Lambda|M)$ is said to be *positive* if for all $p \in M$ and all bases $\{e_i\}$ of T_pM we have $\mu(p)(e_1, \dots, e_n) > 0$.

Denote the positive densities by $|\Lambda|^+ M$.

Let $f : M \to \mathbb{R}$ be a measurable function with $\operatorname{spt}(f) \subset U$ for a chart (U, ψ) with coordinates $x = \psi$. Write $\psi^*\mu =: \mu_\psi |dx^1 \wedge \cdots \wedge dx^n|$ and define

$$\int_U f \ \mathrm{d}\mu := \int_{\psi(U)} \big(f \circ \psi^{-1}\big)\mu_\psi \ \mathrm{d}\mathscr{L} .$$

Consider a diffeomorphism $F : M \to \tilde{M}$ and $\tilde{\mu} \in \mathrm{C}^\infty\Big(\tilde{M}, |\Lambda|\tilde{M}\Big)$. Set $\mu := F^*\tilde{\mu} \in \mathrm{C}^\infty(M, |\Lambda| M)$, where

$$F^*\tilde{\mu}(w_1, \ldots, w_n) := (\tilde{\mu} \circ F)(F_* w_1, \ldots, F_* w_n) .$$

Write $\tilde{\mu} =: \tilde{u}|dx^1 \wedge \cdots \wedge dx^n|$, then

$$F^*\tilde{\mu} = (\tilde{u} \circ F)|\det DF(y)|\big|dy^1 \wedge \cdots \wedge dy^n\big| .$$

This shows that the definition we have made above is well-defined on overlaps.

For general $f : M \to \mathbb{R}$ let $\mathscr{C} := \{(U_i, \psi_i)\}$ be a countable cover of M and $\{\eta_i\}$ a smooth partition of unity subordinate to $\mathscr{C}$.

Define the Radon measure $\mathrm{d}\mu$ on M by

$$\int_M f \ \mathrm{d}\mu := \sum_{i=1}^{\infty} \int_{U_i} (f\eta_i) \ \mathrm{d}\mu .$$

Exercise 2.17.

- Verify independence of the choice of cover and partition of unity.
- $\mathrm{d}\mu$-measurable and $\mathrm{d}\mu$-null agree with measurable and null w.r.t. the density μ.
- Verify that the induced measure is, in fact, a Radon measure.

As a consequence of this construction, we often regard the induced measure $d\mu$ from a density μ and the density itself as the same object.

Definition 2.18 (Measured manifold (M, μ)). A measured manifold is a smooth n-manifold M with boundary along with $\mu \in \mathrm{C}^\infty\big(|\Lambda|^+ M\big)$.

2.6 Induced measure on ∂M

Fix a positive density $\mu \in \mathrm{C}^\infty\big(M, |\Lambda|^+ M\big)$ and an inward pointing $T \in \mathrm{C}^\infty\Big(\partial M, TM|_{\partial M}\Big)$. Then we define a positive density $\nu \in \mathrm{C}^\infty\big(\partial M, |\Lambda|^+ \partial M\big)$ on the boundary by

$$\nu(x)(w_1, \ldots, w_{n-1}) := \mu(x)(T(x), w_1, \ldots, w_{n-1})$$

for $w_i \in T_x\partial M$.

Remark 2.19. Let $d\mu$ be a Radon measure s.t. the notions $d\mu$-null and $d\mu$-measurable agree with our notions from Definition 2.11. Inside a chart (U, ψ) consider the Radon-Nikodym derivative

$$x \mapsto \frac{d\mu}{d\psi^* \mathscr{L}}(x) \in \mathrm{C}^\infty(U)\,.$$

Then $\mu := \frac{d\mu}{d\psi^*\mathscr{L}} |dx^1 \wedge \cdots \wedge dx^n|$ defines a $\mu \in \mathrm{C}^\infty\big(M, |\Lambda|^+ M\big)$.

This shows that we could have equally well begun our discussion with such Radon measures and then attempted to extract the induced measure on the boundary given an inward pointing vectorfield. However, it is unclear the way in which to do this elegantly without an excursion through the density bundle.

2.7 Banach spaces

As in Euclidean analysis, for a systematic study of boundary value problems, we are forced to consider induced function spaces beyond the smooth context. These function spaces naturally emerge from underlying geometric considerations. In application, we will only deal with *Hilbert* spaces. However, the problems we will encounter will force us off the well trodden path of analysis exploiting the natural Hilbert structure and orthogonal reasoning. Despite considering only Hilbert spaces, our methods will be closer to general Banach space analysis. Therefore, we will consider the Banach setting to the extent required and possible, and later, specialise to the Hilbert space setting. The ideas we discuss in this section is explored in great depth in [52] by Yosida and [33] by Kato.

Let $\mathcal{B}$ be a $\mathbb{K}$-vector space, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ (typically $\mathbb{C}$). Let $\|\cdot\| : B \to \mathbb{R}_+$ be a norm on $\mathcal{B}$, which induces a metric by $(x, y) \mapsto \|x - y\|$. Then $(\mathcal{B}, \|\cdot\|)$ is a *Banach space* if it is a complete metric space. $\mathcal{H} := \mathcal{B}$ is a *Hilbert space*, if $\|\cdot,\cdot\|$ polarises, meaning $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$. In this case it induces an inner product by

$$\langle x, y\rangle := \frac{1}{4}\Big(\|x + y\|^2 - \|x - y\|^2 \underbrace{- i\|x + iy\|^2 + i\|x - iy\|^2}_{\text{omitted if } \mathbb{K}=\mathbb{R}}\Big)\,,$$

which then satisfies $\langle x, x\rangle = \|x\|^2$. In the case $\mathbb{K} = \mathbb{C}$, the *adjoint space* of $\mathcal{B}$ is

$$\mathcal{B}^* := \{\varphi : B \to \mathbb{K} \mid \varphi \text{ is conjugate linear and continuous}\}\,,$$

we equip it with the norm $\|\cdot\|_{\mathcal{B}^*}$ given by

$$\|\varphi\|_{\mathcal{B}^*} := \sup_{\|x\|_B = 1} |\varphi(x)|\,.$$

The *dual space* of $\mathcal{B}$ is

$$\mathcal{B}' := \{\varphi : B \to \mathbb{K} \mid \varphi \text{ is linear and continuous}\}\,.$$

We have $\mathcal{B}^* \cong \mathcal{B}'$ by $u \mapsto \bar{u}$.

The *bi-adjoint space* is the space is $\mathcal{B}^{**} := (\mathcal{B}^*)^*$.

The *evaluation functional* $\mathrm{ev} : \mathcal{B} \to \mathcal{B}^{**}$ is defined by $\mathrm{ev}(x)[f] := f(x)$. It is easy to see that its range is a closed subspace of $\mathcal{B}^{**}$. $\mathcal{B}$ is said to be *reflexive* if ev is an isomorphism.

Definition 2.20. Fix a density $\mu \in \mathrm{C}^\infty\big(M, |\Lambda|^+ M\big)$. Let $E \to M$ be a $\mathbb{K}$-vector bundle and h a metric on E.

For $p \in [1, \infty)$ define the $\mathbb{K}$-vector space $u \in \mathrm{L}^p(M, E)$ to mean

(I) $u \in \mathrm{MeasSect}(E)$ and

(II) $\|u\|^p_{\mathrm{L}^p(M,E)} := \int |u(x)|^p_{h(x)} \,\mathrm{d}\mu(x) < \infty$.

For $p = \infty$ we set $\|u\|_{\mathrm{L}^\infty(M,E)} := \operatorname{ess\,sup}_{x \in M} |u(x)|_{h(x)} < \infty$.

When $T \in \mathrm{C}^\infty(\partial M, TM)$ is an inward pointing vectorfield, and ν is the induced density, then we define

$$\mathrm{L}^p(\partial M, E) := \mathrm{L}^p(\partial M, E|_{\partial M}),$$

where ∂M takes the place of M, ν takes the place of μ, and $E \to M$ is replaced with $E|_{\partial M} \to \partial M$.

Remark 2.21. Typically, we will only be concerned with $\mathbb{C}$ vector bundles and their induced L^p spaces as a $\mathbb{C}$ space. If we are given an $\mathbb{R}$-bundle E, we typically complexify it to obtain $E \otimes \mathbb{C}$ and then consider the L^p theory for $E \otimes \mathbb{C}$.

Remark 2.22. The space $\mathrm{L}^\infty(M, E)$ does not require a density μ.

Example 2.23.

1. For $p \in [1, \infty]$, the space $\mathrm{L}^p(M, E)$ is a Banach space.
2. For $p \in (1, \infty)$, the space $\mathrm{L}^p(M, E)$ is reflexive.
3. For $p = 2$, the space $\mathrm{L}^2(M, E)$ is a Hilbert space. Here we have

$$\langle u, v \rangle_{\mathrm{L}^2(M,E)} = \int_M h(x)[u(x), v(x)] \,\mathrm{d}\mu(x).$$

4. For $p \in [1, \infty)$, the space $\mathrm{L}^p(M, E)$ is separable, i.e. has a countable dense subset. $\mathrm{L}^\infty(M, E)$ is not separable.

2.8 Operator Theory

At the heart of differential equations are differential operators. In order to manipulate and understand these latter objects, it is important to fix some notation and ideas surrounding them. As aforementioned, the books [52] and [33] give an extensive description of the material we recall here.

Let $\mathcal{B}_1, \mathcal{B}_2$ be Banach spaces.

Definition 2.24. A linear map $T : \mathrm{dom}(T) \to \mathcal{B}_2$, where $\mathrm{dom}(T) \subset \mathcal{B}_1$ is a linear subspace, is called an *operator*.

The fact that T is not defined on the entire space is sometimes expressed by saying that 'T is unbounded'. It is important to emphasise that an operator is both the map and the domain.

The following are an important list of notation when dealing with operators:

- $\mathrm{dom}(T)$ is the domain of T.
- $\ker(T) := \{u \in \mathrm{dom}(T) \mid Tu = 0\}$.
- $\mathrm{ran}(T) := \{Tu \in \mathcal{B}_2 \mid u \in \mathrm{dom}(T)\}$.
- $\mathrm{graph}(T) := \{(u, Tu) \in \mathcal{B}_1 \times \mathcal{B}_2 \mid u \in \mathrm{dom}(T)\}$.
- The operator T induces the *graph norm* $\|\cdot\|_T$ on $\mathrm{dom}(T)$ given by
$$\|u\|_T^2 := \|u\|_{\mathcal{B}_1}^2 + \|Tu\|_{\mathcal{B}_2}^2 .$$
- We say $S \subset T$ if $\mathrm{dom}(S) \subset \mathrm{dom}(T)$ and $Su = Tu$ for all $u \in \mathrm{dom}(S)$.

2.9 Algebra with unbounded operators

In the situation when operators where operators are defined on the whole space, their algebra simply follows from usual algebraic properties available in a vector space. However, for general unbounded operators, we are required to specify the domain on which the algebraic operators are defined. That is, for unbounded operators $S, T : B \to B$ we define:

$$\begin{aligned} \mathrm{dom}(T+S) &:= \mathrm{dom}(T) \cap \mathrm{dom}(S) , \\ \mathrm{dom}(ST) &:= \{u \in \mathrm{dom}(T) \mid Tu \in \mathrm{dom}(S)\} . \end{aligned}$$

It could happen that $\mathrm{dom}(T+S) = \{0\}$ or $\mathrm{dom}(ST) = \{0\}$ without S or T being 0.

Exercise: The unbounded operators on a Banach space $\mathcal{B}$ with addition form a commutative monoid, i.e. $(S+T)+U = S+(T+U)$, $S+0 = S$, and $S+T = T+S$. Composition is associative, and distributive from the right, i.e. $(ST)U = S(TU)$ and $(S+T)U = SU + TU$. We also have $S - S = 0$, $0S \subset S0 = 0$, and $S(T+U) \supset ST + SU$.

Definition 2.25.
- T is *densely-defined* if $\mathrm{dom}(T)$ is a dense subspace of $\mathcal{B}_1$.
- T is *closed* if $\mathrm{graph}(T) \subset \mathcal{B}_1 \times \mathcal{B}_2$ is a closed subset.

  Notation: $T \in \mathscr{C}(\mathcal{B}_1, \mathcal{B}_2)$.
- $T : \mathcal{B}_1 \to \mathcal{B}_2$ is *bounded* if $\mathrm{dom}(T) = \mathcal{B}_1$ and there is a $C_T < \infty$ s.t. $\|Tx\| \le C_T \|x\|$ for all $x \in \mathcal{B}_1$.

  Notation: $T \in \mathscr{B}(\mathcal{B}_1, \mathcal{B}_2)$, sometimes also $\mathscr{L}(\mathcal{B}_1, \mathcal{B}_2)$.
- T is *injective* if $\ker(T) = 0$.
- T is *invertible* if it is injective with dense range and $\|T^{-1}u\| \le C_T \|u\|$ for all $u \in \mathrm{ran}(T)$.

Exercise 2.26.
- T is closed $\Leftrightarrow$ $(\mathrm{dom}(T), \|\cdot\|_T)$ is a Banach space.
- T is bounded $\Leftrightarrow$ T is continuous.

The following is a central theorem of operator theory. We enlist it here because it shows that closed operators are 'almost' bounded.

Theorem 2.27 (Closed graph theorem). *Let $T : \mathcal{B}_1 \to \mathcal{B}_2$ be closed operator with domain $\mathrm{dom}(T) = \mathcal{B}_1$. Then T is bounded.*

Exercise 2.28. Let $S \in \mathscr{B}(B) := \mathscr{B}(B, B)$ and $T \in \mathscr{C}(B) := \mathscr{C}(B, B)$. Then $S, TS, S^{-1}T, T^{-1} \in \mathscr{C}(B)$ (for the last two if S^{-1} is injective and T^{-1} is injective, respectively).

Theorem 2.29 (Open mapping theorem). *A bounded surjection $T : \mathcal{B}_1 \to \mathcal{B}_2$ is an open map, i.e. $U \subset \mathcal{B}_1$ open implies $T(U) \subset \mathcal{B}_2$ open.*

Important criterion for closedness:

Remark 2.30. T is closed iff $u_n \in \mathrm{dom}(T), u_n \to u$ and $Tu_n \to v$ implies $\exists u \in \mathrm{dom}(T)\colon v = Tu$.

Definition 2.31. An operator T is *closable* if it has a closed extension, i.e. if there is a $\tilde{T} \in \mathscr{C}(\mathcal{B}_1, \mathcal{B}_2)$ with $T \subset \tilde{T}$.

Proposition 2.32. *For T closable there is a $\bar{T}$ corresponding to $\overline{\operatorname{graph}(T)}$, i.e. $\exists \bar{T} \in \mathscr{C}(\mathcal{B}_1, \mathcal{B}_2)$ s.t. $\overline{\operatorname{graph}(T)} = \operatorname{graph}(\bar{T})$.*

Proof. Since $\operatorname{graph}(T) \subset \operatorname{graph}(\tilde{T})$, the closure $\overline{\operatorname{graph}(T)}$ must correspond to an operator which is a restriction of $\tilde{T}$ to a smaller domain. □

In practice, the graphical closure is not a very workable definition to detect closability. Instead, the following is the most commonly used criterion.

Proposition 2.33. *T is closable iff $u_n \in \operatorname{dom}(T), u_n \to 0$ and $Tu_n \to v$ implies $v = 0$.*

Proof. a) '⇒': Easy, exercise.

b) '⇐': Suppose $(x_n, Tx_n) \to (x, y)$ and $(\tilde{x}_n, T\tilde{x}_n) \to (x, z)$. Then $y_n := (x_n - \tilde{x}_n) \to 0$ and $y_n \in \operatorname{dom}(T)$ and $Ty_n \to y - z$. So $y - z = 0$, so $\overline{\operatorname{graph}(T)} = \operatorname{graph}(\bar{T})$. □

2.10 Duality/Perfect Pairings

The geometry of L^2-spaces sits at the heart of our concerns. Often, we are required to consider other spaces when dealing with differential operators. These are typically Hilbert spaces. However, in analysis, the natural structure that emerges forces us to take the L^2-inner product into account. This means that we are forced away from the natural inner product associated with a given Hilbert space, and we are forced to consider it to be *paired* with some other Hilbert space such that on a common dense subset, the pairing is precisely the L^2-inner product. This perspective sits closer to general Banach space analysis rather than the Hilbert space setting, and therefore, we will exposit material surrounding pairings at this level of generality.

Definition 2.34. Consider $\langle \cdot, \cdot \rangle : \mathcal{B}_1 \times \mathcal{B}_2 \to \mathbb{K}$ sesquilinear when $\mathbb{K} = \mathbb{C}$, i.e. linear in the first and conjugate linear in the second entry, or bilinear when $\mathbb{K} = \mathbb{R}$. The triple $(\mathcal{B}_1, \mathcal{B}_2, \langle \cdot, \cdot \rangle)$ is called a *perfect pair* (or duality) if the following hold.

(I) There is a $C_0 < \infty$ s.t. for all $u \in \mathcal{B}_1$ and for all $v \in \mathcal{B}_2$

$$|\langle u, v \rangle| \le C_0 \|u\|_{\mathcal{B}_1} \|v\|_{\mathcal{B}_2}.$$

(II) There is a $C_1 < \infty$ s.t. for every $u \in \mathcal{B}_1$

$$\|u\|_{\mathcal{B}_1} \leq C_1 \sup\Big\{|\langle u, v\rangle| \;\Big|\; \forall v \in \mathcal{B}_2 \colon \|v\|_{\mathcal{B}_2} = 1\Big\}\,.$$

(III) There is a $C_2 < \infty$ s.t. for every $v \in \mathcal{B}_2$

$$\|v\|_{\mathcal{B}_2} \leq C_2 \sup\Big\{|\langle u, v\rangle| \;\Big|\; \forall u \in \mathcal{B}_1 \colon \|u\|_{\mathcal{B}_1} = 1\Big\}\,.$$

The map $\langle\cdot,\cdot\rangle$ is then called a *perfect pairing* or *duality* and it is denoted by $\langle\mathcal{B}_1, \mathcal{B}_2\rangle$.

Remark 2.35. A useful observation is that, in the bound for the norms in (II) or (III), it is possible to pass instead to any dense subspace of $\mathcal{B}_2$ or $\mathcal{B}_1$ respectively. Often in applications, useful formulae hold on dense subspaces, and therefore, bounds on the norms are easily computed for more complicated objects typically by passing to simpler ones.

Example 2.36.
1. A Hilbert space $\mathcal{H}$ together with its inner product $\langle\cdot,\cdot\rangle : \mathcal{H}\times\mathcal{H} \to \mathbb{K}$ is a perfect pair.
2. $\langle\cdot,\cdot\rangle_{\mathrm{L}^2(M,E)}$ induces a perfect pairing $\langle\cdot,\cdot\rangle : \mathrm{L}^p(M,E)\times\mathrm{L}^q(M,E) \to \mathbb{K}$ if $\frac{1}{p}+\frac{1}{q} = 1$ and $p, q \in (1,\infty)$.
3. $\langle\!\langle\cdot,\cdot\rangle\!\rangle : \mathcal{B}^*\times\mathcal{B} \to \mathbb{K}$, $\langle\!\langle T, u\rangle\!\rangle := T[u]$ is called the *canonical pairing*.

Definition 2.37. Let $T : \mathcal{B}_1 \to \mathcal{B}_2$ and $S : \mathcal{B}_2^* \to \mathcal{B}_1^*$ be operators satisfying

$$\langle\!\langle v, Tu\rangle\!\rangle = \langle\!\langle Sv, u\rangle\!\rangle$$

for all $u \in \mathrm{dom}(T) \subset \mathcal{B}_1$ and all $v \in \mathrm{dom}(S) \subset \mathcal{B}_2$. Then we say S *is adjoint* to T.

Notation 2.38. We write $a \lesssim b$ to mean the analyst's inequality. That is, to say there exists a constant such that $a \leq Cb$. We may refer to the constant C here as the *implicit constant*. The dependency on the constant C is usually apparent from context and is typically independent from the terms on the left and right. We write $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$.

Lemma 2.39. *Let $T : \mathcal{B}_1 \to \mathcal{B}_2$ be densely-defined. Suppose that there exist $w, w' \in \mathcal{B}_1^*$ such that for all $u \in \mathrm{dom}(T)$,*

$$\langle\!\langle u, Tu\rangle\!\rangle = \langle\!\langle w, u\rangle\!\rangle \qquad \textit{and} \qquad \langle\!\langle u, Tu\rangle\!\rangle = \langle\!\langle w', u\rangle\!\rangle\,.$$

Then $w = w'$.

Proof. The condition as stated is exactly that $\langle\!\langle w - w', u\rangle\!\rangle = 0$ for all $u \in \mathrm{dom}(T)$. Using property (II), and on noting that $\mathrm{dom}(T)$ is dense in $\mathcal{B}_1$ and using Remark 2.35, we obtain that

$$\|w - w'\| \lesssim \sup\{|\langle\!\langle w - w', u\rangle\!\rangle| \mid u \in \mathrm{dom}(T)\} = 0\,.$$

I.e. $w = w'$. □

As a consequence of this lemma, we are able to formulate the existence of the maximal adjoint as follows.

Definition 2.40 (The canonical adjoint). Let T be densely-defined, and let

$$\mathrm{dom}(T^{*,\mathrm{can}}) := \{v \in \mathcal{B}_2^* \mid \exists w \in \mathcal{B}_1^*\, \forall u \in \mathrm{dom}(T)\colon \langle\!\langle v, Tu\rangle\!\rangle = \langle\!\langle w, u\rangle\!\rangle\}\,.$$

For $v \in \mathrm{dom}(T^{*,\mathrm{can}})$ and we set $T^{*,\mathrm{can}}v := w$, where $w \in \mathcal{B}_1^*$ satisfying $\langle\!\langle v, Tu\rangle\!\rangle = \langle\!\langle w, u\rangle\!\rangle$. The operator $T^{*,\mathrm{can}}$ is *the canonical* adjoint.

The c here in the definition refers to the fact that this operator is obtained with respect to the canonical pairing $\langle\!\langle \cdot, \cdot\rangle\!\rangle$. It is maximal in the sense that, whenever S is adjoint to T, then $S \subset T^{*,\mathrm{can}}$.

Our discussion so far has been only with respect to the canonical pairing. Given a perfect paring $\langle \mathcal{B}_1, \mathcal{B}_2\rangle$ and a densely-defined operator $T : \mathcal{B}_2 \to \mathcal{B}_2$ whether an adjoint map exists with respect to this pairing. The following (non)example highlights that this is not always the case.

Example 2.41 (Non example). The space of zero sequences

$$c_0 := \left\{(x_n)_{n\in\mathbb{N}} \;\middle|\; x_n \to 0\right\}$$

is a Banach space with respect to the norm $\|\cdot\|_\infty$.

We first show that its dual space is isomorphic to

$$\ell_1 := \left\{(x_n)_{n\in\mathbb{N}} \;\middle|\; \sum_{n\in\mathbb{N}} |x_n| < \infty\right\}.$$

For $u \in \ell_1$ and $v \in c_0$, note that $u[v] := \sum_{i=1}^\infty u_i v_i$ is a well-defined bilinear map. Conversely, if $u \in c_0^*$, then on taking $x_i := (y_j)_{j\in\mathbb{N}}$ where $y_j = 1$ when $j = i$ and $y_j = 0$ for $j \neq i$, we obtain that

$$u[v] = u\left[\sum_{i=1}^\infty v_i x_i\right] = \sum_{i=1}^\infty v_i u[x_i]\,.$$

and setting $u_i := u[x_i]$, it is easy to see that $(u_i)_{i\in\mathbb{N}} \in \ell_1$. This shows that $c_0^* \cong \ell_1$ and therefore, we have an induced perfect pairing $\langle \ell_1, c_0\rangle$ given by

$$\langle u, v\rangle = \sum_{i=1}^\infty u_i v_i\,.$$

Let $T : \ell_1 \to \ell_1$ be the map defined by

$$Tu = \left(\sum_{j=1}^{\infty} u_i, 0, \dots \right).$$

Suppose that an adjoint map with respect to $\langle \ell_1, c_0 \rangle$ exists for T. That is, suppose there is $T' : c_0 \to c_0$ such that $\langle v, Tu \rangle = \langle T'v, u \rangle$. Let $T'v = (w_i)_{i \in \mathbb{N}}$.

$$\langle v, Tu \rangle = \sum_{i=1}^{\infty} v_1 u_i = v_1 \sum_{i=1}^{\infty} u_i$$

whereas

$$\langle T'v, u \rangle = \sum_{i=1}^{\infty} w_i u_i \,.$$

From this, we see that $w_i = v_1$ for all i, which yields that $\lim_{i \to \infty} w_i \neq 0$. Therefore, T' cannot be a map into c_0.

It is also possible to show that $\ell_1^* \cong \ell_\infty \neq c_0$, which in particular shows that c_0 and ℓ_1 are not reflexive space.

Let $\langle \mathcal{B}_1, \mathcal{B}_2 \rangle$ be a perfect pair.

Definition 2.42. Define

$$\begin{aligned} &\Phi_1 : \mathcal{B}_1 \to \mathcal{B}_2^* &&\text{by} &&(\Phi_1 u)[v] := \langle u, v \rangle &&\text{and} \\ &\Phi_2 : \mathcal{B}_2 \to \mathcal{B}_1^* &&\text{by} &&(\Phi_2 x)[y] := \overline{\langle y, x \rangle} \,. \end{aligned}$$

Lemma 2.43. *The ranges $\Phi_1(\mathcal{B}_1) \subset \mathcal{B}_2^*$ and $\Phi_2(\mathcal{B}_2) \subset \mathcal{B}_1^*$ are closed linear subspaces.*

Proof. The proof is left as an exercise. □

The following is one of the cornerstone theorems of functional analysis, of which there are many generalisations beyond the Banach space setting. For our purposes we produce the following incarnation.

Theorem 2.44 (Hahn-Banach). *Let $\mathcal{B}$ be a Banach space and $X \subset \mathcal{B}$ a linear subspace. Suppose that $f_X \in X^*$, that is, a linear functional on X^*. Then, there exists an extension $f \in \mathcal{B}^*$ such that $f = f_X$ on X and $\|f\|_{X^*} = \|f\|_{\mathcal{B}^*}$.*

Corollary 2.45. *Let $X \subset \mathcal{B}$ be a proper closed subspace. Fix $x_0 \in \mathcal{B} \setminus X$ (so that in particular $\operatorname{dist}(x_0, X) > 0$). Then there exists $f \in \mathcal{B}^*$ with $\|f\| = 1$, $X \subset \ker(f)$ and $f(x_0) = \operatorname{dist}(x_0, X)$.*

The most significant consequence of the Hahn-Banach theorem in our context is the following.

Proposition 2.46. *Let $\langle \mathcal{B}_1, \mathcal{B}_2 \rangle$ (i.e. a perfect paring). Then $\mathcal{B}_2$ is reflexive iff $\mathcal{B}_1$ is reflexive and $\Phi_1(\mathcal{B}_1) = \mathcal{B}_2^*$ and $\Phi_2(\mathcal{B}_2) = \mathcal{B}_1^*$.*

Proof. Assume that $\mathcal{B}_2$ is reflexive and suppose that $\Phi_1 \mathcal{B}_1 \subset \mathcal{B}_2^*$ is a proper subspace. So, let $\xi \in \mathcal{B}_2^* \setminus \Phi_1 \mathcal{B}_1$. By Corollary 2.45, we can find an $f \in \mathcal{B}_2^{**}$ such that $\|f\| = 1$ and $f(\Phi_1 b_1) = 0$ for all $b_1 \in \mathcal{B}_1$. Reflexivity of $\mathcal{B}_2$ is exactly that $\mathcal{B}_2^{**} = \operatorname{ev} \mathcal{B}_2$. For $b_2^* \in \mathcal{B}_2^*$,

$$f(b_2^*) = \operatorname{ev}\big(\operatorname{ev}^{-1} f\big)[b_2^*] = b_2^*\big[\operatorname{ev}^{-1} f\big] \,.$$

Since $\Phi_1 \mathcal{B}_1 \subset \mathcal{B}_2^*$, for a choice of $b_2^* = \Phi_1 b_1$,

$$0 = f(\Phi_1 b_1) = (\Phi_1 b_1)\big[\operatorname{ev}^{-1} f\big] = \big\langle b_1, \operatorname{ev}^{-1} f \big\rangle \,.$$

That is, for all b_1, we have that $\langle b_1, \operatorname{ev}^{-1} f \rangle = 0$. Using (III), we note that

$$\big\|\operatorname{ev}^{-1} f\big\| \lesssim \sup\big\{ \big|\big\langle b_1, \operatorname{ev}^{-1} f \big\rangle\big| \,\big|\, b_1 \in \mathcal{B}_1,\ \|b_1\| = 1 \big\} = 0 \,.$$

But $\operatorname{ev} : \mathcal{B}_2 \to \mathcal{B}_2^{**}$ is a Banach space isomorphism, which yields $f = 0$. This contradicts that $\|f\| = 1$. Therefore, $\Phi_1 \mathcal{B}_1 = \mathcal{B}_2^*$.

A Banach space is reflexive if and only if its adjoint space is reflexive. Moreover, if two Banach spaces are isomorphic and one is reflexive, so is the other. Since $\mathcal{B}_2$ is reflexive, so is $\mathcal{B}_2^*$ and since we have shown $\Phi_1 : \mathcal{B}_1 \to \mathcal{B}_2$ is an isomorphism, this shows that $\mathcal{B}_1$ is reflexive.

The equality $\Phi_2 \mathcal{B}_2 = \mathcal{B}_1^*$ is argued similarly. □

Definition 2.47. We say that $\langle \mathcal{B}_1, \mathcal{B}_2 \rangle$ is *reflexive* if either $\mathcal{B}_1$ or $\mathcal{B}_2$ is reflexive.

The canonical pairing is explored to great lengths in [33], i.e. in Chapter III and Section 4. Proposition 2.46 allows us to 'import' all these results into the setting of reflexive pairings $\langle \mathcal{B}_1, \mathcal{B}_2 \rangle$.

Proposition 2.48. *If $\mathcal{B}_2$ is reflexive and $\langle \mathcal{B}_1, \mathcal{B}_2 \rangle$, then for $T : \mathcal{B}_2 \to \mathcal{B}_2$ densely-defined there is a unique closed maximal $T^* : \mathcal{B}_1 \to \mathcal{B}_2$ satisfying*

$$\langle T^* u, v \rangle = \langle u, T v \rangle$$

for all $v \in \operatorname{dom}(T)$ and $u \in \operatorname{dom}(T^*) := \Phi_1^{-1}(\operatorname{dom}(T^{*,can}))$. We have

$$T^* = \Phi_1^{-1} T^{*,can} \Phi_1 \,.$$

Proof. The fact that T^* is an adjoint to T is easily seen. We leave it as an exercise to show that T^* is maximal. □

Example 2.41 demonstrates this proposition is sharp. Moreover, the proposition illustrates in a concrete way the way in which an adjoint can fail to exist. In the absence of reflexivity, it is possible that the subspace $\Phi_1^{-1}\operatorname{dom}(T^{*,\mathrm{can}}) = \{0\}$ despite $\operatorname{dom}(T^{*,\mathrm{can}}) \neq \{0\}$.

Definition 2.49. For $\langle \mathcal{B}_1, \mathcal{B}_2 \rangle$ and $\mathcal{B}_2$ reflexive, the closed operator $T^* : \mathcal{B}_1 \to \mathcal{B}_1$ obtained for a densely-defined operator $T : \mathcal{B}_2 \to \mathcal{B}_2$ is called the adjoint of T with respect to $\langle \mathcal{B}_1, \mathcal{B}_2 \rangle$.

Often, this is simply called the adjoint when the pairing is fixed and the map is clear from context.

Corollary 2.50. *If $\mathcal{B}_1 = \mathcal{B}_2 = H$ is a Hilbert space, then T^* is the usual adjoint w.r.t. the Hilbert inner product.*

Definition 2.51 (Annihilator). Let $\langle \mathcal{B}_1, \mathcal{B}_2 \rangle$ and subspace $S \subset \mathcal{B}_1$. Then

$$S^{\perp,\langle\cdot,\cdot\rangle} := \{y \in \mathcal{B}_2 \mid \forall x \in S \colon \langle x, y \rangle = 0\}$$

is called *annihilator* of S w.r.t. $\langle \cdot, \cdot \rangle$.

Lemma 2.52. *$S^{\perp,\langle\cdot,\cdot\rangle} \subset \mathcal{B}_2$ is a closed subspace and*

$$\bar{S} \subset \left(S^{\perp,\langle\cdot,\cdot\rangle}\right)^{\perp,\langle\cdot,\cdot\rangle} = \left(\bar{S}^{\perp,\langle\cdot,\cdot\rangle}\right)^{\perp,\langle\cdot,\cdot\rangle}.$$

Proposition 2.53. *Let $\mathcal{B}_1$ be reflexive and $\langle \mathcal{B}_1, \mathcal{B}_2 \rangle$. Then*

I) $\Phi_2\left(S^{\perp,\langle\cdot,\cdot\rangle}\right) = S^{\perp,\langle\!\langle \mathcal{B}_1^*, \mathcal{B}_1 \rangle\!\rangle}$ and

II) $\left(S^{\perp,\langle \mathcal{B}_1, \mathcal{B}_2 \rangle}\right)^{\perp,\langle \mathcal{B}_1, \mathcal{B}_2 \rangle} = \bar{S}$.

Proof. a) Ad I).

$$\begin{aligned} S^{\perp,\langle\!\langle \mathcal{B}_1^*, \mathcal{B}_1 \rangle\!\rangle} &= \{y \in \mathcal{B}_1^* \mid \forall x \in S \colon \langle\!\langle x, y \rangle\!\rangle = 0\} \\ &= \left\{y \in \mathcal{B}_1^* \,\middle|\, \forall x \in S \colon \Phi_2 \circ \left(\Phi_2^{-1} y\right)[x] = 0\right\} \\ &= \left\{y \in \mathcal{B}_1^* \,\middle|\, \forall x \in S \colon \overline{\langle x, \Phi_2^{-1} y \rangle} = 0\right\} \\ &= \Phi_2^{-1} S^{\perp,\langle\cdot,\cdot\rangle}. \end{aligned}$$

b) Ad II). Recalling the evaluation functional, $\mathrm{ev} : B \to \mathcal{B}^{**}$, $\mathrm{ev}(x)[y] = y[x]$, we have in general that

$$\left(S^{\perp,\langle\!\langle \mathcal{B}_1^*, \mathcal{B}_1 \rangle\!\rangle}\right)^{\perp,\langle\!\langle \mathcal{B}_1^{**}, \mathcal{B}_1^* \rangle\!\rangle} \cap \mathrm{ev}\,\mathcal{B} = \mathrm{ev}\,\overline{S}.$$

Reflexive means exactly that $\mathrm{ev}\,\mathcal{B} = \mathcal{B}^{**}$. The conclusion follows from this. $\square$

Remark 2.54. Identify $(\mathcal{B}_1 \times \mathcal{B}_2)^* \cong \mathcal{B}_1^* \times \mathcal{B}_2^*$. $T : \mathcal{B}_1 \to \mathcal{B}_2$ densely-defined, $T^{*,\mathrm{can}} : \mathcal{B}_2^* \to \mathcal{B}_1^*$ is given by

$$\begin{aligned}\operatorname{inv}\operatorname{graph}(-T^{*,\mathrm{can}}) :=& \{(-T^{*,\mathrm{can}}u, u) \mid u \in \operatorname{dom}(T^{*,\mathrm{can}})\} \\ =& \operatorname{graph}(T)^{\perp,\langle\langle(\mathcal{B}_1\times\mathcal{B}_2)^*,\mathcal{B}_1\times\mathcal{B}_2\rangle\rangle} .\end{aligned}$$

Applying this to the case that $\mathcal{B}_1 = \mathcal{B}_2$ and where we have an another *reflexive* Banach space $\mathcal{B}_3$ where we have a perfect pairing $\langle \mathcal{B}_3, \mathcal{B}_2 \rangle$, we obtain that the adjoint $T : \mathcal{B}_3 \to \mathcal{B}_3$ satisfies

$$\operatorname{inv}\operatorname{graph}(-T^*) = \operatorname{graph}(T)^{\perp,\langle\mathcal{B}_3,\mathcal{B}_2\rangle} .$$

2.11 Projectors and decompositions

Of significance is to understand how to break up a given Banach space into subspaces in a useful manner. We will see here that projectors are a systematic language in which we can understand how to topologically decompose a Banach space. Later, in application, we will see that the projectors are naturally associated with information emerging from differential operators. As a consequence, even in the Hilbert setting, these subspaces will not generally be orthogonal. Again, we find that the techniques that we are forced to use are closer to the general Banach space setting, and therefore, we give an account of decompositions for Banach spaces.

Definition 2.55. Let $\mathcal{B}$ be a Banach space. For not necessarily closed subspaces $\mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{B}$ we consider the vector space

$$\mathcal{B}_1 + \mathcal{B}_2 := \{u + v \mid u \in \mathcal{B}_1, v \in \mathcal{B}_2\} .$$

If $\mathcal{B}_1 \cap \mathcal{B}_2 = 0$, then we say that $\mathcal{B}_1, \mathcal{B}_2$ are *algebraically complementary* and we write

$$\mathcal{B}_1 \oplus_{\mathrm{a}} \mathcal{B}_2 := \mathcal{B}_1 + \mathcal{B}_2 ,$$

where the ‘a’ signifies *algebraic sum.*

If $\mathcal{B}_1$ and $\mathcal{B}_2$ are closed, algebraically complementary and $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2$ we say that $\mathcal{B}_1, \mathcal{B}_2$ are *complementary.*

If $\mathcal{B}_1 \subset \mathcal{B}$ is closed and there is a $\mathcal{B}_2 \subset \mathcal{B}$ closed s.t. $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2$, we say that $\mathcal{B}_1$ is *complemented.*

In the literature, it is sometimes written $\mathcal{B}_1 \oplus \mathcal{B}_2$ without the sum being a Banach space. In such situations, it is often written that the sum is not closed. These definitions are not made arbitrarily. It can be the case that a Banach space and even a Hilbert space is an algebraic sum of a closed subspace and a non-closed space, or that the sum of two algebraically complementary closed subspaces might not be closed. In general, in infinite dimensions, understanding the closedness of sums of spaces is a key and highly non-trivial issue. The following examples highlight the various situations that can arise.

Example 2.56. 1. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{H}_1 \subset \mathcal{H}$ a subspace. Then

$$\mathcal{H}_1^{\perp} := \{u \in \mathcal{H} \mid \forall v \in \mathcal{H}_1 \colon \langle u, v\rangle = 0\}$$

is a closed subspace and $\mathcal{H} = \overline{\mathcal{H}_1} \oplus \mathcal{H}_1^{\perp}$. So in a Hilbert space, every closed subspace is complemented.

2. If $\mathcal{H}_1, \mathcal{H}_2 \subset \mathcal{H}$ are closed algebraically complementary subspaces i.e. $\mathcal{H}_1 \cap \mathcal{H}_2 = 0$, then $\mathcal{H}_1 \oplus \mathcal{H}_2$ is not necessarily closed.

   We demonstrate this with the following example due to Robert Israel. Let $\mathcal{H} := \ell^2$ and define

$$\begin{aligned} \mathcal{H}_1 &:= \left\{(h_n)_{n\in\mathbb{N}} \,\middle|\, \forall n \in \mathbb{N} \colon h_{2n} = 0\right\}, \\ \mathcal{H}_2 &:= \left\{(h_n)_{n\in\mathbb{N}} \,\middle|\, \forall n \in \mathbb{N} \colon h_{2n+1} = n h_{2n}\right\}. \end{aligned}$$

   a) $\mathcal{H}_1, \mathcal{H}2$ are algebraically complementary.

   Fix $h \in \mathcal{H}_1 \cap \mathcal{H}_2$. This means that implies $h_{2n} = 0$ and $h_{2n+1} = n h_{2n} = 0$, so $h = 0$.

   b) $\mathcal{H}_1$ is closed.

   Let $(x_i) := (h_{n,i})_n \in \mathcal{H}_1$ with $x_i \to x = (h_n) \in \mathcal{H}$. Then,

$$\sum_n |h_{n,i} - h_n|^2 = \sum_j |h_{2j}|^2 + \sum_j |h_{2j+1,i} - h_{2j+1}|^2 ,$$

   and since the left hand side tends to 0 as $i \to \infty$, so do each of the terms on the the right hand side. This implies $h_{2j} = 0$ and thus $x \in \mathcal{H}_1$.

   c) $\mathcal{H}_2$ is closed. As before, let $(x_i) = (h_{n,i})_n \in \mathcal{H}_2$ and suppose that $x_i \to x = (h_n)_n \in \mathcal{H}$. Then,

$$\sum_n |h_{n,i} - h_n|^2 = \sum_j |h_{2j,i} - h_{2j}|^2 + \sum_j |n h_{2j,i} - h_{2j+1}|^2 ,$$

   and again, again the left hand side goes to 0 implying $h_{2j+1} = n h_{2j}$ and thus $x \in \mathcal{H}_2$.

   d) $\mathcal{H}_1 \oplus_{\mathrm{a}} \mathcal{H}_2 \neq \mathcal{H}$.

   Consider $x = (h_n) \in \mathcal{H} = \ell^2$ with $h_n := \frac{1}{n+1}$. Suppose there are $u \in \mathcal{H}_1, v \in \mathcal{H}_2$ s.t. $x = u + v$. Then $u_{2n} = 0$, implying $v_{2n} = \frac{1}{2n+1}$ and thus $v_{2n+1} = \frac{n}{2n+1}$. We get

$$\sum_n |v_n|^2 = \sum_n \frac{n^2}{(2n+1)^2} = \infty .$$

e) $\mathcal{H}_1 \oplus_{\mathrm{a}} \mathcal{H}_2$ is dense in $\mathcal{H}$, and hence, not closed.

Let $\ell_{\mathrm{F}} := \{x = (x_j) \mid \exists N = N(x)\, \forall j > n\colon x_j = 0\}$. It is well known that $\ell_{\mathrm{F}} \subset \ell^2$ is dense in $\mathcal{H}$. Since $\ell_{\mathrm{F}} \subset \mathcal{H}_1 \oplus_{\mathrm{a}} \mathcal{H}_2$, the latter space is also dense.

Throughout, we will consider separable Hilbert spaces. Recall that for a separable Hilbert space $\mathcal{H}$, we can choose an orthonormal basis $\{h_i\}_{i\in\mathbb{N}}$, and by mapping $h_i \mapsto e_i$, we obtain an isometry between $\mathcal{H}$ and ℓ^2. Therefore, we are able to pull the subspaces we just constructed via this map which shows that every separable Hilbert space always admits closed subspaces whose sum is non-closed.

3. We can have $\mathcal{H} = \mathcal{H}_1 \oplus_{\mathrm{a}} \mathcal{H}_2$ where $\mathcal{H}_2$ is closed and finite dimensional but $\mathcal{H}_1$ is not closed.

   To see this, again let $\mathcal{H} := \ell^2$. Take an $f : \ell^2 \to \mathbb{R}$ which is linear and unbounded (an unbounded functional). Certainly, we can find $x_0 \in \ell^2$ s.t. $f(x_0) = 1$. Write $x \in \ell^2$ as

$$x = \underbrace{(x - f(x)x_0)}_{\in \ker(f)} + \underbrace{f(x)x_0}_{\in \mathbb{R}x_0},$$

   where we see that $f(x - f(x)x_0) = f(x) - f(x)x_0 = 0$. Clearly, $\mathbb{R}x_0$ is one dimensional, and hence, it is closed.

   It remains to argue $\ker(f)$ is not closed. This follows from the following more general fact: for a Banach space $\mathcal{B}$,

$$\varphi : \mathcal{B} \to \mathbb{R} \text{ unbounded} \quad \Rightarrow \quad \ker(\varphi) \text{ is not closed}.$$

   To see this, since φ is assumed to be unbounded, there exists $x_n \in \mathcal{H}$ s.t. $|\varphi(x_n)| \geq n\|x_n\| > 0$. For any $x \in \mathcal{B}$, define

$$y_n := x - \frac{x_n}{\varphi(x_n)}.$$

   By the choice of x_n, it is clear that

$$\left\| \frac{x_n}{\varphi(x_n)} \right\| = \frac{\|x_n\|}{|\phi(x_n)|} \leq \frac{1}{n} \to 0.$$

   Now, we choose x such that $\varphi(x) = 1$. Then

$$\varphi(y_n) = \varphi(x) - \frac{\varphi(x_n)}{\phi(x_n)} = 1 - 1 = 0,$$

   so $y_n \in \ker(\varphi)$.

   Since $y_n \to x$, if $\ker(\varphi)$ closed, we would obtain $x \in \ker(\varphi)$. That is, $\phi(x) = 0$ but this contradicts $\varphi(x) = 1$.

4. $\ell_0 \subset \ell^\infty$, $\ell_0 := \{(x_n) \mid \lim_{n\to\infty} x_n = 0\}$ is not complemented.

5. Banach spaces for which every closed subspace is complemented is essentially a Hilbert space. This follows from an important theorem of Lindenstrauss and Tzafriri 1971 [37], where they prove that a Banach space is isomorphic to a Hilbert space iff every closed subspace is complemented.

We want to study a systematic way of decomposing spaces via classes of operators. To that end, we first note the following.

Proposition 2.57. *Let $P \in \mathscr{B}(\mathcal{B})$ be an idempotent (i.e. $P^2 = P$). Then $P\mathcal{B}$ and $(I - P)\mathcal{B}$ are closed and complementary, i.e.*

$$\mathcal{B} = P\mathcal{B} \oplus (I - P)\mathcal{B}\,.$$

Proof. a) $\mathcal{B} = P\mathcal{B} + (I - P)\mathcal{B}$.

Fix $b \in \mathcal{B}$, then trivially, $b = Pb + (I - P)b \in P\mathcal{B} + (I - P)\mathcal{B}$.

b) $P\mathcal{B}$ and $(I - P)\mathcal{B}$ are closed.

Let $x_n \in P\mathcal{B}$, $x_n \to x$. Then there are $\tilde{x}_n$ s.t. $x_n = P\tilde{x}_n$. Then

$$Px_n = P^2\tilde{x}_n = P\tilde{x}_n = x_n\,. \qquad \square$$

Since P is bounded, we have that $Px_n \to Px$ from the fact that $x_n \to x$. As we have just shown $x_n = Px_n$ and therefore, we have that $x_n \to Px$. Limits are unique in a Hausdorff space, and hence, $x = Px \in P\mathcal{B}$.

The corresponding argument for $(I-P)\mathcal{B}$ comes from showing that the map $(I-P)$ is an idempotent and bounded. Therefore, the same argument works on replacing P by $(I - P)$.

c) $P\mathcal{B}$ and $(I - P)\mathcal{B}$ are complementary.

Let $x \in P\mathcal{B} \cap (I - P)\mathcal{B}$. Then, $x = Pb = (I - P)Pb = (P - P^2)b = 0\,.$

This proposition justifies the following definition.

Definition 2.58. Let $P \in \mathscr{B}(\mathcal{B})$ idempotent, i.e. $P^2 = P$. Then P is called a *projector* or *projection.*

We say that P projects to $P\mathcal{B}$ along $(I - P)\mathcal{B}$.

Remark 2.59. Here the boundedness of $P : \mathcal{B} \to \mathcal{B}' \subset \mathcal{B}$ was vital.

Combining Proposition 2.57 and the following, we are able to characterise all decompositions of Banach spaces in terms of projectors.

Proposition 2.60. *$\mathcal{B} = B_1 \oplus B_2$. Then there is a projection P to B_1 along B_2.*

Proof. Left as an exercise (Hint: closed graph theorem). □

Remark 2.61. When a projector P projects to $\mathcal{B}_1$ along $\mathcal{B}_2$, it is easy to check that $\ker P = (I - P)\mathcal{B} = \mathcal{B}_2$. For a projector, it is not enough to specify its range, but rather, both the range and kernel. More precisely, if $\mathcal{B}_1$ is a closed subspace complemented by $\mathcal{B}_2 \neq \mathcal{B}_3$, we have two distinct projections P_1 which projectors to $\mathcal{B}_1$ along $\mathcal{B}_2$ and P_2 which projects to $\mathcal{B}_1$ along $\mathcal{B}_3$. The projectors $P_1 \neq P_2$.

Proposition 2.62. *Let $\langle \mathcal{B}_1, \mathcal{B}_2 \rangle$ be reflexive. If $P \in \mathscr{B}(\mathcal{B}_2)$ is a projector, then the adjoint map $P^* \in \mathscr{B}(\mathcal{B}_1)$ is also a projector.*

Proof. Let $P^{*,\mathrm{can}} \in \mathscr{B}(\mathcal{B}_2^*)$ be the canonical adjoint. This exists and has domain $\mathrm{dom}(P^{*,\mathrm{can}}) = \mathcal{B}_2^*$ (this does not require reflexivity).

We have $\mathrm{dom}(P^*) = \Phi_1^{-1}\,\mathrm{dom}(P^{*,\mathrm{can}}) = \Phi_1^{-1}\mathcal{B}_2^* = \mathcal{B}_1$ by Proposition 2.48. Moreover, P^* is closed and on invoking the closed graph theorem, Theorem 2.27, we obtain $P^* \in \mathscr{B}(\mathcal{B}_1)$. Now,

$$\left\langle (P^*)^2 v, u \right\rangle = \langle P^* v, Pu \rangle = \left\langle v, P^2 u \right\rangle = \langle v, Pu \rangle = \langle P^* v, u \rangle \,,$$

so $(P^*)^2 = P^*$. □

Proposition 2.63. *Let $\langle \mathcal{B}_1, \mathcal{B}_2 \rangle$ be reflexive and $P : \mathcal{B}_2 \to \mathcal{B}_2$ a projection. Then $\langle P^*\mathcal{B}_1, P\mathcal{B}_2 \rangle$ is reflexive.*

Proof. It is a fact that any closed subspace of a reflexive space is reflexive. The rest is left as an instructive exercise in calculating with perfect pairings. □

Proposition 2.62 yields a decomposition of $\mathcal{B}_1$ when $\langle \mathcal{B}_1, \mathcal{B}_2 \rangle$ is reflexive, and so together, we obtain the following decompositions:

$$\begin{aligned} \mathcal{B}_2 &= P\mathcal{B}_2 \oplus (I - P)\mathcal{B}_2 \,, \\ \mathcal{B}_1 &= P^*\mathcal{B}_1 \oplus (I - P^*)\mathcal{B}_1 \,, \end{aligned}$$

Proposition 2.63 then asserts that the first summand (and also the second summand) are, in fact, a reflexive perfect pairing. From this, it is easy to see that

$$PB_2 = ((I - P^*)B_1)^{\perp, \langle \mathcal{B}_1, \mathcal{B}_2 \rangle} \,,$$

and similarly for the other spaces.

3 Differential operators and their extensions

3.1 Density of $\mathrm{C}^\infty_{\mathrm{cc}}(M,E)$ in $\mathrm{L}^p(M,E)$

Lemma 3.1. *Let (M',μ) be a measured manifold in the usual sense (i.e. $\partial M' = \varnothing$ and $\mu \in |\Lambda|^+ M$). Let $E \to M'$ a vector bundle with metric h.*

Then $\mathrm{C}^\infty_{\mathrm{c}}(M',E)$ is dense in $\mathrm{L}^p(M',E)$ for $p \in [1,\infty)$.

Proof. a) First note that we can find a countable open covering $\mathscr{C} := \{(U_j,\psi_j)\}$ s.t. $E|_{U_j}$ is trivial and for which there are $V_j \subset U_j$ open and $\bar{V}_j \subset U_j$ is compact with $M' = \bigcup_j V_j$.

b) Let $\mathscr{P} := \{\eta_j\}$ be a smooth partition of unity subordinate to $\mathscr{C}$.

c) Let $\mathrm{L}^p_{\mathrm{c}}(M',E) := \{u \in \mathrm{L}^p(M',E) \mid \mathrm{spt}(u) \text{ compact}\}$. Then, $\mathrm{L}^p_{\mathrm{c}}(M',E)$ is dense in $\mathrm{L}^p(M',E)$. Proof: Exercise.

d) Let $u \in \mathrm{L}^p_{\mathrm{c}}(M',E)$. Then there is $u_n \in \mathrm{C}^\infty_{\mathrm{c}}(M',E)$ s.t. $u_n \to u$ in $\mathrm{L}^p(M',E)$, $p \in [1,\infty)$.

Fix $u \in \mathrm{L}^p_{\mathrm{c}}(M',E)$. Then there exists an $N > 0$ s.t.

$$u = \sum_{i=1}^{N} (\eta_i u)\,.$$

Let ϱ be the standard symmetric mollifier on $\mathbb{R}^n$, i.e.

$$\varrho(x) = \begin{cases} c_n \mathrm{e}^{\frac{1}{1-|x|^2_{\mathbb{R}^n}}} & \text{if } |x| \leq 1\,, \\ 0 & \text{if } |x| \geq 0\,, \end{cases}$$

where c_n is a normalisation constant s.t. $\int_{\mathbb{R}^n} \varrho \, \mathrm{d}\mathscr{L} = 1$.

Fix $\varepsilon > 0$ to be chosen later. Let $u_i := \eta_i u$ and note that $\mathrm{spt}(u_i) \subset V_i$.

Choose $\delta_i := \delta_i(\varepsilon) > 0$ s.t.

$$\left\| u_i \circ \psi_i^{-1} - \left(\eta_i \circ \psi_i^{-1}\right) \varrho^{\delta_i(\varepsilon)} * \underbrace{\left(u_i \circ \psi_i^{-1}\right)}_{\in \mathrm{L}^p(\psi_i(U_i),E)} \right\|_{\mathrm{L}^p(\psi_i(U_i),E)} < \frac{\varepsilon}{2N}\,, \tag{3.1}$$

where $\varrho^\delta(x) := \delta^{-n}\varrho(\delta^{-1}x)$ and

$$\big(\varrho^\delta * f\big)(x) = \left(\int \varrho^\delta(x) f^\alpha(x-y)\, \mathrm{d}\mathscr{L}(x)\right) e_\alpha\,,$$

where e_α is an orthonormal frame for E in U.

We remark that mollifiers converge against the Lebesgue measure and our integrals are against μ. Therefore, to obtain the estimate (3.1), we use the fact that μ is smooth and positive, and therefore, inside V_i there is a $c_i \geq 1$ s.t.

$$c_i^{-1} \leq \mu_\psi \leq c_i\,.$$

The constant c_i, as well as potential other constants in V_i are all absorbed, since we are free to choose δ_i as arbitrarily close to 0.

For $\varepsilon := \frac{1}{n}$ define

$$u^n := \sum_{i=1}^N \eta_i \Big(\varrho^{\delta_i(\frac{1}{n})} * \big(u_i \circ \psi_i^{-1}\big)\Big) \circ \psi_i \in \mathrm{C}^\infty_{\mathrm{c}}(M', E)\,.$$

Then, we note that

$$\|u - u^n\| \leq \sum_{i=1}^N \|u_i - \eta_i u^n\| \leq \sum_{i=1}^N \frac{1}{n}\frac{1}{2^N} < \frac{1}{n}\,,$$

where the penultimate inequality follows from (3.1).

That is, we have proved that $\mathrm{C}^\infty_{\mathrm{c}}(M', E)$ is dense in $\mathrm{L}^2_{\mathrm{c}}(M', E)$.

e) $\mathrm{C}^\infty_{\mathrm{c}}(M', E)$ is dense in $\mathrm{L}^p(M', E)$. Proof: Exercise. □

Proposition 3.2. *Let (M, μ) be a measured manifold with boundary. Then $\mathrm{C}^\infty_{\mathrm{cc}}(M, E)$ is dense in $\mathrm{L}^p(M, E)$ for $p \in [1, \infty)$.*

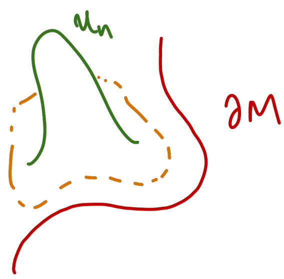
Proof. Clearly, $M' := \mathring{M}$ is a manifold without boundary and $\mu|_{M'} \in \mathrm{C}^\infty(|\Lambda|^+ M')$ and $\mathring{u} := u|_{\mathring{M}} \in L^p(\mathring{M}, E)$.

By Lemma 3.1, we obtain a sequence $u_n \in \mathrm{C}^\infty_{\mathrm{c}}(\mathring{M}, E)$ s.t. $\mathring{u}_n \to \mathring{u}$ in $\mathrm{L}^p(\mathring{M}, E)$. Define

$$u_n(x) := \begin{cases} \mathring{u}_n(x) & \text{if } x \in \mathring{M}\,, \\ 0 & \text{if } x \in \partial M\,, \end{cases}$$

then $u_n \in \mathrm{C}^\infty_{\mathrm{cc}}(M, E)$ and

$$\begin{aligned} \|u_n - u\|^p_{\mathrm{L}^p} &= \int_M |u_n - u|^p_n \,\mathrm{d}\mu \\ &= \int_{\mathring{M}} |u_n - u|^p_n \,\mathrm{d}\mu + \underbrace{\int_{\partial M} |u_n - u|^p_n \,\mathrm{d}\mu}_{=0} \\ &= \|\mathring{u}_n - \mathring{u}\|^p_{\mathrm{L}^p} \to 0\,. \end{aligned}$$



This completes the proof. □

3.2 Differential operators on vector bundles

Let $E, F \to M$ be vector bundles and $D : \mathrm{C}^\infty(M, E) \to \mathrm{C}^\infty(M, F)$ linear.

Definition 3.3. D is of order at most $m \in \mathbb{N}$, if in every trivialising chart (U, ψ), on fixing frames $\{e_\beta\}$ and $\{f_\vartheta\}$ for $E|_U$ and $F|_U$ respectively, there are smooth functions $x \mapsto A^{\alpha,\vartheta}_\beta(x)$ s.t.

$$Du|_U(x) = \sum_{|\alpha| \le m} A^{\alpha,\vartheta}_\beta(x) \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} u^\beta \right)(x) f_\vartheta(x) \,,$$

where $\alpha \in \mathbb{N}^n_+$ is a multi-index and $|\alpha| := \sum_{i=1}^n \alpha_i$.

Let $\mathrm{Diff}_m(E, F)$ be the set of such operators.

For $m > 0$, D is of order m if $D \in \mathrm{Diff}_m(E, F) \setminus \mathrm{Diff}_{m-1}(E, F)$.

We point out the following important properties regarding differential operators, each of which are readily verified from their definition.

Proposition 3.4. *I) Differential operators are local. That is, if $D \in \mathrm{Diff}_m(E, F)$, then $\operatorname{spt} Du \subset \operatorname{spt} u$.*

II) $\mathrm{Diff}_m(E, F) \subset \mathrm{Diff}_{m+1}(E, F)$.

To make the notation more compact, for $D \in \mathrm{Diff}_m(E, F)$ write $Du = \sum_{|\alpha|=m} A^\alpha(x) \frac{\partial^m}{\partial x^\alpha} u +$ l.o.t. where $A^\alpha(x) : E_x \to F_x$ is defined by $A^\alpha(x) e_\beta(x) := A^{\alpha,\vartheta}_\beta(x) f_\vartheta(x)$.

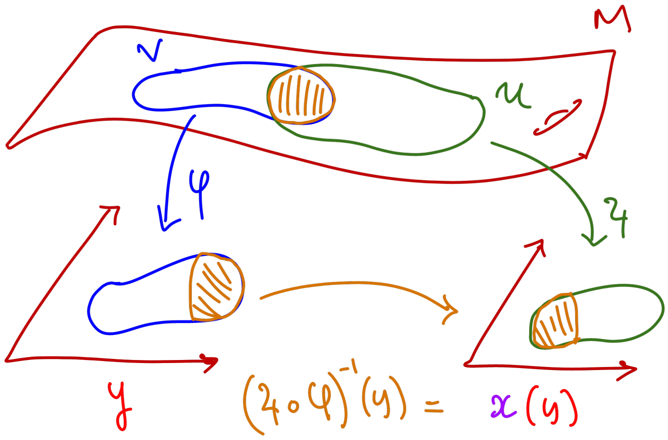
In what follows, we would like to extract a part of the operator which yields a section of a certain bundle. To explicitly compute this in a simple situation, let us restrict ourselves to $m = 1$. Inside a trivialising chart (U, ψ), for $x \in \psi(U)$,

$$D\big(u \circ \psi^{-1}\big)(x) = \sum_{i=1}^n \big(A^i \circ \psi^{-1}\big)(x) \frac{\partial}{\partial x^i} \big(u \circ \psi^{-1}\big)(x) + A_0(x) \big(u \circ \psi^{-1}\big)(x) \,. \tag{3.2}$$

Let (V, φ) be another trivialising chart near $\psi^{-1}(x)$. Then

$$D\big(u \circ \varphi^{-1}\big)(y) = \sum_{i=1}^n \big(B^i \circ \varphi^{-1}\big)(y) \frac{\partial}{\partial y^i} \big(u \circ \varphi^{-1}\big)(y) + B_0(y) \big(u \circ \varphi^{-1}\big)(y) \,. \tag{3.3}$$

We remark that, implicitly, we are using the same frame in the definition of A^i and B^i.



We then calculate the behaviour of the expression in (3.2) in coordinates x when we change to coordinates y:

$$\begin{aligned}\left(A^i \circ \psi^{-1}\right)(x)\frac{\partial}{\partial x^i}\left(u \circ \psi^{-1}\right)(x) &= \left(A^i \circ \psi^{-1} \circ \left(\psi \circ \varphi^{-1}\right)\right)(x)\frac{\partial}{\partial x^i}\left(u \circ \psi^{-1}\right)\left(\left(\psi \circ \varphi^{-1}\right)(y)\right) \\ &= \left(A^i \circ \varphi^{-1}\right)(y)\frac{\partial y^j}{\partial x^i}\frac{\partial}{y^j}\left(u \circ \varphi^{-1}\right)(y)\end{aligned}$$

Now define:

$$\sigma_D(x) := \sigma_{D,1}(x) := A^{i,\vartheta}_{\beta}(x) \otimes \left.\frac{\partial}{\partial x^i}\right|_x \otimes e^{\beta}(x) \otimes f_{\vartheta}(x)\,,$$

where $e^{\beta}(x)$ is the induced basis, i.e. $e^{\beta}(x)[e_{\alpha}(x)] = \delta^{\beta}_{\alpha}$.

From the transformation behaviour and considering the expression (3.3), we see that this defines a $\sigma_D \in \mathrm{C}^{\infty}(T^*M \otimes E^* \otimes F)$. More generally, we define the following.

Definition 3.5. Let $D \in \mathrm{Diff}_m(E, F)$. Let $\xi \in T^*_x M$ and $f \in \mathrm{C}^{\infty}(M)$ satisfying $f(x) = 0$ and $df(x) = \xi$. Fix $v \in E_x$ and let $\tilde{v} \in \mathrm{C}^{\infty}\left(U, E|_U\right)$ be an extension of v on an open neighbourhood U of x, i.e. $\tilde{v}(x) = v$. Define

$$\sigma_{D,m}(x,\xi)v := \frac{1}{m!}D(f^m \tilde{v})\,.$$

Remark 3.6. I) Using the bundle projection $\pi : T^*M \to M$ we can write

$$\sigma_{D,m} \in \mathrm{C}^{\infty}(M, \mathrm{Hom}(\pi^* E, \pi^* F))\,.$$

II) $\forall x \in M, \xi \in T^*_x M\colon \sigma_{D,m}(x,\xi) = 0$ happens iff $D \in \mathrm{Diff}_{m-1}(E, F)$.

Definition 3.7. The *principal symbol* σ_D is the $\sigma_{D,m}$ for the smallest m s.t. $D \in \mathrm{Diff}_m(E,F)$.

The more globally inclined might have given the following equivalent definitions.

Remark 3.8. Let $E, F \to M$ be vector bundles and $\nabla : C^\infty(M,E) \to C^\infty(M, T^*M \otimes E)$ a connection on E. We say $P \in \mathrm{Diff}_k(E,F)$ (linear differential operator of order $\leq k$) if

$$P = \sum_{j=0}^{k} A_j \circ \nabla^j$$

for suitable homomorphism fields $A_j \in C^\infty(M, \mathrm{Hom}(T^*M^{\otimes j} \otimes E, F))$ (note $\cong C^\infty(M, TM^{\otimes j} \otimes \mathrm{Hom}(E,F))$, they eat the vector entries and leave homomorphisms from E to F).

We say that P is of order k if $A_k \neq 0$. In that case we call the k-homogeneous section $\sigma_P := \big(\xi \mapsto A_k\big(\xi^{\otimes k}\big)\big) \in C^\infty(T^*M, \mathrm{Hom}(E,F))$ the principal symbol of P.

Exercise:

- For D of order 1, verify the expression

$$\sigma_D(x) = A_\beta^{i,\vartheta}(x) \otimes \left.\frac{\partial}{\partial x^i}\right|_x \otimes e^\beta(x) \otimes f_\vartheta(x)\,.$$

- For D of order 1, $f \in C^\infty(M)$ (we do not assume $f(x) = 0$)) and $v \in C^\infty(M,E)$, show

$$\sigma_D(x, df(x))(v(x)) = ([D, fI]v)(x)\,.$$

Proposition 3.9. *For $D_1 \in \mathrm{Diff}_\ell(E,F)$ and $D_2 \in \mathrm{Diff}_m(F,G)$ we have*

$$\sigma_{D_2 \circ D_1, \ell+m}(x,\xi) = \sigma_{D_2,m}(x,\xi) \circ \sigma_{D_1,\ell}(x,\xi)\,.$$

Definition 3.10. $D \in \mathrm{Diff}_m(E,F)$ is *elliptic* if $\sigma_D(x,\xi)$ is invertible for all $x \in M$ and $\xi \in T_x^*M \setminus \{0\}$.

Remark 3.11. If D is elliptic, then $\mathrm{rk}(E) = \mathrm{rk}(F)$.

Example 3.12.

- $d : C^\infty(M, \Lambda M) \to C^\infty(M, \Lambda M)$ exterior derivative, then

$$\sigma_d(x,\xi)\omega = d(f\tilde{\omega})(x) = (df \wedge \tilde{\omega})(x) + \underbrace{(f d\tilde{\omega})(x)}_{=0} = \xi \wedge \omega\,.$$

  The operator d is non-elliptic (Exercise).

- $\nabla : \mathrm{C}^\infty(M,E) \to \mathrm{C}^\infty(M, T^*M \otimes E)$ connection,

$$\sigma_\nabla(x,\xi)v = \nabla(f\tilde{v})\big|_x = (df \otimes \tilde{v} + f\nabla\tilde{v})\big|_x = \xi \otimes v\,.$$

  This is not elliptic, and indeed since $\mathrm{rk}(E) \neq \mathrm{rk}(T^*M \otimes E)$.

3.3 Formal adjoint

Let h^E, h^F be metrics on the vector bundles E and F respectively. We fix $D \in \mathrm{Diff}_m(E,F)$ be of order m and a positive density μ.

Proposition 3.13. *There is a unique $D^\dagger \in \mathrm{Diff}_m(F,E)$ satisfying*

$$\langle Du, v\rangle_{\mathrm{L}^2(M,F)} = \left\langle u, D^\dagger v\right\rangle_{\mathrm{L}^2(M,E)}$$

for all $u \in \mathrm{C}^\infty_{\mathrm{cc}}(M,E)$ and all $v \in \mathrm{C}^\infty_{\mathrm{cc}}(M,F)$.

Remark 3.14. D and $D^\dagger$ are automatically densely-defined in L^2.

Proof. a) Uniqueness.

Suppose $D_1^\dagger$ and $D_2^\dagger$ are both adjoints of D on the domains as given. Then

$$\begin{aligned}\left\langle u, \left(D_1^\dagger - D_2^\dagger\right)v\right\rangle &= \left\langle u, D_1^\dagger v\right\rangle - \left\langle u, D_2^\dagger v\right\rangle \\ &= \langle Du, v\rangle - \langle Du, v\rangle \\ &= 0\,.\end{aligned}$$

Recall that $\langle\cdot,\cdot\rangle : \mathrm{L}^2(M,E) \times \mathrm{L}^2(M,E) \to \mathbb{K}$ is a perfect pairing. Therefore

$$\begin{aligned}\left\|\left(D_1^\dagger - D_2^\dagger\right)v\right\|_{\mathrm{L}^2(M,E)} &\lesssim \sup_{\substack{u \in \mathrm{L}^2(M,E)\\ u \neq 0}} \frac{\left|\left\langle u, \left(D_1^\dagger - D_2^\dagger\right)v\right\rangle\right|}{\|u\|_{\mathrm{L}^2}} \\ &= \sup_{\substack{u \in \mathrm{C}^\infty_{\mathrm{cc}}(M,E)\\ u \neq 0}} \frac{\left|\left\langle u, \left(D_1^\dagger - D_2^\dagger\right)v\right\rangle\right|}{\|u\|_{\mathrm{L}^2}} \\ &= 0\,,\end{aligned}$$

where the second equality follows from Proposition 3.2, where we showed that $\mathrm{C}^\infty_{\mathrm{cc}}(M,E)$ is a dense subset of $\mathrm{L}^2(M,E)$. This shows $D_1^\dagger v = D_2^\dagger v$ almost-everywhere. In the existence of $D^\dagger$, we will show it is an operator with smooth coefficients, and hence will give equality everywhere.

b) Existence.

If $u \in \mathrm{C}^{\infty}_{\mathrm{cc}}(M, E)$ with $\mathrm{spt}(u) \subset U$ for a chart (U, ψ) corresponding to trivialisations for E, F and $\{e_\beta\}, \{f_\vartheta\}$ are orthonormal frames of E, F respectively, then

$$\begin{aligned} \langle Du, v\rangle_{\mathrm{L}^2(M,F)} &= \int h^F \left[\sum_{|\alpha|\leq m} A^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha} u, v \right] \mu_\psi \, \mathrm{d}\psi^* \mathscr{L} \\ &= \int h^E \left[u, \frac{1}{\mu_\psi} \sum_{|\alpha|\leq m} (-1)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x^\alpha} \Big(\mu_\psi (A^\alpha)^T v \Big) \right] \mu_\psi \, \mathrm{d}\psi^* \mathscr{L} \, . \end{aligned}$$

Here, we have used integration by parts and the divergence theorem. For general $u \in \mathrm{C}^{\infty}_{\mathrm{cc}}(M, E)$ patch these together using a partition of unity. A direct calculation, using the product rule, yields that

$$D^\dagger v := (-1)^m A^\alpha(x)^T \frac{\partial^m}{\partial x^\alpha} v + \text{l.o.t.} \, .$$

Note that the adjoint certainly contains metric information of h^E and h^F, but this is encoded in the definition of the the matrix A^α since it is defined it using orthonormal frames in each bundle. Since $x \mapsto A^\alpha(x)$ is smooth, it follows that $x \mapsto A^\alpha(x)^T$, and therefore, this upgrades uniqueness from almost-everywhere to everywhere. □

Definition 3.15. For $D \in \mathrm{Diff}_m(E, F)$ of order m, the unique $D^\dagger \in \mathrm{Diff}_m(F, E)$ is called the *formal adjoint.*

Proposition 3.16. *Let $D \in \mathrm{Diff}_m(E, F)$. Then $\sigma_{D^\dagger}(x, \xi) = (-1)^m \sigma_D(x, \xi)^\dagger$.*

Remark 3.17. For $E = F$ and $m = 1$ and $D = D^\dagger$ we have

$$\sigma_{D^\dagger}(x, \xi) = \sigma_D(x, \xi) = -\sigma_D(x, \xi)^\dagger \, ,$$

i.e. $\sigma_D(x, \xi)$ is skew-adjoint when we have a formally self-adjoint operator D. We define the symbol in this way for consistency with $\mathbb{K} = \mathbb{R}$. To understand this explicitly, consider the operator $D = \frac{d}{dx}$ on $\mathbb{R}$. There we have

$$\left\langle \frac{d}{dx} u, v \right\rangle = \left\langle u, -\frac{d}{dx} v \right\rangle \qquad \text{and} \qquad \sigma_{\frac{d}{dx}}(x, \xi) = \xi \, .$$

Clearly,

$$\sigma_{-\frac{d}{dx}}(x, \xi) v = -\frac{d}{dx}(fv)\big|_x = -\xi \, .$$

Also,

$$\sigma_{\left(\frac{d}{dx}\right)^\dagger}(x, \xi) = \sigma_{-\frac{d}{dx}}(x, \xi) = (-1) \sigma_{\frac{d}{dx}}(x, \xi)^\dagger = -\xi \, .$$

Therefore, we see in $\mathbb{K} = \mathbb{R}$ that including the (-1) in the definition is important.

If we are working on $\mathbb{K} = \mathbb{C}$, then we are able to obtain a formally self-adjoint version of $\frac{d}{dx}$ by instead considering $\imath \frac{d}{dx}$. I.e.,

$$\left\langle \imath \frac{d}{dx} u, v \right\rangle = \left\langle u, \imath \frac{d}{dx} v \right\rangle .$$

In some places in the literature, formally self-adjoint operators have Hermitian-symmetric symbols. More precisely, their symbols are obtained by taking the symbol we defined and multiplying it by $\imath$, but this essentially restricts us to only consider $\mathbb{C}$-valued bundles. It is important to always verify the precise definition of symbol as used by various authors when consulting the literature.

Example 3.18.

- Fix a Riemannian metric g on M. Define the cut product $\llcorner$ by

  $$g(\xi \llcorner \eta, \omega) = g(\eta, \xi \wedge \omega) \qquad \text{for } \xi, \omega, \eta \in \Lambda M.$$

  Note, this is not the interior product $\lrcorner$, which is the insertion of a vector into a covector, which does not require a metric. Here, we are defining the adjoint of $\wedge$ with respect to a metric g. Indeed, $\llcorner$ can be related to $\lrcorner$ via appropriately composing with musical isomorphisms or through computation in an orthonormal frame once a metric is fixed.

  Note that, for a frame $\{e_i\}$ for ΛM, $d\omega = e^i \wedge \nabla_{e_i} \omega$. Further assuming that e_i is synchronous at x, the following is readily verified:

  $$g(d\omega, \eta)\big|_x = g(\nabla_{e_i} \omega, e_i \llcorner \eta)\big|_x = -g(\omega, e^i \llcorner \nabla_{e_i} \eta)\big|_x + \operatorname{div} X_{g,\omega,\eta}(x) \,, \tag{3.4}$$

  where $X_{g,\omega,\eta}$ is a vectorfield depending on g, ω and η. Let $d_g^\dagger \in \operatorname{Diff}_1(\Lambda M)$ denote the formal adjoint of d. Then, from (3.4), since the divergence term integrates to zero and the formal adjoint is unique, we see that

  $$\sigma_{d_g^\dagger}(x, \xi)\omega = -\xi \llcorner \omega \,.$$

- $\Delta_g := d_g^\dagger d = -\operatorname{div} \circ \operatorname{grad}$, the Laplace-Beltrami operator on M. Then

  $$\sigma_{\Delta_g}(x, \xi) = -|\xi|_g^2 \,,$$

  this is an elliptic second-order operator.

- $D = d + d_g^\dagger \in \operatorname{Diff}_1(\Lambda M)$, the Hodge-Dirac operator, then

  $$\sigma_D(x, \xi)\omega = \xi \wedge \omega - \xi \llcorner \omega \,,$$

  this is an elliptic first-order operator.

3.4 The operator theoretic perspective of weak and distributional derivatives

Consider $\partial_j := \frac{\partial}{\partial x^j}$ on a smooth bounded domain $\Omega \subset \mathbb{R}^n$. In this classical setting, we often consider the weak derivative as well as the distributional derivative from a functional analytic perspective, usually framed in the context of the so-called 'test functions'. For the purposes of the general differential operators we discuss, where we are forced to frame the objects of interest from an operator theoretic point of view, we rephrase these classical notions in order to understand their operator theoretic essence.

For $u, v \in \mathrm{C}^\infty_{\mathrm{cc}}(\Omega)$, a direction calculation yields that

$$\langle \partial_j u, v\rangle = \int \partial_j u \bar{v}\ \mathrm{d}\mathscr{L} = -\int u \partial_j \bar{v}\ \mathrm{d}\mathscr{L} = \langle u, -\partial_j v\rangle\,. \tag{3.5}$$

In context of our discussion in the previous section, this precisely is saying that $-\partial_j$ is the formal adjoint of ∂_j.

We recall the construction of the distributional derivative. This is a map

$$(\partial_j)_{\mathrm{dist}} : \mathrm{L}^2(\Omega) \to (\mathrm{C}^\infty_{\mathrm{cc}})^{\mathrm{dual}} := \{\mathbb{C}\text{-linear functionals on } \mathrm{C}^\infty_{\mathrm{cc}}\}\,,$$

defined using (3.5) as

$$\big((\partial_j)_{\mathrm{dist}} u\big)[v] := \langle u, -\partial_j v\rangle\,,$$

for $u \in \mathrm{L}^2(\Omega)$.

Now let us recall the weak derivative, which is constructed as follows. Given a $u \in \mathrm{L}^2(\Omega)$ suppose there exists $f_u \in \mathrm{L}^2(\Omega)$ s.t.

$$\langle u, -\partial_j v\rangle = -\int u\partial_j \bar{v} = \int f_u \bar{v} = \langle f, v\rangle\,. \tag{3.6}$$

The weak derivative $(\partial_j)_{\mathrm{weak}}$ is then defined as

$$(\partial_j)_{\mathrm{weak}} u := f_u\,.$$

This is, in fact, an unbounded yet closed operator

$$(\partial_j)_{\mathrm{weak}} : \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)\,.$$

To see this, we need to understand the set of all possible $f_u \in \mathrm{L}^2(\Omega)$ for a given $u \in \mathrm{L}^2(\Omega)$ which satisfies the formula (3.6). Suppose such an f_u exists given a $u \in \mathrm{L}^2(\Omega)$ satisfying (3.6). Invoking the Cauchy-Schwarz inequality, we find that

$$|\langle u, -\partial_j v\rangle| = |\langle f_u, v\rangle| \leq \|f_u\|_{\mathrm{L}^2}\|v\|_{\mathrm{L}^2}$$

for all $v \in \mathrm{C}^\infty_{\mathrm{cc}}(\Omega)$.

Conversely suppose that $|\langle u, -\partial_j v\rangle| \leq c_u\|v\|_{\mathrm{L}^2}$ for all $v \in \mathrm{C}^\infty_{\mathrm{cc}}(\Omega)$. Since $\mathrm{C}^\infty_{\mathrm{cc}}(\Omega)$ is a dense subspace of $\mathrm{L}^2(\Omega)$, we can invoke the Riesz representation theorem and obtain a unique $f_u \in \mathrm{L}^2(\Omega)$ s.t.

$$\langle u, -\partial_j v\rangle = \langle f_u, v\rangle\,.$$

Let $(-\partial_j)_{\mathrm{c}} = \partial_j$ with domain

$$\mathrm{dom}\big((-\partial_j)_{\mathrm{c}}\big) := \mathrm{C}^\infty_{\mathrm{cc}}(\Omega)\,.$$

This is a densely-defined operator in $\mathrm{L}^2(\Omega)$ and hence, it admits a unique *maximal* adjoint $(\partial_j)_{\mathrm{c}}^*$. Its domain is precisely

$$\mathrm{dom}\big((-\partial_j)_{\mathrm{c}}^*\big) = \big\{u \in \mathrm{L}^2 \;\big|\; \exists f \in \mathrm{L}^2\, \forall v \in \mathrm{C}_{\mathrm{cc}}^\infty \colon \big\langle u, (-\partial_j)_{\mathrm{c}} v\big\rangle = \langle f, v\rangle\big\}$$

But this is precisely the domain of the operator $(\partial_j)_{\mathrm{weak}}$. In other words, the weak derivative, from an operator theory point of view, is none other than the L^2-adjoint of the *formal adjoint* of itself. I.e.,

$$(\partial_j)_{\mathrm{weak}} = (-\partial_j)_{\mathrm{c}}^*\,.$$

It is also worthwhile to consider the relationship of this weak derivative to the distributional derivative. We say that $T \in (\mathrm{C}_{\mathrm{cc}}^\infty(\Omega))^{\mathrm{dual}} \cap \mathrm{L}^2(\Omega)$ if there exists a unique $T_f \in \mathrm{L}^2$ satisfying

$$T[v] = \langle T_f, v\rangle$$

for all $v \in \mathrm{C}_{\mathrm{cc}}^\infty(\Omega)$. From this, we can see that

$$\mathrm{dom}((\partial_j)_{\mathrm{weak}}) = \mathrm{dom}((-\partial_j)_{\mathrm{c}}^*) = \big\{u \in \mathrm{L}^2(\Omega) \;\big|\; (\partial_j)_{\mathrm{dist}} u \in \mathrm{L}^2(\Omega)\big\}$$

That is,

$$\mathrm{dom}((\partial_j)_{\mathrm{weak}}) = (\partial_j)_{\mathrm{dist}}^{-1}(\mathrm{C}_{\mathrm{cc}}^\infty \cap \mathrm{L}^2(\Omega))\,.$$

3.5 Maximal and minimal extensions

Let $D_{\mathrm{cc}} := D$ with $\mathrm{dom}(D_{\mathrm{cc}}) := \mathrm{C}_{\mathrm{cc}}^\infty(M,E)$ and $D_{\mathrm{cc}}^\dagger := \big(D^\dagger\big)_{\mathrm{cc}}$, i.e. $D^\dagger$ with $\mathrm{dom}\big(D_{\mathrm{cc}}^\dagger\big) = \mathrm{C}_{\mathrm{cc}}^\infty(M,F)$. Recall that † denotes the formal adjoint, i.e.

$$\langle D_{\mathrm{cc}} u, v\rangle_{\mathrm{L}^2(M,F)} = \big\langle u, D_{\mathrm{cc}}^\dagger v\big\rangle_{\mathrm{L}^2(M,E)} \tag{3.7}$$

for all $u \in \mathrm{dom}(D_{\mathrm{cc}}) = \mathrm{C}_{\mathrm{cc}}^\infty(M,E)$ and all $v \in \mathrm{dom}\big(D_{\mathrm{cc}}^\dagger\big) = \mathrm{C}_{\mathrm{cc}}^\infty(M,F)$.

Definition 3.19. We set

$$D_{\max} := \big(D_{\mathrm{cc}}^\dagger\big)^* \qquad \text{and} \qquad D_{\max}^\dagger := \big(D^\dagger\big)_{\max} = (D_{\mathrm{cc}})^*\,.$$

To be completely explicit, the following is a precise description of the maximal extension:

$$\begin{aligned}\mathrm{dom}(D_{\max}) &= \big\{u \in \mathrm{L}^2(M,E) \;\big|\; \exists w \in \mathrm{L}^2(M,F), v \in \mathrm{C}_{\mathrm{cc}}^\infty(M,F) \colon \big\langle u, D_{\mathrm{cc}}^\dagger v\big\rangle = \langle w, v\rangle\big\}\\ &= \big\{u \in \mathrm{L}^2(M,E) \;\big|\; \mathrm{C}_{\mathrm{cc}}^\infty \cap \mathrm{L}^2(M,F) \ni v \mapsto \big\langle u, D_{\mathrm{cc}}^\dagger v\big\rangle \text{ is } \mathrm{L}^2\text{-continuous}\big\}\\ &= \big\{u \in \mathrm{L}^2(M,E) \;\big|\; \exists c = c(n,D)\, \forall v \in \mathrm{C}_{\mathrm{cc}}^\infty(M,F) \colon \big|\big\langle u, D_{\mathrm{cc}}^\dagger v\big\rangle\big| \le c\|v\|_{\mathrm{L}^2}\big\}\,.\end{aligned}$$

In this calculation, the penultimate equality is via the Riesz Representation Theorem.

Remark 3.20. It is worth emphasising that that the notation $D^\dagger_{\max}$ is not ambiguous. That is, we cannot have an interpretation $D^\dagger_{\max} = (D_{\max})^\dagger$. This is due to the fact that our notion of maximal extension is the unique adjoint obtained with respect to the formal adjoint operator restricted to $\mathrm{C}^\infty_{\mathrm{cc}}$. At the heart of this lies the fact that our notion of formal adjoint is a notion only defined for differential operators.

From (3.7), it is clear that $D_{\mathrm{cc}} \subset D_{\max}$ and $D^\dagger_{\mathrm{cc}} \subset D^\dagger_{\max}$ by the construction of the maximal extensions. That is, $\mathrm{dom}(D_{\mathrm{cc}}) \subset \mathrm{dom}(D_{\max})$ and $\mathrm{dom}\big(D^\dagger_{\mathrm{cc}}\big) \subset \mathrm{dom}\big(D^\dagger_{\max}\big)$ with $\mathrm{graph}(D_{\mathrm{cc}}) \subset \mathrm{graph}(D_{\max})$. Therefore, $\overline{\mathrm{graph}(D_{\mathrm{cc}})} \subset \mathrm{graph}(D_{\max})$ since the adjoint is always a closed operator. Moreover, we obtain that $\overline{\mathrm{graph}(D_{\mathrm{cc}})} = \mathrm{graph}\big(\overline{D_{\mathrm{cc}}}\big)$, i.e. D_{cc} is closable.

Definition 3.21. We set

$$D_{\min} := \overline{D_{\mathrm{cc}}} \qquad \text{and} \qquad D^\dagger_{\min} := \overline{D^\dagger_{\mathrm{cc}}}\,.$$

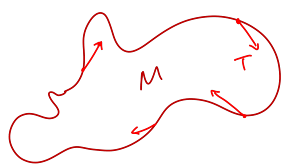
Exercise: Show that

$$D^*_{\min} := (D_{\min})^* = D^\dagger_{\max} \qquad \text{and} \qquad \big(D^\dagger_{\max}\big)^* = D_{\min}\,.$$

3.5.1 Boundary Value Problems

Let us fix an inward pointing vectorfield T along the boundary. A *boundary value problem* is typically phrased as follows: given $f \in \mathrm{MeasSect}(M,E)$, solve for $u \in \mathrm{MeasSect}(M,E)$ satisfying

$$\begin{aligned} Du &= f \\ (\partial^k_T u)|_{\partial M} &= f_k \end{aligned}$$



where $k = 0, 1, \ldots, m-1$ and $f_k \in \mathrm{MeasSect}(\partial M, E)$.

There are two important aspects to such a formulation that we need to identify for a systematic treatment of BVPs. The first is concerning the *boundary value* part, and the second is concerning the *problem* part.

1. *Boundary values*: where do the f_k live? Ideally, we would have some function space
$$B \subset \mathrm{MeasSect}(\partial M, E)$$
which induces an extension D_B satisfying
$$D_{\min} \subset D_B \subset D_{\max}\,,$$

where

$$\mathrm{dom}(D_B) = \left\{ u \in \mathrm{dom}(D_{\max}) \;\middle|\; \left(u|_{\partial M}, (\partial_T^1 u)|_{\partial M}, \ldots, (\partial_T^1 u)|_{\partial M} \right) \in B \right\}.$$

With this setup, we would then have $f_k \in B$. The subspace B would be a *boundary condition*.

2. *Problem*: Having fixed a boundary condition B, for which $f \in \mathrm{L}^2(M, E)$, possibly living in some subspace F, can we 'invert' D_B?

Our primary concern here would be to address the first aspect 1, regarding boundary conditions. Questions surrounding solvability, i.e., 2, is a whole different kettle of fish. It is beyond the scope of our considerations.

From here on, we will be attempting to formalise and rigourise 1. To do so, we need to pull the 'interior' operators, $D_{\min}$ and $D_{\max}$, to the boundary ∂M.

Recall that $D_{\min} \subset D_{\max}$, both are closed operators. Equivalently, the spaces $\mathrm{dom}(D_{\min})$ and $\mathrm{dom}(D_{\max})$ are Banach spaces with respect to the graph norm $\|\cdot\|_D$. Moreover, $\mathrm{dom}(D_{\min}) \subset \mathrm{dom}(D_{\max})$ is a closed subspace. This tells us that in order to consider extensions D_{ext} (possibly not closed) satisfying $D_{\min} \subset D_{\mathrm{ext}} \subset D_{\max}$, we can equivalently consider consider subspaces $\mathrm{dom}(D_{\min}) \subset X_{\mathrm{ext}} \subset \mathrm{dom}(D_{\max})$.

To rigourise this thinking, we note the following abstract result.

Proposition 3.22. *Let $\mathcal{B}_1, \mathcal{B}_2$ be Banach spaces and $\gamma : \mathcal{B}_1 \to \mathcal{B}_2$ a bounded surjection with kernel $B_0 := \ker(\gamma)$. Then, the following hold.*

I) *γ induces a Banach space isomorphism $\tilde{\gamma} : \mathcal{B}_1/B_0 \to \mathcal{B}_2$.*

II) *If $B_0 \subset S \subset \mathcal{B}$ is a subspace S, then $\gamma S \subset \mathcal{B}_2$ is a subspace. γS is closed if S is closed.*

III) *If $S' \subset \mathcal{B}_2$ is subspace, then $\gamma^{-1} S'$ is a subspace satisfying $B_0 \subset \gamma^{-1} S' \subset \mathcal{B}_1$. $\gamma^{-1} S'$ is closed if S' is closed.*

IV) *$S = \gamma^{-1}(\gamma S)$.*

Proof. a) Ad I). The induced map $\tilde{\gamma} : \mathcal{B}_1/B_0 \to \mathcal{B}_2$ is canonically given by $\tilde{\gamma}(u + B_0) = \gamma u$. The induced norm is

$$\|u + B_0\|_{\mathcal{B}_1/B_0} = \inf\left\{ \|u + b_0\|_{\mathcal{B}_1} \;\middle|\; b_0 \in B_0 \right\}.$$

For any $b_0 \in \mathcal{B}_0$, we have, using the boundedness of γ, that

$$\|\tilde{\gamma} u\| = \|\gamma(u + b_0)\| \lesssim \|u + b_0\|.$$

Therefore, on taking an infimum over $b_0 \in \mathcal{B}_0$ in this expression, we see that

$$\|\tilde{\gamma}(u + B_0)\|_{\mathcal{B}_2} \lesssim \|u + B_0\|_{\mathcal{B}_1/B_0}.$$

This shows that $\tilde{\gamma} : \mathcal{B}_1/B_0 \to \mathcal{B}_2$ is bounded. Since γ is a surjection, so is $\tilde{\gamma}$ and therefore, we conclude it is an open map from the open mapping theorem. In addition, by construction $\tilde{\gamma}$ is injective, and therefore it is a bijection so the fact that $\tilde{\gamma}$ is an open map yields that the inverse $\tilde{\gamma}^{-1} : \mathcal{B}_2 \to \mathcal{B}_1/\mathcal{B}_0$ is bounded.

b) The proofs of parts II) and III) are left as exercises.

c) Ad IV). $S \subset \gamma^{-1}(\gamma S)$ is clear. We show $\gamma^{-1}(\gamma S) \subset S$. Let $x \in \gamma^{-1}(\gamma S)$, this means $x \in \gamma^{-1}y$ with $y = \gamma s$ for a $s \in S$. Then $\gamma x = y = \gamma s$, implying $\gamma(x - s) = 0$, i.e. $x - s \in \mathcal{B}_0$. So we can write $x = (x - s) + s \in \mathcal{B}_0 + S \subset S + S = S$. □

Example 3.23. $B_0 = \mathrm{dom}(D_{\min})$, $\mathcal{B}_1 = \mathrm{dom}(D_{\max})$, $\mathcal{B}_2 = {}^{\mathrm{dom}(D_{\max})}\!\big/_{\mathrm{dom}(D_{\min})}$ and γ the quotient map.

So all subspaces and therefore extensions $D_{\min} \subset D_{\mathrm{ext}} \subset D_{\max}$ can be understood as subspaces of ${}^{\mathrm{dom}(D_{\max})}\!\big/_{\mathrm{dom}(D_{\min})}$. The operator D_{ext} is closed if and only if the subspace $\gamma D_{\mathrm{ext}} \subset {}^{\mathrm{dom}(D_{\max})}\!\big/_{\mathrm{dom}(D_{\min})}$ is closed.

3.5.2 Interior to boundary via the boundary restriction map

We continue to utilise Proposition 3.22 to obtain a complete understanding of boundary conditions. It is not hard to see that the spaces $\mathcal{B}_0$, $\mathcal{B}_1$ are automatically determined in this setup as $\mathrm{dom}(D_{\min})$ and $\mathrm{dom}(D_{\max})$ respectively. There is also a canonical map γ that is automatically determined in this setup. Consequently, $\mathcal{B}_2$ is also determined as the range of this map.

Having fixed an inward pointing $T \in \mathrm{C}^\infty(\partial M, TM)$ vectorfield along ∂M, we build the map γ by first considering the following map, the *boundary restriction map*:

$$\begin{aligned}\gamma_{\mathrm{c}} : \mathrm{C}_{\mathrm{c}}^\infty(M, E) &\to \bigoplus_{j=0}^{m-1} \mathrm{C}_{\mathrm{c}}^\infty(\partial M, E), \\ u &\mapsto \left(u|_{\partial M}, (\partial_T u)|_{\partial M}, \dots, (\partial_T^{m-1} u)|_{\partial M}\right).\end{aligned}$$

In order to formulate boundary conditions using Proposition 3.22, we desire (and more seriously, require) the following.

[Req 1] The map γ_{c} should be extended to γ, in some bounded manner, acting on the whole of $\mathrm{dom}(D_{\max})$.

[Req 2] We then want $\ker(\gamma) = \mathrm{dom}(D_{\min})$, which we expect to be automatic since

$$\gamma_{\mathrm{c}} \, \mathrm{dom}(D_{\mathrm{cc}}) = \gamma_{\mathrm{c}} \mathrm{C}_{\mathrm{cc}}^\infty(M, E) = 0$$

and $D_{\min} = \overline{D_{\mathrm{cc}}}$.

Having such a γ satisfying [Req 1] and [Req 2], we define the Czech space

$$\check{\mathrm{H}}(D) := \gamma \, \mathrm{dom}(D_{\max}) \subset \mathrm{MeasSect}(\partial M, E).$$

We topologise this space by pulling across the topology of ${}^{\mathrm{dom}(D_{\max})}\!\big/_{\mathrm{dom}(D_{\min})}$ via γ. Then

$$\gamma : {}^{\mathrm{dom}(D_{\max})}\!\big/_{\mathrm{dom}(D_{\min})} \to \check{\mathrm{H}}(D)$$

is a bounded surjection with $\ker(\gamma) = \mathrm{dom}(D_{\min})$.

At this point, we note that we have accomplished the goal of 'pulling' the interior problem to the boundary. However, all we have done is to rephrase ${}^{\mathrm{dom}(D_{\max})}\!\diagup_{\mathrm{dom}(D_{\min})}$ as $\check{\mathrm{H}}(D)$. In order to truly understanding ${}^{\mathrm{dom}(D_{\max})}\!\diagup_{\mathrm{dom}(D_{\min})}$ from data only emerging from the boundary would require the following desire to be fulfilled.

[Req 3] Describe $\check{\mathrm{H}}(D)$, and in particular its topology, from data 'only' living on ∂M.

Having [Req 1], [Req 2] and [Req 3], and at the risk of repetition and labouring the point, let us explicitly state the way in which we can understand extensions D_{ext} in terms of $\check{\mathrm{H}}(D)$. Invoking Proposition 3.22, we have

I) For a subspace $B \subset \check{\mathrm{H}}(D)$ define

$$\begin{aligned} \mathrm{dom}(D_B) &:= \gamma^{-1} B \subset \mathrm{dom}(D_{\max}), \\ D_B u &:= D_{\max} u \qquad \text{for } u \in \mathrm{dom}(D_B). \end{aligned}$$

Then we have $D_{\min} \subset D_B \subset D_{\max}$ and D_B is closed iff B is closed.

II) If $D_{\min} \subset D_{\mathrm{ext}} \subset D_{\max}$, define

$$B_{\mathrm{ext}} := \gamma\, \mathrm{dom}(D_{\mathrm{ext}}) \subset \check{\mathrm{H}}(D),$$

then we have $D_{B_{\mathrm{ext}}} = D_{\mathrm{ext}}$ by 3.22 IV). We also have

$$D_{\mathrm{ext}} \text{ closed} \quad \Leftrightarrow \quad B_{\mathrm{ext}} \text{ closed}.$$

This leads us to the following definitions.

Definition 3.24. We call a closed subspace $B \subset \check{\mathrm{H}}(D)$ a *boundary condition* for D.

We call a general subspace $B \subset \check{\mathrm{H}}(D)$ (not necessarily closed) a *generalised boundary condition.*

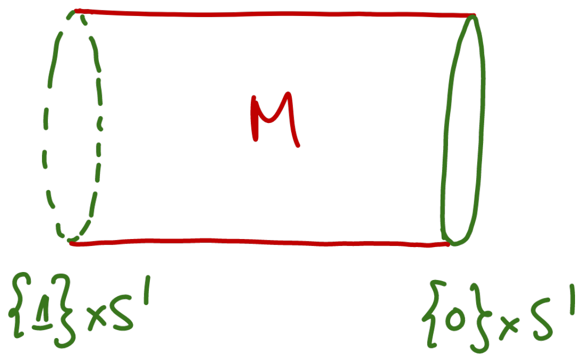
A boundary condition is then a closed generalised boundary condition.

In either case, the associated operator D_B is given by

$$\begin{aligned} \mathrm{dom}(D_B) &= \gamma^{-1} B, \\ D_B u &= D_{\max} u \qquad \text{for } u \in \mathrm{dom}(D_B). \end{aligned}$$

Obtaining the desirable conditions [Req 1], [Req 2] and [Req 3] requires us to understand more sophisticated technical machinery. In addition, there are certainly a number of obstacles, geometric and otherwise. The viability of these desires, in terms of examples, are pointed out below.

1. Ad [Req 1]. Let $N := \mathcal{S}^1$, $M := [0,1) \times \mathcal{S}^1$, $g := dt \otimes dt + g_{\mathcal{S}^1}$.



   M is a manifold with boundary $\partial M = \{0\} \times \mathcal{S}^1$. Consider the Hodge-Dirac operator $D_{\mathrm{H}} = d + d_g^{\dagger}$, then $\gamma u = u|_{\partial M}$. But we cannot have a surjection $\mathrm{dom}(D_{\max}) \to \left\{ u|_{\partial M} \,\middle|\, u \in \mathrm{dom}(D_{\max}) \right\}$. This arises from the 'incompleteness' of the metric near the end $\{1\} \times \mathcal{S}^1$. Intuitively, the issue is that a section can take a nonzero value near the end $\{1\} \times \mathcal{S}^1$, which cannot be seen by the boundary restriction map. In this situation, we really need to 'add' the topological boundary $\{1\} \times \mathcal{S}^1$ and consider the map $u|_{\{0,1\}\times\mathcal{S}^1}$.

   This is a geometric concern, not a topological one. For instance, take the space $M' = [0,\infty) \times \mathcal{S}^1$ with the same metric $g = dt \otimes dt + g_{\mathcal{S}^1}$. In this case, it is possible to control ${\mathrm{dom}(D_{\max})}/{\mathrm{dom}(D_{\min})}$ purely in terms of the map $u \mapsto u|_{0\times\mathcal{S}^1}$. However, it is easy to see that M and M' are diffeomorphic to each other. It is their metric structures that are different.

   A condition to prevent this situation is to ask for $\mathrm{C}^\infty_{\mathrm{c}}(M,E)$ to be dense in $\mathrm{dom}(D_{\max})$.

2. Ad [Req 2]. Also $\ker(\gamma) = \mathrm{dom}(D_{\min})$ follows from and requires the density of $\mathrm{C}^\infty_{\mathrm{cc}}(M,E)$ in $\mathrm{dom}(D_{\max})$.

3. Ad [Req 3]. This is possible to a large extent for general-order operators, and even more so for first-order operators. A significant portion of this text will be dedicated to this point.

Exercise 3.25. If M is compact with boundary, then $\mathrm{C}^\infty_{\mathrm{c}}(M,E)$ is dense in $\mathrm{dom}(D_{\max})$.

Let us finish this section with the following result, which is a consequence of an important theorem due to Chernoff [16]. It illustrates the way in which completeness of a metric guarantees favourable conditions on powers of first-order operators.

Theorem 3.26. *Suppose (M,g) is a complete Riemannian manifold without boundary and $D \in \mathrm{Diff}_1(E)$ is formally self-adjoint. Furthermore, suppose that there exists a constant $C < \infty$ such that*

$$|\sigma_D(x,\xi)| \leq C|\xi|_{g(x)} ,$$

where the first is the operator norm. Then,

$$\operatorname{dom}\big((D^k)_{\max}\big)\big/\operatorname{dom}\big((D^k)_{\min}\big) = 0\,.$$

In this case, the operator $\overline{D^k_{\mathrm{cc}}}$ is self-adjoint. In the literature, when a formally self-adjoint differential operator has a unique closure that is self-adjoint as in this situation, it is sometimes said to be essentially self-adjoint.

3.6 Sobolev spaces on vector bundles

To avoid notational and technical complications, from here on, we will restrict our attention to $\mathbb{K} = \mathbb{C}$. However, as a remark, let us point out that many results pertaining to the case $\mathbb{K} = \mathbb{R}$ can be obtained through complexification and then restriction of the results in the $\mathbb{C}$ case. Moreover, our discussion of Sobolev spaces will be constrained purely to the L^2 context, although, a similar construction can be carried out in L^p.

Let us first start by recalling the L^2 Sobolev spaces for *systems* over $\mathbb{R}^n$, i.e., the case $M := \mathbb{R}^n$ and $E := \mathbb{R}^n \times \mathbb{C}^N$. For $k \in \mathbb{N}$, these spaces are defined as

$$\mathrm{H}^k(\mathbb{R}^n, \mathbb{C}^N) := \Big\{ u \in \mathrm{L}^2\big(\mathbb{R}^n, \mathbb{C}^N\big) \;\Big|\; \forall j \leq n\colon\, \big(\nabla^j\big)_{\max} u \in \mathrm{L}^2\Big(\mathbb{R}^n, \mathbb{C}^{n^j N}\Big) \Big\}\,,$$

with norm

$$\|u\|^2_{\mathrm{H}^k} = \sum_{j=1}^{k} \big\|\nabla^j u\big\|^2_{\mathrm{L}^2} + \|u\|^2_{\mathrm{L}^2}\,.$$

Here, the powers of 'gradient' operators ∇^j of $\mathbb{R}^n$ and $\mathbb{C}^N$ are

$$\nabla^j u = \sum_{|\alpha|=j} \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} u^i \right) e^\alpha \otimes e_i\,,$$

where e_i is the canonical basis of $\mathbb{C}^N$ and e^α the basis for $\mathbb{C}^{n^j}$ written in multi-index form. It is easily seen that these operators are obtained from the Levi-Civita connection of the standard Euclidean metric, as well as the canonical compatible metric on $\mathbb{C}^N$ with respect to the standard Hermitian product there. Therefore, within this definition, the implicit flat geometry of $\mathbb{R}^n$ and $\mathbb{C}^N$ is hidden.

Now let M be any manifold with boundary and $E \to M$ a vector bundle. Having identified the underlying geometry of the classical Sobolev scales, the additional data required to formulate Sobolev spaces on E is clear. We require a metric h^E and a connection ∇^E on E, and a metric g on M and a connection ∇^{T^*M} on T^*M. Moreover, since we require these spaces to sit inside $\mathrm{L}^2(M, E)$, we require a smooth positive density μ on M. It is worth emphasising that we do not assume compatibility between the respective metrics and connections.

For $u \in \mathrm{C}^\infty(E)$ we have $\nabla^E u \in \mathrm{C}^\infty(T^*M \otimes E)$ and recall the induced connection $\nabla^{T^*M \otimes E}$ on $T^*M \otimes E$ obtained by enforcing the Leibniz rule for every direction $X \in \mathrm{C}^\infty(M, TM)$, i.e.

$$\nabla_X^{T^*M \otimes E}(u \otimes v) = \nabla_X^{T^*M} u \otimes v + u \otimes \nabla_X^E v\,.$$

By induction get $\nabla^{T^{(j,0)}M \otimes E}$.

In what is to follow, to emphasise both connections in the notation, we define

$$\left(\nabla^E, \nabla^{T^*M}\right)^j : \mathrm{C}^\infty(E) \to \mathrm{C}^\infty\left(T^*M^{\otimes j} \otimes E\right)$$

by

$$\begin{aligned} \left(\nabla^E, \nabla^{T^*M}\right)^1 &:= \nabla^E\,, \\ \left(\nabla^E, \nabla^{T^*M}\right)^{j+1} &:= \nabla^{T^*M^{\otimes j} \otimes E} \circ \left(\nabla^E, \nabla^{T^*M}\right)^j\,. \end{aligned}$$

Then $\left(\nabla^E, \nabla^{T^*M}\right)^j \in \mathrm{Diff}_j\left(E, T^{(j,0)}M \otimes E\right)$.

Definition 3.27. We define the following *Sobolev spaces*:

$$\mathrm{H}^k\left(M, E; h^E, g, \nabla^E, \nabla^{T^*M}, \mu\right) := \bigcap_{j=1}^k \mathrm{dom}\left(\left(\left(\nabla^E, \nabla^{T^*M}\right)^j\right)_{\max}\right)$$

with norm

$$\|u\|_{\mathrm{H}^k}^2 := \|u\|_{\mathrm{L}^2(M,E)}^2 + \sum_{j=1}^k \left\| \left(\nabla^E, \nabla^{T^*M}\right)^j u \right\|_{\mathrm{L}^2\left(M, T^{(j,0)}M \otimes E\right)}^2,$$

and

$$\mathrm{H}_0^k\left(M, E; h^E, g, \nabla^E, \nabla^{T^*M}, \mu\right) := \overline{\mathrm{C}_{\mathrm{cc}}^\infty(E)}^{\|\cdot\|_{\mathrm{H}^k}}\,.$$

It is clear from inspection that

$$\mathrm{H}_0^k\left(M, E; h^E, g, \nabla^E, \nabla^{T^*M}, \mu\right) = \bigcap_{j=1}^k \mathrm{dom}\left(\left(\left(\nabla^E, \nabla^{T^*M}\right)^j\right)_{\min}\right).$$

Furthermore, it is immediate from construction that:

$$\mathrm{H}^{k+1}\left(M, E; h^E, g, \nabla^E, \nabla^{T^*M}, \mu\right) \subset \mathrm{H}^k\left(M, E; h^E, g, \nabla^E, \nabla^{T^*M}, \mu\right)$$

and

$$\mathrm{H}_0^{k+1}\left(M, E; h^E, g, \nabla^E, \nabla^{T^*M}, \mu\right) \subset \mathrm{H}_0^k\left(M, E; h^E, g, \nabla^E, \nabla^{T^*M}, \mu\right).$$

In practice, all the data which we have identified explicitly is fixed, and therefore, to make the notation less verbose, these spaces are simply denoted by $\mathrm{H}^k(M, E)$ and $\mathrm{H}_0^k(M, E)$.

Often, ∇^{T^*M} is induced by the Levi-Civita connection to g. If the connection on T^*M is clear, the shorthand $\left(\nabla^E\right)^j := \left(\nabla^E, \nabla^{T^*M}\right)^j$ is used.

Example 3.28. 1. (M,g) Riemannian manifold, $E := T^{(p,q)}M$, ∇^{T^*M} induced from Levi-Civita, $\nabla^E = \nabla^{T^{(p,q)}M}$ induced connection, and $d\mu = d\,\mathrm{vol}_g$ is the induced volume measure from g. Then

$$\mathrm{H}^k\big(M, T^{(p,q)}M\big) = \big\{u \in \mathrm{L}^2\big(M, T^{(p,q)}M\big) \mid \forall j \le n \colon \big(\nabla^j u\big)_{\max} \in \mathrm{L}^2\big(M, T^{(p+j,q)}M\big)\big\}\,.$$

2. For $p = q = 0$, i.e. for $E = M \times \mathbb{C}$ whose sections are functions, write $\mathrm{H}^k(M)$ and $\mathrm{H}^k_0(M)$. These spaces are studied in [27] by Hebey.

3. Let us now explicitly consider the way that geometry manifests itself in Sobolev spaces as they have been defined.

   a) If $M = \mathbb{R}^n$, $E = \mathbb{R}^n \times \mathbb{C}$, then

   $$\mathrm{H}^k(\mathbb{R}^n) = \mathrm{H}^k_0(\mathbb{R}^n)\,.$$

   It is a classical fact, which can be easily seen using the Fourier transform, that

   $$\mathrm{H}^{2k}(\mathbb{R}^n) = \mathrm{dom}\big(\nabla^k_{\max}\big) = \mathrm{dom}\big(\Delta^k_{\max}\big) = \mathrm{dom}\big(\Delta^k_{\min}\big)\,.$$

   b) Now let (M,g) be a complete Riemannian manifold without boundary. Then it is easily verified, by using the fact that completeness is equivalent to the precompactness of arbitrarily large geodesic balls, that $\mathrm{H}^1_0(M) = \mathrm{H}^1(M)$ (on functions).

   Do we have $\mathrm{H}^2(M) = \mathrm{H}^2_0(M)$? In general, without additional assumptions, this is unlikely to be true. Understanding when we have equality is an ongoing research question.

   Intuitively, from a geometric perspective, we expect curvature at 'infinity' to interfere with such an equality. Analytically, this can be seen by understanding why $\mathrm{H}^1_0(M) = \mathrm{H}^1(M)$. The crux of the matter here is that, by using the precompactness of arbitrarily large geodesic balls, we are able to find a smooth cutoff functions with appropriate control on its gradient. This allows for a $\mathrm{H}^1(M)$ function to be approximated by a sequence of compactly supported functions, which can now be smoothed and remain compactly supported. To replicate this argument for a $\mathrm{H}^2_0(M)$ function would require controlling the second derivative of the smooth cutoff, and it is here where we anticipate curvature considerations near infinity to dominate.

   Let us now try and understand a sufficient curvature condition to obtain this equality. Through the results in [16], we obtain

   $$\mathrm{dom}\big(\Delta^k_{\max}\big) \big/ \mathrm{dom}\big(\Delta^k_{\min}\big) = 0\,.$$

   The case that is of interest to us in this example, $k = 1$, dates even further back to the works of Gaffney [20], Wolf [50] and Cordes [17].

Another useful gadget is Bochner's formula:

$$\frac{1}{2}\Delta\left(|\nabla u|_g^2\right) = g(\nabla\Delta u, \nabla u) + \left|\nabla^2 u\right|^2 + \mathrm{Ric}_g(\nabla u, \nabla u)\,.$$

Now, if there is an $\eta \in \mathbb{R}$ with $\mathrm{Ric}_g \geq \eta g$ (as bilinear forms), which in the literature is often called a *uniform lower bound on Ricci curvature*,

$$\mathrm{H}^2_0(M) = \mathrm{H}^2(M)\,.$$

Exercise 3.29. Using these ingredients, prove this equality. Hint: since we are in the setting where ∇ is the Levi-Civita connection, compare ∇^2 to Δ and study their domain containments.

Our goal is to now build 'local' versions of Sobolev spaces. In order to do this, we first present the following perturbation result. Note that by $\mathrm{dom}(D_{\max}; h_i, \mu_i)$, we mean the domain of the maximal extension of D with respect to a metric h_i and measure μ_i.

Proposition 3.30. *Let $D \in \mathrm{Diff}_m(E)$ and let h_1, h_2 be metrics on E and μ_1, μ_2 smooth positive densities on M. Let*

$$\langle u, v\rangle_i := \int_M h_i[u, v]\, \mathrm{d}\mu_i\,.$$

If $U \subset M$ open, $\overline{U} \subset M$ compact, $\xi \in \mathrm{MeasSect}(E)$ with $\mathrm{spt}(\xi) \subset U$, then the following hold.

I) $\xi \in \mathrm{L}^2(M, E; h_1, \mu_1) \quad \Leftrightarrow \quad \xi \in \mathrm{L}^2(M, E; h_2, \mu_2)$.

II) $\xi \in \mathrm{dom}(D_{\max}; h_1, \mu_1) \quad \Leftrightarrow \quad \xi \in \mathrm{dom}(D_{\max}; h_2, \mu_2)$.

Moreover, there is a $C = C(U, h_i, \mu_i) \geq 1$ s.t.

$$C^{-1}|D\xi|_{h_1} \leq |D\xi|_{h_2} \leq C|D\xi|_{h_1}\,.$$

Proof. a) Ad I). Since h_i, μ_i are smooth, there is a $C \geq 1$ s.t. for all $x \in \overline{U}$ we have

$$C^{-1}|v|_{h_1(x)} \leq |v|_{h_2(x)} \leq C|v|_{h_1(x)}$$

for all $v \in E_x$, and

$$\frac{d\mu_1}{d\mu_2}(x) \leq C \qquad \text{and} \qquad \frac{d\mu_2}{d\mu_1}(x) \leq C\,.$$

Therefore, for ξ as in hypothesis, we have I).

b) Ad I). First, note that by fibrewise considerations, we can find $B \in \mathrm{End}(E)$ s.t. $h_1[u, v] = h_2[Bu, v]$ for all $u, v \in E_x$. Moreover, using the precompactness of U, we can find V open, $\overline{V}$ compact, $\overline{U} \subset V$. As in the proof of I), we can find $C_2 = C_2(V)$ such that inside $\overline{V}$,

$$|B|_{h_i} \leq C_2 \qquad \text{and} \qquad \frac{d\mu_1}{d\mu_2} + \frac{d\mu_2}{d\mu_1} \leq C_2 \,.$$

Let $D_i^\dagger$ be the formal adjoint w.r.t. h_i. Suppose $\xi \in \mathrm{dom}(D_{\max}, h_1, \mu_1)$ and $w \in \mathrm{C}^\infty_{\mathrm{cc}}(M, E)$.

Then, setting $\tilde{B} := B\frac{d\mu_1}{d\mu_2}$, we obtain

$$\left\langle \xi, D^{\dagger,1} w\right\rangle_1 = \langle D\xi, w\rangle_1 = \left\langle \tilde{B} D\xi, w\right\rangle_2 = \left\langle D\xi, \tilde{B}^* w\right\rangle_2 = \left\langle \xi, D^{\dagger,2}\tilde{B}^* w\right\rangle_2 .$$

Let $\chi \in \mathrm{C}^\infty(M, [0,1])$ such that $\chi|_U = 1$ and $\chi|_{M\setminus V} = 0$. Given $\tilde{w} \in \mathrm{C}^\infty_{\mathrm{cc}}(M, E)$, $w := \left(\tilde{B}^*\right)^{-1} \tilde{w} \in \mathrm{C}^\infty_{\mathrm{cc}}(M, E)$ we have

$$\left\langle \xi, D^{\dagger,2}\tilde{w}\right\rangle_2 = \left\langle \chi\xi, D^{\dagger,1}\left(\tilde{B}^*\right)^{-1} w\right\rangle_1 .$$

Therefore,

$$\begin{aligned} \left|\left\langle \xi, D^{\dagger,2}\tilde{\omega}\right\rangle_2\right| \leq C_2 \left\| \left(\tilde{B}^*\right)^{-1}\tilde{w}\right\|_{\mathrm{L}^2(V,E,h_1,\mu_1)} &\lesssim \|\tilde{w}\|_{\mathrm{L}^2(V,E,h_1,\mu_1)} \\ &\lesssim \|\tilde{w}\|_{\mathrm{L}^2(V,E,h_2,\mu_2)} \lesssim \|\tilde{w}\|_{\mathrm{L}^2(M,E,h_2,\mu_2)} . \end{aligned}$$

Therefore, $\xi \in \mathrm{dom}(D_{\max}, h_2, \mu_2)$. The reverse argument follows mutatis mutandis.

The estimate in the conclusion is then immediate. □

Corollary 3.31. *Let $U \subset M$ open with $\overline{U} \subset M$ compact. Suppose that $g_i, h_i^E, \nabla_i^E, \nabla_i^{T^*M}, \mu_i$ for $i = 1, 2$ are two sets of metrics, connections, and smooth positive densities. Then $u \in \mathrm{L}^2\big(M, E, h_1^E\big)$ with $\mathrm{spt}(u) \subset U$ satisfies*

$$u \in \mathrm{H}^k\big(M, E, g_1, h_1^E, \nabla_1^E, \nabla_1^{T^*M}, \mu_1\big) \quad \Leftrightarrow \quad u \in \mathrm{H}^k\big(M, E, g_2, h_2^E, \nabla_2^E, \nabla_1^{T^*M}, \mu_2\big)$$

and there is a constant $C_k \geq 1$ s.t.

$$\left|\big(\nabla_1^E, \nabla_1^{T^*M}\big)^k u\right|_{(g_1, h_1^E)} \simeq_{C_k} \left|\big(\nabla_2^E, \nabla_2^{T^*M}\big)^k u\right|_{(g_2, h_2^E)} .$$

Proof. We apply Proposition 3.30, with $E \oplus \big(T^{(k,0)}M \otimes E\big)$ in place of E and

$$D = \begin{bmatrix} 0 & 0 \\ (\nabla_1^E, \nabla_1^{T^*M})^k & 0 \end{bmatrix} .$$

Directly applying Proposition 3.30 to $\nabla_1^E = \nabla_2^E$ and $\nabla_1^{T^*M} = \nabla_2^{T^*M}$ gives the desired for changing metrics and measures. For allowing a change of connections, observe that

$$\big(\nabla_1^E, \nabla_1^{T^*M}\big)^k - \big(\nabla_2^E, \nabla_2^{T^*M}\big)^k \in \mathrm{Diff}_{j-1}\big(E, T^{(k,0)}M \otimes E\big) .$$

□

Lemma 3.32. *If* $u \in \mathrm{dom}(D_{\max})$ *and* $\chi \in \mathrm{C}^\infty_\mathrm{c}(M)$, *then* $\chi u \in \mathrm{dom}(D_{\max})$. *Similarly for* $\mathrm{dom}(D_{\min})$.

In particular,

$$u \in \mathrm{H}^k\big(M, E, h^E, g, \nabla^E, \nabla^{T^*M}\big) \quad \Rightarrow \quad \chi u \in \mathrm{H}^k\big(M, E, h^E, g, \nabla^E, \nabla^{T^*M}\big)\,.$$

By Corollary 3.31, for any choices of metrics, connections and densities, we have that

$$\chi u \in \mathrm{H}^k\big(M, E, h_1^E, g_1, \nabla_1^E, \nabla_1^{T^*M}, \mu_1\big) \quad \Leftrightarrow \quad \chi u \in \mathrm{H}^k\big(M, E, h_2^E, g_2, \nabla_2^E, \nabla_2^{T^*M}, \mu_2\big)\,.$$

This leads us to the following definition.

Definition 3.33. We say $u \in \mathrm{H}^k_\mathrm{loc}(M,E)$ if $u \in \mathrm{MeasSect}(E)$ and there are $h^E, g, \nabla^E, \nabla^{T^*M}, \mu$ s.t. for all $\chi \in \mathrm{C}^\infty_\mathrm{c}(M)$ with $\mathrm{spt}\,\chi$ having nonempty interior,

$$\chi u \in H^k\big(M, E, h^E, g, \nabla^E, \nabla^{T^*M}, \mu\big)\,.$$

We equip $\mathrm{H}^k_\mathrm{loc}(M,E)$ with a family (ϱ_χ) of semi-norms, indexed over $\chi \in \mathrm{C}^\infty_\mathrm{c}(M)$ with $\mathrm{spt}\,\chi$ having nonempty interior, given by

$$\varrho_\chi(u) := \|\chi u\|_{\mathrm{H}^k}\,.$$

The space $\mathrm{H}^k_\mathrm{loc}(M,E)$, with the semi-norms ϱ_χ is then a locally convex topological vector space.

Proposition 3.34. *We have* $u \in \mathrm{H}^k_\mathrm{loc}(M,E)$ *iff for all trivialising charts* (U, ψ, Ψ)

$$\Psi \circ u|_U \circ \psi^{-1} \in \mathrm{H}^k_\mathrm{loc}\big(\underbrace{\psi(U)}_{\subset \mathbb{R}^n_+}, \mathbb{C}^N\big)\,.$$

Proposition 3.35. *Let* M' *be compact and* $E \to M'$ *a vector bundle. Then for any* $h^E, g, \nabla^E, \nabla^{T^*M}, \mu$ *we have*

$$\mathrm{H}^k\big(M, E, h^E, g, \nabla^E, \nabla^{T^*M}\big) = \mathrm{H}^k_\mathrm{loc}(M,E)$$

and

$$\mathrm{H}^k_0\big(M, E, h^E, g, \nabla^E, \nabla^{T^*M}\big) = \Big\{ u \in \mathrm{H}^k_\mathrm{loc}(M,E) \;\Big|\; \mathrm{spt}(u) \subset \mathring{M} \Big\}\,.$$

Proposition 3.36. *If* M' *is compact and* $\partial M' = \varnothing$, *then*

$$\mathrm{H}^k_\mathrm{loc}(M,E) = \mathrm{H}^k(M,E) = \mathrm{H}^k_0(M,E)\,.$$

3.7 Fractional Sobolev spaces

To understand the space of boundary conditions, it is of vital importance to be able to define Sobolev spaces with a non integer exponent. Although we will only consider this for the case M' is compact without boundary, we need to take care in ensuring that these spaces are consistent with the Euclidean picture.

Let us first consider the case of $\mathbb{R}^n$. Recall that the Fourier transform maps partial derivatives to an appropriate multiplication in the frequency space. I.e.,

$$(\mathscr{F}\partial_j u)(\xi) = \xi_j(\mathscr{F}u)(\xi)\,.$$

With this, we can see that

$$\begin{aligned}
\mathrm{H}^k(\mathbb{R}^n) &= \bigcap_{j=1}^{k} \mathrm{dom}\big(\nabla^k\big) \\
&= \big\{u \in \mathrm{L}^2(\mathbb{R}^n) \;\big|\; \forall i \leq k, j \leq n\colon \big(\xi \mapsto \xi_j^i(\mathscr{F}u)(\xi)\big) \in \mathrm{L}^2(\mathbb{R}^n)\big\} \\
&= \Big\{u \in \mathrm{L}^2(\mathbb{R}^n) \;\Big|\; \Big(\xi \mapsto |\xi|^k(\mathscr{F}u)(\xi)\Big) \in \mathrm{L}^2(\mathbb{R}^n)\Big\}\,.
\end{aligned} \tag{3.8}$$

Recall $\Delta u = -\sum_i \partial_i^2 u$. Therefore, for an appropriately chosen normalising constant c to match the flavour of Fourier transform used,

$$(\mathscr{F}\Delta u)(\xi) = c|\xi|^2(\mathscr{F}u)(\xi)\,.$$

Therefore, we can write the Laplacian as

$$\Delta u(x) = c\int |\xi|^2(\mathscr{F}u)(\xi)e^{2\pi i\langle x,\xi\rangle f}\;\mathrm{d}\xi\,,$$

with domain

$$\mathrm{dom}(\Delta) = \big\{u \in \mathrm{L}^2(\mathbb{R}^n) \;\big|\; \big(\xi \mapsto |\xi|^2(\mathscr{F}u)(\xi)\big) \in \mathrm{L}^2(\mathbb{R}^n)\big\}\,.$$

This gives a method of constructing a *fractional power* of the Laplacian by defining:

$$\begin{aligned}
\mathrm{dom}\Big(\Delta^{\frac{k}{2}}\Big) &:= \Big\{u \in \mathrm{L}^2(\mathbb{R}^n) \;\Big|\; \Big(\xi \mapsto |\xi|^k(\mathscr{F}u)(\xi)\Big) \in \mathrm{L}^2(\mathbb{R}^n)\Big\}\,, \\
\Delta^{\frac{k}{2}}u &:= \mathscr{F}^{-1}\Big(\xi \mapsto |\xi|^k(\mathscr{F}u)(\xi)\Big)\,.
\end{aligned}$$

Given $\alpha \in \mathbb{R}_{\geq 0}$ we obtain the *fractional* Sobolev space

$$\mathrm{H}^\alpha(\mathbb{R}^n) := \big\{u \in \mathrm{L}^2 \;\big|\; (\xi \mapsto |\xi|^\alpha\mathscr{F}(u)(\xi)) \in \mathrm{L}^2(\mathbb{R}^n)\big\} = \mathrm{dom}\big(\Delta^{\frac{\alpha}{2}}\big)\,.$$

Clearly, this is consistent with Fourier transform characterisation of the integer Sobolev spaces describe in (3.8).

Taking this a step further, for a Borel measurable function $f : \mathbb{R} \to \mathbb{R}$, define

$$\begin{aligned}
\mathrm{dom}(f(\Delta)) &:= \big\{u \in \mathrm{L}^2(\mathbb{R}^n) \;\big|\; \big(\xi \mapsto f\big(|\xi|^2\big)(\mathscr{F}u)(\xi)\big) \in \mathrm{L}^2(\mathbb{R}^n)\big\}\,, \\
f(\Delta)u &:= \mathscr{F}^{-1}\big(\xi \mapsto f\big(|\xi|^2\big)\mathscr{F}(u)(\xi)\big)\,.
\end{aligned}$$

3.8 Operator theoretic characterisation

The non-local nature of fractional powers of the Laplacian on $\mathbb{R}^n$ means that it is not obvious whether we can obtain fractional Sobolev spaces through localisation. In any case, it will be to our advantage to see the Sobolev spaces from a global point of view as well as a local point of view. For that, we require an operator theoretic characterisation of fractional powers, with a tool equivalent to the Fourier transform, but which lends itself to generalisation to the manifold context.

Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ self-adjoint, i.e. T is densely-defined and $T^* = T$. That is, T is symmetric, i.e. $\forall u, v \in \mathrm{dom}(T)\colon \langle Tu, v\rangle = \langle u, Tv\rangle$, and additionally $\mathrm{dom}(T^*) = \mathrm{dom}(T)$ and $Tu = T^*u$. For such an operator, there is a 'measure' valued in $\mathcal{H}$ s.t.

$$Tu = \int_{\mathbb{R}} \lambda \underbrace{\mathrm{d}E_T(\lambda)[u]}_{\in \mathcal{H}} \qquad \text{for all } u \in \mathrm{dom}(T).$$

The measure here is called the *spectral measure*, and the existence of this measure, often called *the spectral theorem*, is a cornerstone result in the theory of self-adjoint operators. A detailed discussion of it can be found in Chapter 13 in [42] by Rudin. In later parts, we shall return to this topic, including the construction of this measure, but from a different point of view.

For a continuous function $f : \mathbb{R} \to \mathbb{R}$, define

$$\begin{aligned} \mathrm{dom}(f(T)) &:= \left\{ u \in \mathcal{H} \;\middle|\; \forall R > 0\colon \int_{[-R,R]} |f(\lambda)|^2 \, \|\mathrm{d}E_T(\lambda)[u]\|^2 < C \right\}, \\ f(T)u &:= \int_{\mathbb{R}} f(\lambda) \, \mathrm{d}E_T(\lambda)[u]\,. \end{aligned}$$

We remark that, in fact, the measure $\mathrm{d}E_T$ is a Borel measure, and the above definition could be extended to Borel functions. However, for the moment, we shall only require this for the continuous case.

Note that if $\|f\|_{\mathrm{L}^\infty(\mathbb{R})} < \infty$, then $f(T) \in \mathscr{B}(\mathcal{H})$ is self-adjoint. In fact, for $f = \mathrm{id}$ we have $f(T) = T$ and therefore

$$\|f(T)\|_{\mathcal{H}\to\mathcal{H}} \le \|f\|_{\mathrm{L}^\infty(\mathbb{R})}\,.$$

Example 3.37. For the Laplacian Δ on $\mathbb{R}^n$, for an appropriately chosen normalisation constant $c > 0$,

$$c\mathscr{F}^{-1}\big((\xi \mapsto f\big(|\xi|^2\big))\mathscr{F}(u)\big) = \int_{\mathbb{R}} f\big(\lambda^2\big) \, \mathrm{d}E_T(\lambda)[u]\,.$$

That is, the functions of the operator we take using the spectral theoretic approach is consistent with our earlier approach using the Fourier transform. In particular,

$$\Delta^{\frac{k}{2}} u = \int \lambda^{\frac{k}{2}} \, \mathrm{d}E_\Delta(\lambda)[u]\,.$$

In what is to follow, here and in later parts, we need the following important result pertaining to the regularity of elliptic differential operators.

Theorem 3.38 (Elliptic regularity). *Let M' be a compact manifold without boundary, $\left(E, h^E\right)$ and $\left(F, h^F\right)$ Hermitian vector bundles over M', and $D \in \mathrm{Diff}_m(E,F)$ elliptic. Then*

$$\mathrm{dom}(D_{\max}) = \mathrm{dom}(D_{\min}) = \mathrm{H}^m(M', E)$$

and

$$\|u\|_D \simeq \|u\|_{\mathrm{L}^2} + \|Du\|_{\mathrm{L}^2} \simeq \|u\|_{\mathrm{H}^m}.$$

We do not provide a detailed proof of this fact, but rather a sketch of a proof to highlight the salient properties underpinning its validity.

Proof sketch. For $D \in \mathrm{Diff}_m(E,F)$, the direction $\mathrm{H}^m(M', E) \subset \mathrm{dom}(D_{\max})$ is clear.

Recall that for a differential operator D, we for u supported inside a trivialising chart,

$$Du = \sum_{|\alpha|=m} A^\alpha(x) \frac{\partial^m}{\partial x^\alpha} u + L$$

where the L are the lower order terms. It is easy to see that for $u \in \mathrm{dom}(D_{\max})$, we obtain that that $u \in \mathrm{dom}(A^\alpha \partial^m_{x^\alpha})$ for all $|\alpha| = m$. If D is elliptic, then A^α is invertible. From this, we obtain that $u \in A^\alpha\, \mathrm{dom}(\partial^m_{x^\alpha})$. Again using the smoothness and invertibility of A^α, we can assert that $u \in \cap_{|\alpha|=m} \mathrm{dom}(\partial^m_{x^\alpha})$.

We are left with understanding why $\cap_{|\alpha|=m} \mathrm{dom}(\partial^m_{x^\alpha}) = \mathrm{H}^m(\mathbb{R}^n)$. For the sake of simplicity, consider assume that $m = 2$. Now, take $v \in \mathrm{dom}(\partial^2_{x_i^2})$. Since by construction $v \in \mathrm{L}^2(\mathbb{R}^n)$, we have that

$$\left\| \frac{\partial}{\partial x_i} v \right\|^2 = \left\langle \frac{\partial}{\partial x_i} v, \frac{\partial}{\partial x_i} v \right\rangle = -\left\langle \frac{\partial^2}{\partial x_i^2} v, v \right\rangle \leq \left\| \frac{\partial^2}{\partial x_i^2} v \right\| \|v\|.$$

I.e. $v \in \cap_{i=1}^n \mathrm{dom}(\partial_{x_i}) = \mathrm{H}^1(\mathbb{R}^n)$. The main point here is that, as long as $v \in \mathrm{L}^2(\mathbb{R}^n)$ to begin with, when higher derivatives of v are in L^2, so are lower derivatives. There are Sobolev spaces known as *homogeneous Sobolev spaces* where we would not be able to make the assertion that $v \in \mathrm{L}^2$, but they are a totally different school of dolphins, and we shall never be concerned with them in the manifold context.

Together with this, we can now assert that $u \in \mathrm{H}^m(\mathbb{R}^n)$. Since M' is compact, we can institute a finite and smooth partition of unity subordinate to some chosen set of trivialising charts and conclude that $\mathrm{dom}(D_{\max}) = \mathrm{H}^m(M', E)$ using Proposition 3.36.

The conclusion $\mathrm{dom}(D_{\min}) = \mathrm{dom}(D_{\max})$ follows from showing that $\mathrm{dom}(D_{\min}) = \mathrm{H}^m_0(M', E)$ and invoking Proposition 3.36. □

Armed with elliptic regularity, let us now construct fractional Sobolev scales. Fix connections ∇ on E and M'. Let

$$\Delta_{\mathrm{c}} := \nabla_{\mathrm{c}}^{\dagger}\nabla_{\mathrm{c}} \in \mathrm{Diff}_2(E)\,,$$

which is easily verified to be elliptic. Let

$$\Delta := \Delta_{\max} = (\nabla_{\mathrm{c}})^{*}\overline{(\nabla_{\mathrm{c}})}\,.$$

Then, by elliptic regularity, $\mathrm{dom}(\Delta) = \mathrm{H}^2(M', E)$, $\Delta_{\mathrm{c}}^k \in \mathrm{Diff}_{2k}(E)$ is elliptic, and

$$\Delta^k = (\Delta_{\max})^k = \left(\Delta^k\right)_{\max}.$$

Again, by elliptic regularity

$$\mathrm{dom}\left(\Delta^k\right) = \mathrm{H}^{2k}(M', E) \qquad \text{and} \qquad \|u\|_{\Delta^k} \simeq \|u\|_{\mathrm{H}^{2k}}\,.$$

The function $f_\alpha(x) = x^{\frac{\alpha}{2}}$ is continuous, and therefore, we obtain $f_{\frac{\alpha}{2}}(\Delta)$. It is readily verified that $f_m(\Delta) = \Delta^m$ and therefore, the following definition is consistent.

Definition 3.39. For $\alpha \in \mathbb{R}_{>0}$ define

$$\begin{aligned} \mathrm{H}^{\alpha}_{\Delta}(M', E) &:= \mathrm{dom}\left(\Delta^{\frac{\alpha}{2}}\right), \\ \|u\|^2_{\mathrm{H}^{\alpha}_{\Delta}} &:= \left\|\Delta^{\frac{\alpha}{2}}u\right\|_{\mathrm{L}^2(M',E)} + \|u\|^2_{\mathrm{L}^2(M',E)}\,. \end{aligned}$$

As we expect, the space of higher order fractional derivatives (with respect to this reference connection and induced Laplacian) are contained in the space of lower orders.

Proposition 3.40. *We have that $\Delta^\beta \subset \Delta^\alpha$ for $\beta \geq \alpha$.*

Proof. This is evident from examining the integral defining the fractional power, as well as its domain. □

By elliptic regularity, $\mathrm{H}^{2m}_{\Delta}(M', E) = \mathrm{H}^{2m}(M', E)$ for $m \in \mathbb{N}$. That is, at even integer points along the fractional scale, we obtain spaces which are independent of the operator. In the forthcoming section, we show the whole scale can be made independent of initial connection.

However, this is not a straightforward exercise of localisation into Euclidean charts. A priori, localisation is possible when the operator is *local*. Recall that T is a local operator on $\mathrm{L}^2(M', E)$ iff $\mathrm{spt}(Tu) \subset \mathrm{spt}(u)$ for all $u \in \mathrm{dom}(T)$. We asserted in Proposition Proposition 3.4 that all differential operators are local. In particular, Δ^k is local for $k \in \mathbb{N}$. However, when $\alpha \in \mathbb{R}_{>0} \setminus \mathbb{N}$, this is no longer true. This is best understood by examining the concrete case of the Euclidean Laplacian.

3.9 Interpolation theory for Banach spaces

The strategy we undertake to divorce the space $\mathrm{H}^\alpha_\Delta(M', E)$ from the operator Δ is to obtain this space as an 'intermediate space' between two even exponents. An important and powerful mathematical tool to achieve this is the theory of interpolation for Banach spaces. The method we demonstrate here is the so-called *complex method* applicable to complex Banach spaces, due to Lions in [38] and Calderón in [14]. Modern treatments of this material are abundant, with two references being the book [49] by Triebel as well as [26] by Haase.

We emphasise, as we did before, and in particular since we only discuss the complex method, that $\mathbb{K} = \mathbb{C}$.

Definition 3.41. Let $\mathcal{B}$ be a complex Banach space and $\Omega \subset \mathbb{C}$ an open set. Then $f : \Omega \to \mathcal{B}$ is

- *holomorphic*, if
$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h},$$
exists for all $z \in \Omega$, and
- *weakly holomorphic*, if $(\varrho \circ f) : \Omega \to \mathbb{C}$ is holomorphic for all $\varrho \in \mathcal{B}^*$.

Obviously holomorphic implies weakly holomorphic. However, the converse is also true.

Theorem 3.42. *Weakly holomorphic implies holomorphic.*

This is a particularly useful tool as it allows many of the results pertaining to holomorphic complex valued functions to be 'lifted' to the Banach space holomorphic setting.

Definition 3.43. For Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$, the pair $(\mathcal{B}_1, \mathcal{B}_2)$ is called an *interpolation couple*, if there exists a Banach space $\mathcal{B}_3$ s.t. $\mathcal{B}_i \subset \mathcal{B}_3$ and $\mathcal{B}_i \hookrightarrow \mathcal{B}_3$ is continuous $(i = 1, 2)$.

An interpolation couple $(\mathcal{H}_1, \mathcal{H}_2)$ of Hilbert spaces is called an *Hilbert interpolation couple.*

Remark 3.44.
1. In the literature, the notion of an interpolation couple is slightly different to our definition here (albeit superficially). Typically, it is only assumed that $\mathcal{B}_1$ and $\mathcal{B}_2$ can be continuously embedded into a space $\mathcal{B}_3$, rather than assuming they are, in fact, subspaces of $\mathcal{B}_3$ as we have required. We assume they are subspaces for convenience, and since this is always true

in the applications we have in mind. Moreover, since $\mathcal{B}_i$ can be identified with its image under the embedding, this is a moot point.

2. The role of the space $\mathcal{B}_3$ is purely of an auxiliary nature. The continuous embedding divorces any significance to a particular $\mathcal{B}_3$, and simply provides an 'ambient' vector space structure to form the sum $\mathcal{B}_1 + \mathcal{B}_2$.

Example 3.45. 1. For $\mathcal{B}_1 = \mathrm{H}^j(M', E)$ and $\mathcal{B}_2 = \mathrm{H}^{j+k}(M', E)$, we have that for a choice of $\mathcal{B}_3 = \mathrm{L}^2(M', E)$,

$$\|u\|_{\mathrm{L}^2} \lesssim \|u\|_{\mathrm{L}^2} + \sum_{\ell=1}^{\ell'} \left\|\nabla^\ell u\right\| \simeq \|u\|_{\mathrm{H}^1},$$

for $\ell' = j$ or $j + k$.

2. $\mathcal{B}_1 := \mathrm{H}^{2j}(M', E)$, $\mathcal{B}_2 := \mathrm{H}^{2(j+1)}(M', E)$, $\mathcal{B}_3 := \mathrm{L}^2(M', E)$. Clearly $\mathrm{H}^k(M, E) \subset \mathrm{L}^2(M', E)$ and

$$\|u\|_{L^2(M',E)} \lesssim \|u\| + \left\|\Delta^{j-m+1} u\right\| \simeq \|u\|_{H^{2(j-m+1)}(M',E)}$$

due to elliptic regularity.

3. For $\mathcal{B}_1 = \mathrm{H}^j(M', E)$ and $\mathcal{B}_2 = \mathrm{H}^{j+k}(M', E)$ for $k > 0$, we can choose $\mathcal{B}_3 = \mathcal{B}_1$.

4. For $\mathcal{B}_1 = \mathrm{H}^\alpha_\Delta(M', E)$ and $\mathcal{B}_2 = \mathrm{H}^\beta_\Delta(M', E)$ where $\alpha \leq \beta$, we can choose $\mathcal{B}_3 = \mathrm{L}^2(M', E)$ or $\mathcal{B}_3 = \mathrm{H}^\vartheta_\Delta(M', E)$, for $\vartheta \leq \alpha$ due to Proposition 3.40.

The goal is to obtain Banach spaces $[\mathcal{B}_1, \mathcal{B}_2]_\vartheta$, sandwiched as $\mathcal{B}_1 \cap \mathcal{B}_2[\mathcal{B}_1, \mathcal{B}_2]_\vartheta \subset \mathcal{B}_1 \cap \mathcal{B}_2 \subset \mathcal{B}_3$, in some manner controlled by a parameter ϑ. In applications, it is significant that the 'endpoints' $\mathcal{B}_1 \cap \mathcal{B}_2$ and $\mathcal{B}_1 + \mathcal{B}_2$ as well as the sandwiched spaces $[\mathcal{B}_1, \mathcal{B}_2]_\vartheta$ should be continuously embedded in $\mathcal{B}_3$. In order to obtain such a construction, we topologise the endpoint spaces in the following manner.

Definition 3.46. For $\mathcal{B}_1 + \mathcal{B}_2 = \{b_1 + b_2 \in \mathcal{B}_3 \mid b_i \in \mathcal{B}_i\}$ define

$$\|x\|_{\mathcal{B}_1+\mathcal{B}_2} := \inf\left\{\|b_1\|_{\mathcal{B}_1} + \|b_2\|_{\mathcal{B}_2} \;\middle|\; x = b_1 + b_2\right\}.$$

For $\mathcal{B}_1 \cap \mathcal{B}_2$, define

$$\|x\|_{\mathcal{B}_1\cap\mathcal{B}_2} := \|x\|_{\mathcal{B}_1} + \|x\|_{\mathcal{B}_2}.$$

Proposition 3.47. *Then $\left(\mathcal{B}_1 + \mathcal{B}_2, \|\cdot\|_{\mathcal{B}_1+\mathcal{B}_2}\right)$ and $\left(\mathcal{B}_1 \cap \mathcal{B}_2, \|\cdot\|_{\mathcal{B}_1\cap\mathcal{B}_2}\right)$ are Banach spaces which are continuously embedded in $\mathcal{B}_3$.*

Proof. This follows from the fact that $\mathcal{B}_1$ and $\mathcal{B}_2$ are continuously embedded in $\mathcal{B}_3$. □

Note that the spaces $\mathcal{B}_1 + \mathcal{B}_2$ and $\mathcal{B}_1 \cap \mathcal{B}_2$ are typically not closed subspaces of $\mathcal{B}_3$. In fact, we will see that for closed subspaces, the obtained intermediate spaces are trivial.

Now, let

$$S := \{x + iy \in \mathbb{C} \mid x \in [0,1],\ y \in \mathbb{R}\} \subset \mathbb{C}\,.$$

be the closed vertical strip in the complex plane between 0 and 1.

Definition 3.48. Let $\mathcal{F}(\mathcal{B}_1, \mathcal{B}_2)$ be the space of *interpolation functions* $f : S \to \mathcal{B}_1 + \mathcal{B}_2$ s.t.

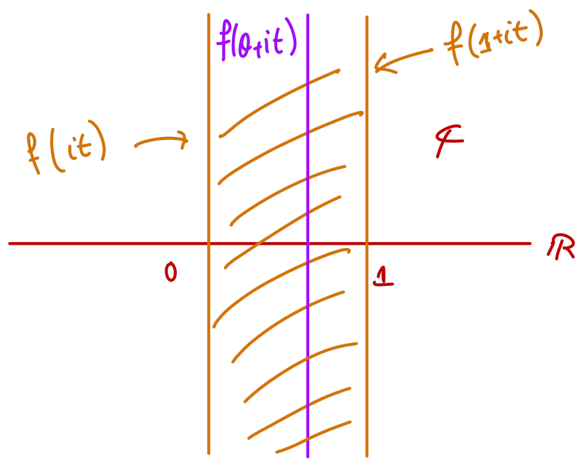
(I) $f \in \mathrm{C_b}(S, \mathcal{B}_1 + \mathcal{B}_2)$, where $\mathrm{C_b}$ denotes continuous and bounded,

(II) $f|_{\mathring{S}} : \mathring{S} \to \mathcal{B}_1 + \mathcal{B}_2$ is holomorphic, and

(III) $t \mapsto f(it) \in \mathrm{C_b}(\mathbb{R}, \mathcal{B}_1)$ and $t \mapsto f(1 + it) \in \mathrm{C_b}(\mathbb{R}, \mathcal{B}_2)$.

The norm on $\mathcal{F}(\mathcal{B}_1, \mathcal{B}_2)$ is

$$\|f\|_{\mathcal{F}(\mathcal{B}_1,\mathcal{B}_2)} := \max\left\{\sup_{t\in\mathbb{R}} \|f(it)\|_{\mathcal{B}_1},\ \sup_{t\in\mathbb{R}} \|f(1+it)\|_{\mathcal{B}_2}\right\}.$$



Armed with this function space, which recovers the endpoint spaces in an appropriate manner, we define the interpolation scale between $\mathcal{B}_1 \cap \mathcal{B}_2$ and $\mathcal{B}_1 + \mathcal{B}_2$.

Definition 3.49. For $\vartheta \in [0,1]$, define the *interpolation spaces*

$$[\mathcal{B}_1, \mathcal{B}_2]_\vartheta := \{f(\vartheta) \mid f \in \mathcal{F}(\mathcal{B}_1, \mathcal{B}_2)\}\,,$$
$$\|b\|_{[\mathcal{B}_1,\mathcal{B}_2]_\vartheta} := \inf\left\{\|f\|_{\mathcal{F}(\mathcal{B}_1,\mathcal{B}_2)} \;\middle|\; f(\vartheta) = b\right\}.$$

Theorem 3.50. *Let $(\mathcal{B}_1, \mathcal{B}_2)$ and $\left(\widetilde{\mathcal{B}_1}, \widetilde{\mathcal{B}_2}\right)$ be interpolation couples. Then:*

I) The space $\left([\mathcal{B}_1, \mathcal{B}_2]_\vartheta, \|\cdot\|_{[\mathcal{B}_1,\mathcal{B}_1]_\vartheta}\right)$ is a Banach space satisfying

$$\mathcal{B}_1 \cap \mathcal{B}_2 \subset [\mathcal{B}_1, \mathcal{B}_2]_\vartheta \subset \mathcal{B}_1 + \mathcal{B}_2\,.$$

The inclusion maps here are all continuous.

II) The space $\mathcal{B}_1 \cap \mathcal{B}_2$ is dense in $[\mathcal{B}_1, \mathcal{B}_2]_\vartheta$ for all $\vartheta \in [0,1]$.

III) If the operator $T : \mathcal{B}_1 + \mathcal{B}_2 \to \widetilde{\mathcal{B}_1} + \widetilde{\mathcal{B}_2}$ restricts to bounded maps $T : \mathcal{B}_1 \to \widetilde{\mathcal{B}_2}$ and $T : \mathcal{B}_2 \to \widetilde{\mathcal{B}_2}$, then

$$T : [\mathcal{B}_1, \mathcal{B}_2]_\vartheta \to \left[\widetilde{\mathcal{B}_1}, \widetilde{\mathcal{B}_2}\right]_\vartheta$$

is also bounded. We have the norm estimate

$$\|T\|_{[\mathcal{B}_1,\mathcal{B}_2]_\vartheta \to [\mathcal{B}_1,\mathcal{B}_2]_\vartheta} \leq \|T\|_{\mathcal{B}_1}^{1-\vartheta} \|T\|_{\mathcal{B}_2}^{\vartheta}\,.$$

Remark 3.51. Suppose that $\mathcal{B}_1 \subset \mathcal{B}_2 = \mathcal{B}_3$. Then, $\mathcal{B}_1 \subset [\mathcal{B}_1, \mathcal{B}_2]_\vartheta \subset \mathcal{B}_2$. This is the usual situation that we will encounter in applications.

However, if the space $\mathcal{B}_1$ is closed in $\mathcal{B}_2$, interpolation will not yield any interesting spaces. This is because by Theorem 3.50 II), we have that $\mathcal{B}_1$ is dense in $[\mathcal{B}_1, \mathcal{B}_2]_\vartheta$. This means that if $\mathcal{B}_1$ is a closed subspace of $\mathcal{B}_2$, we have for all $\vartheta \in [0,1]$ that $[\mathcal{B}_1, \mathcal{B}_2]_\vartheta = \mathcal{B}_1$. In particular, interpolation theory is not an effective tool in finite dimensions.

For our purposes, what is vital is to connect interpolation scales to fractional powers of self-adjoint operators in Hilbert spaces.

Proposition 3.52. *Let $(\mathcal{H}_1, \mathcal{H}_2)$ be a Hilbert interpolation couple continuously embedding into a Hilbert space $\mathcal{H}_3$, and S a non-negative self-adjoint operator on $\mathcal{H}_3$ with $\mathrm{dom}(S) = \mathcal{H}_1$, $\mathrm{dom}(S^2) = \mathcal{H}_2$. Then*

$$[\mathcal{H}_1, \mathcal{H}_2]_\vartheta = \mathrm{dom}\left((S^2)^{\frac{1}{2}(1+\vartheta)}\right) = \mathrm{dom}(S^{1+\vartheta})\,.$$

Corollary 3.53. *Let Δ_1, Δ_2 be two Laplacians (i.e. w.r.t. two connections) on $E \to M'$. Then*

$$\mathrm{H}^\alpha_{\Delta_1}(M', E) = \mathrm{H}^\alpha_{\Delta_2}(M', E)\,.$$

Proof. Fix α s.t. $2m+2>\alpha>2m$.

$$\mathrm{H}^{2m+2}_{\Delta_i}(M',E)=\mathrm{dom}\big(\Delta_i^{2m+2}\big)$$
$$\mathrm{H}^{2m}(M',E)=\mathrm{dom}\big(\Delta_i^{2m}\big)$$

By elliptic regularity, $\mathrm{dom}\big(\Delta_1^k\big)=\mathrm{dom}\big(\Delta_2^k\big)$ for the choices of $k=m$ and $k=m+1$. Choosing $\mathcal{H}_i$ appropriately and $\mathcal{H}_3=\mathrm{L}^2(M',E)$, for $\vartheta\in(0,1)$, we obtain

$$\underbrace{\big[\mathrm{H}^{2m}_{\Delta_1}(M',E),\mathrm{H}^{2m+2}_{\Delta_1}(M',E)\big]_\vartheta}_{=\mathrm{H}^{2(m+\vartheta)}_{\Delta_1}(M',E)}=\underbrace{\big[\mathrm{H}^{2m}_{\Delta_2}(M',E),\mathrm{H}^{2m+2}_{\Delta_2}(M',E)\big]_\vartheta}_{=\mathrm{H}^{2(m+\vartheta)}_{\Delta_2}(M',E)}.$$

In particular, $\mathrm{H}^\alpha_{\Delta_1}(M',E)=\mathrm{H}^\alpha_{\Delta_2}(M',E)$. □

It now makes sense to define the fractional Sobolev spaces independent of an operator.

Definition 3.54. For any Laplacian Δ on $E\to M'$ and $\alpha\in\mathbb{R}_{>0}$ let

$$\mathrm{H}^\alpha(M',E):=\mathrm{H}^\alpha_\Delta(M',E)\,.$$

Remark 3.55. We could have, from the very start, defined fractional Sobolev spaces through interpolation. However, this approach would have been disadvantageous for two reasons. Firstly, it would have been unclear that, for instance,

$$\big[\mathrm{H}^1(M',E),\mathrm{H}^3(M',E)\big]_{\vartheta=\frac{1}{2}}=\mathrm{H}^2(M',E)\,.$$

Secondly, the spaces we obtain would be abstract, and it would be unclear what precise relationship they have to measuring differentiability. The upshot of the approach we have taken is to build these spaces, at least conceptually, as spaces measuring a notion of fractional differentiability with respect to a fixed differential operator. The interpolation scales are inevitable, as they allow us to divorce the dependency on the operator.

Before we conclude this section, let us return to the question of localisability. As aforementioned, Δ^α on $\mathbb{R}^n$ is not local for $\alpha\notin\mathbb{N}$. Therefore, it is unclear whether we can locally relate fractional Sobolev spaces over vector bundles in a meaningful manner to the fractional Sobolev spaces over $\mathbb{R}^n$. The advantage to localisation would be that we can understand from a multitude of different perspectives and it would potentially provide a mechanism to import Euclidean results to the manifold setting. Let us broach this topic by starting with the following definition.

Definition 3.56. $R\in\mathscr{B}(\mathcal{B}_1,\mathcal{B}_2)$ is a *retraction* if there is a $S\in\mathscr{B}(\mathcal{B}_2,\mathcal{B}_1)$ s.t. $RS=\mathrm{id}\in\mathscr{B}(\mathcal{B}_2)$. The operator S is called a *coretraction* associated to the retraction R.

The following is a fundamentally important result in the study of Sobolev and other kinds of function spaces over vector bundles. It can be found as Theorem *) in [49].

Theorem 3.57 (The retraction-coretraction theorem). *Let* $(\mathcal{B}_1, \mathcal{B}_2)$ *and* $\left(\widetilde{\mathcal{B}_1}, \widetilde{\mathcal{B}_2}\right)$ *be interpolation couples and* $S \in \mathscr{B}\left(\widetilde{\mathcal{B}_1} + \widetilde{\mathcal{B}_2}, \mathcal{B}_1 + \mathcal{B}_2\right)$ *a coretraction to* $R \in \mathscr{B}\left(\mathcal{B}_1 + \mathcal{B}_2, \widetilde{\mathcal{B}_1} + \widetilde{\mathcal{B}_2}\right)$ *which restrict to bounded coretractions and retractions on* $\widetilde{\mathcal{B}_i} \to \mathcal{B}_i$ *and* $\mathcal{B}_i \to \widetilde{\mathcal{B}_i}$ *respectively. Then*

$$S_\vartheta := S|_{\left[\widetilde{\mathcal{B}_1}, \widetilde{\mathcal{B}_2}\right]_\vartheta} : \left[\widetilde{\mathcal{B}_1}, \widetilde{\mathcal{B}_2}\right]_\vartheta \to [\mathcal{B}_1, \mathcal{B}_2]_\vartheta$$

has closed range and $P_\vartheta := S_\vartheta R_\vartheta$ *is a projection s.t.*

$$S_\vartheta \left[\widetilde{\mathcal{B}_1}, \widetilde{\mathcal{B}_2}\right]_\vartheta = P_\vartheta [\mathcal{B}_1, \mathcal{B}_2]_\vartheta \,.$$

Remark 3.58. Note that $S_\vartheta : \left[\widetilde{\mathcal{B}_1}, \widetilde{\mathcal{B}_2}\right]_\vartheta \to S_\vartheta \left[\widetilde{\mathcal{B}_1}, \widetilde{\mathcal{B}_2}\right]_\vartheta$ is actually a Banach space isomorphism. We first have that $\|S_\vartheta u\|_{\left[\widetilde{\mathcal{B}_1}, \widetilde{\mathcal{B}_2}\right]_\vartheta} \lesssim \|u\|_{\left[\widetilde{\mathcal{B}_1}, \widetilde{\mathcal{B}_2}\right]_\vartheta}$, and since it has closed range, it is an open map. However, S is a coretraction and in particular an injection. Therefore, S_ϑ has a continuous inverse.

Let $(V_i, \psi_i, \Psi_i)_{i=1}^k$ be a covering of M' by trivialising charts such that $(U_i, \psi_i, \Psi_i)_{i=1}^k$ is also a cover with $\overline{U_i} \subset V_i$. Furthermore, let $\{\eta_i\}_{i=1}^k$ be a smooth partition of unity subordinate to this cover.

Define $S : \mathrm{H}^{2m}(M', E) \to \mathrm{H}^{2m}\left(\mathbb{R}^n, \mathbb{C}^N\right)^k = \bigtimes_{j=1}^k \mathrm{H}^{2m}\left(\mathbb{R}^n, \mathbb{C}^N\right)$ with

$$Su := \left(\Psi_1 \circ (\eta_1 u) \circ \psi_1^{-1}, \dots, \Psi_k \circ (\eta_k u) \circ \psi_k^{-1}\right).$$

Now, let $\chi_i : \mathbb{R}^n \to [0,1]$ be a smooth function such that $\operatorname{spt} \chi_i \subset \psi_i(V_i)$ and $\chi_i = 1$ on $\psi_i(U_i)$. Define

$$R(v_1, \dots, v_k) := \sum_{i=1}^k \Psi_i^{-1} \circ (\chi_i v_i) \circ \psi_i \,.$$

By the choices of the trivialisations and the cutoffs χ_i, we see that $RS = \mathrm{id}$. Therefore, S is a coretraction to R, and by Theorem 3.57,

$$\begin{aligned} S_\vartheta \mathrm{H}^{2m\vartheta}(M', E) &\subset \mathrm{H}^{2m\vartheta}\left(\mathbb{R}^n, \mathbb{C}^N\right)^k, \\ R_\vartheta S_\vartheta \mathrm{H}^{2m}(M', E) &= \mathrm{H}^{2m}(M', E) \,. \end{aligned}$$

This shows that, despite the non-locality of fractional operators, we are able to localise fractional Sobolev spaces in a meaningful manner. In particular, this allows for Euclidean results in the fractional Sobolev scales to be imported to the manifold setting with relative ease.

A similar setup can be done for Sobolev spaces on a compact manifold with boundary, where the localisations need to take place in $\mathbb{R}^n_+$. This is particularly useful in applications, particularly in the context of boundary value problems. One such result is the following cornerstone theorem.

Theorem 3.59. *Let (M, μ) be a measured manifold with ∂M compact and $E \to M$ a vector bundle with metric h^E. Then the boundary trace map* $\mathrm{Tr}_\partial := \left(u \mapsto u|_{\partial M}\right) : \mathrm{C}^\infty_{\mathrm{c}}(M, E) \to \mathrm{C}^\infty_{\mathrm{c}}(\partial M, E)$ *extends to a bounded map*

$$\mathrm{Tr}_\partial : \mathrm{H}^k_{\mathrm{loc}}(M, E) \to \mathrm{H}^{k-\frac{1}{2}}(\partial M, E)$$

for $k \in \mathbb{N}_{k \geq 1}$.

The proof of this in the case of $M = \mathbb{R}^n_+$ is an exercise involving the Fourier transform. See, for instance, Proposition 1.6 in [45]. As aforementioned, the Euclidean result can then be imported using the retraction-coretraction theorem.

Remark 3.60.

1. The map Tr_∂ is known by many names in the literature. Some include *trace map*, *boundary trace map*, *restriction map* and *boundary restriction map*.

2. If M is compact with boundary, we can assert that

$$\mathrm{Tr}_\partial : \mathrm{H}^\alpha(M, E) \to \mathrm{H}^{\alpha-\frac{1}{2}}(\partial M, E)$$

is bounded for $\alpha > \frac{1}{2}$. In other words,

$$\|\mathrm{Tr}_\partial u\|_{\mathrm{H}^{\alpha-\frac{1}{2}}} \leq C_\alpha \|u\|_{\mathrm{H}^\alpha}.$$

In general we cannot get this to the critical exponent $\alpha = \frac{1}{2}$ as the constant $C_\alpha \to \infty$ as $\alpha \to \frac{1}{2}$. For more general problems, we require *negative order* Sobolev spaces, which we will develop in the forthcoming section.

3.10 Negative order Sobolev spaces

On a manifold M with boundary, mirroring the setup of a Euclidean domain, we are able to setup a space of distributions. This is done in light of Corollary 3.31 and Proposition 3.34, which allow us to duplicate the $\mathbb{R}^n$ setup in the manifold context. More precisely, the space of distributions $\mathcal{D}'(M, E) = \mathrm{C}^\infty_{\mathrm{cc}}(M, E)^{\mathrm{dual}}$, where the dual is the topological dual of continuous linear functions over the space $\mathcal{D}(M, E) = \mathrm{C}^\infty_{\mathrm{cc}}(M, E)$. The space $\mathcal{D}(M, E)$ is topologised via the inductive limit topology. In the bundle context, this is obtained as follows. Choosing a connection ∇ on M and E, as well as two metrics g and h^E, and consider the semi-norms:

$$\varrho_{K,m}(f) = \sup\left\{ |\nabla^s f(x)|_{g(x), h^E(x)} \;\middle|\; \forall s \leq m \, \forall x \in K \right\}.$$

on the space $\mathcal{D}_K(M,E)$, which are $f \in \mathcal{D}(M,E)$ with $\operatorname{spt} f \subset K$, and where K is a precompact open set. Topologised via these semi-norms, this turns $\mathcal{D}_K(M,E)$ into a locally convex linear space. The topology on $\mathcal{D}(M,E)$ is then obtained as the inductive limit topology by considering the collection of subspaces $\mathcal{D}_K(M,E)$. Corollary 3.31 and Proposition 3.34 ensure that the construction is independent of ∇, g and h^E. For details in the $\mathbb{R}^n$ case, see Chapter 1, Section 1 in [52].

Given a metric h^E and measure μ, we consider

$$\mathrm{L}^2(M,E) \subset \mathcal{D}'(M,E)\,,$$

by letting $f \in \mathrm{L}^2(M,E)$ act on $v \in \mathrm{C}^\infty_{\mathrm{cc}}(M,E)$ as

$$f(v) := \langle f, v\rangle_{\mathrm{L}^2(M,E)}\,.$$

Note that, in the case of a compact manifold M', we have that $\mathrm{L}^2(M',E) = \mathrm{L}^2_{\mathrm{loc}}(M',E)$ as a set. I.e. as a set $\mathrm{L}^2(M,E)$ is independent of a metric and measure. Obviously, the a choice of inner product on $\mathrm{L}^2(M',E)$ is certainly dependent on a choice of metric h^E and density μ.

In what is to follow, let us restrict ourselves to M' compact with $\partial M' = \varnothing$.

Definition 3.61. For $\alpha > 0$ define $u \in \mathrm{H}^{-\alpha}(M',E)$ if

$$\|u\|_{\mathrm{H}^{-\alpha}(M',E)} := \sup_{v \in \mathrm{H}^\alpha_0} \frac{|u[v]|}{\|v\|_{\mathrm{H}^\alpha_0}} < \infty\,.$$

Remark 3.62. Although the compactness and lack of boundary for M' means $\mathrm{C}^\infty_{\mathrm{cc}}(M',E) = \mathrm{C}^\infty(M',E)$, we suggestively use the notation $\mathrm{C}^\infty_{\mathrm{cc}}(M',E)$ as this lends itself to the correct generalisation on manifolds with boundary.

Proposition 3.63. *The map $(u,v) \mapsto \langle u,v\rangle := u[v]$ is a reflexive perfect-pairing $\langle \mathrm{H}^{-\alpha}(M',E), \mathrm{H}^\alpha(M',E)\rangle$. In particular,*

$$\mathrm{H}^\alpha(M',E)^* \cong \mathrm{H}^{-\alpha}(M',E)\,.$$

Proof. This is evident by construction. It is reflexive since $\mathrm{H}^\alpha(M',E)$ is a reflexive space. □

Remark 3.64. We have seen that the space of distributions $\mathcal{D}'(M,E)$ is a topological dual space of $\mathrm{C}^\infty_{\mathrm{cc}}(M,E)$, a subspace which is contained in all the spaces of interest to us. However, $\mathcal{D}'(M,E)$ is independent of connections and metrics and therefore, it does not 'see' the underlying geometry of the space. In contrast, the Sobolev spaces, whether they be negative or positive, do. This is difficult to appreciate in the compact setting as these spaces apparently become divorced from the geometry. However, this is not entirely true. In the compact case, what is interesting are the ways in which to equivalently compute the norms of these spaces, and this will become apparent in later parts.

Through careful consideration, this setup could be replicated in the noncompact setting. We have already seen that the positive order Sobolev spaces are affected by geometry in the noncompact setting. When appropriately defined, the negative order Sobolev spaces are then their dual spaces. From this point of view, the negative order Sobolev spaces can be thought of as 'Sobolev distributions'. Much like the space of distributions, they provide a very large ambient space where many interesting function spaces are continuously embedded. However, unlike the space of distributions, in addition to their ability to encode geometry, they are a Banach space. This latter fact makes them very useful in applications.

We now want to understand the norm $\|\cdot\|_{\mathrm{H}^{-\alpha}}$ from an operator point of view. To do this, we first note the following.

Lemma 3.65. $\mathrm{C}^\infty(M', E)$ *is a dense subspace of* $\mathrm{H}^{-\alpha}(M', E)$.

Proof. We leave this as an exercise. Hint: use the Hahn-Banach theorem along with Proposition 3.63. □

Since this Lemma affords us with a good dense subspace of $\mathrm{H}^{-\alpha}(M', E)$, which is also a dense subspace of $\mathrm{L}^2(M', E)$, we consider the norm $\|\cdot\|_{\mathrm{H}^{-\alpha}}$ there. So, on $\mathrm{C}^\infty(M', E)$ we compute the norm $\|\cdot\|_{\mathrm{H}^{-\alpha}}$.

Proposition 3.66. *For* $u \in \mathrm{C}^\infty(M', E)$,

$$\|u\|_{\mathrm{H}^{-\alpha}} = \left\|(I + \Delta^\alpha)^{-1} u\right\|_{\mathrm{L}^2}.$$

Proof. We have that Δ is non-negative, so Δ^α is non-negative also and this implies that $(I + \Delta^\alpha)$ is invertible. This is a fact we will visit in later parts. We have $\ker(I + \Delta^\alpha) = 0$, $\mathrm{ran}(I + \Delta^\alpha)$ is dense and there is a $c_\alpha < \infty$ s.t.

$$\|(I + \Delta^\alpha)u\| \geq c_\alpha \|u\|,$$

we also have

$$\|u\|_{\mathrm{H}^\alpha} \simeq \|u\| + \|\Delta^\alpha u\| \simeq \|(I + \Delta^\alpha)u\|.$$

Fix $u \in \mathrm{C}^\infty(M', E) \subset \mathrm{L}^2(M', E)$. Then

$$\|u\|_{\mathrm{H}^{-\alpha}} = \sup_{v \in \mathrm{H}^\alpha} \frac{|\langle u, v\rangle|}{\|v\|_{\mathrm{H}^\alpha}} \simeq \sup_{v \in \mathrm{H}^\alpha} \frac{\left|\left\langle (I + \Delta^\alpha)^{-1} u, (I + \Delta^\alpha v)\right\rangle\right|}{\|(I + \Delta^\alpha)v\|_{\mathrm{L}^2}}.$$

Set $w := (I + \Delta^\alpha)v \in \mathrm{L}^2$ and substituting back into this expression, we obtain

$$\|u\|_{\mathrm{H}^{-\alpha}} = \sup_{v \in \mathrm{H}^\alpha} \frac{|\langle u, v\rangle|}{\|v\|_{\mathrm{H}^\alpha}} \sup_{w \in \mathrm{L}^2} \frac{\left|\left\langle (I+\Delta^\alpha)^{-1} u, w\right\rangle\right|}{\|w\|_{\mathrm{L}^2}} \simeq \left\|(I+\Delta^\alpha)^{-1} u\right\|_{\mathrm{L}^2}. \qquad \square$$

Note that, by Lemma 3.65, we can regard

$$(I+\Delta^\alpha)^{-1} : \mathrm{H}^{-\alpha}(M', E) \to \mathrm{L}^2(M', E).$$

Proposition 3.67. *Given an* L^2*-inner product* $\langle \mathrm{L}^2(M', E), \mathrm{L}^2(M', E)\rangle$*, i.e. metrics* g *and* h^E*, as well as a connection* ∇*, there is an induced reflexive perfect-pairing* $\langle \mathrm{H}^\alpha(M', E), \mathrm{H}^{-\alpha}(M', E)\rangle$ *which agrees with* $\langle \mathrm{L}^2(M', E), \mathrm{L}^2(M', E)\rangle$ *on restriction to* $\mathrm{L}^2(M', E)$*. More precisely, the pairing is given by* $\langle u, v\rangle = \langle (I+\Delta^\alpha)^{-1}u, (I+\Delta^\alpha)v\rangle$ *for the induced Laplacian* Δ *from* ∇*.*

Remark 3.68. Fixing a connection ∇ on E, a natural inner product on $\mathrm{H}^\alpha(M', E)$ is

$$\langle u, v\rangle_{\mathrm{H}^\alpha} := \langle (I+\Delta^\alpha)u, (I+\Delta^\alpha)v\rangle.$$

This can be seen more generally in our language as a reflexive pairing $\langle \mathrm{H}^\alpha(M', E), \mathrm{H}^\alpha(M', E)\rangle$. If we were to require the restriction of this pairing to $\mathrm{C}^\infty(M', E)$ to equal the $\mathrm{L}^2(M', E)$ inner product, it is clear that this is not the correct pairing. The correct pairing is $\langle \mathrm{H}^\alpha(M', E), \mathrm{H}^{-\alpha}(M', E)\rangle$ appearing in the proposition.

The moral of the story is that, in application, the $\mathrm{L}^2(M', E)$ geometry plays a special role, and it places constraints on the relationships between the other induced spaces. In practical terms, this means that, even in the Hilbert space setting, it forces us to consider general perfect pairings rather than the natural inner products alone.

3.11 Back to Boundary Conditions

Let us first recall the relevant boundary trace map for an order m differential operator. Fix an interior pointing vectorfield T along the boundary. Then,

$$\gamma_{\mathrm{c}} : \mathrm{C}^\infty_{\mathrm{c}}(M, E) \to \bigoplus_{j=0}^{m-1} \mathrm{C}^\infty(\partial M, E),$$
$$u \mapsto \left(u|_{\partial M}, (\partial_T u)|_{\partial M}, \dots, (\partial_T^{m-1} u)|_{\partial M} \right).$$

Here, care needs to be taken when considering $\partial_T u$. This is really done by fixing a connection ∇ in a neighbourhood of the boundary. We shall be concerned with compact boundary and there, to understanding the mapping properties of γ_{c}, the specific choice of connection will be irrelevant. Given that the vectorfield T is global, by $\partial_T u$, we will fix this notation to mean 'flat' differentiation in this direction, given by

$$\partial_T u = (du^\alpha(T))e^\alpha \,.$$

Theorem 3.69. *Let M be a compact manifold with boundary, $D \in \mathrm{Diff}_m(E,F)$ elliptic, T an inward pointing vector field along ∂M. Then γ_{c} extends uniquely to a bounded map*

$$\gamma : \mathrm{dom}(D_{\max}) \to \bigoplus_{j=0}^{m-1} \mathrm{H}^{-\frac{1}{2}-j}(\partial M, E)$$

with dense range. Moreover,

$$\ker(\gamma) = \mathrm{dom}(D_{\min}) = \mathrm{H}^m_0(M,E)\,.$$

This yields what we wanted in [Req 1] and [Req 2] when M is compact with boundary. That is, we have that

$$\check{\mathrm{H}}(D) = \gamma\,\mathrm{dom}(D_{\max}) \underset{\text{densely}}{\subset} \bigoplus_{j=0}^{m-1} \mathrm{H}^{-\frac{1}{2}-j}(\partial M, E)\,,$$

with $\ker\gamma = \mathrm{dom}(D_{\min})$. Therefore, we can topologise $\check{\mathrm{H}}(D)$ such that it is isomorphic to ${}^{\mathrm{dom}(D_{\max})}\!\diagup_{\mathrm{dom}(D_{\min})}$.

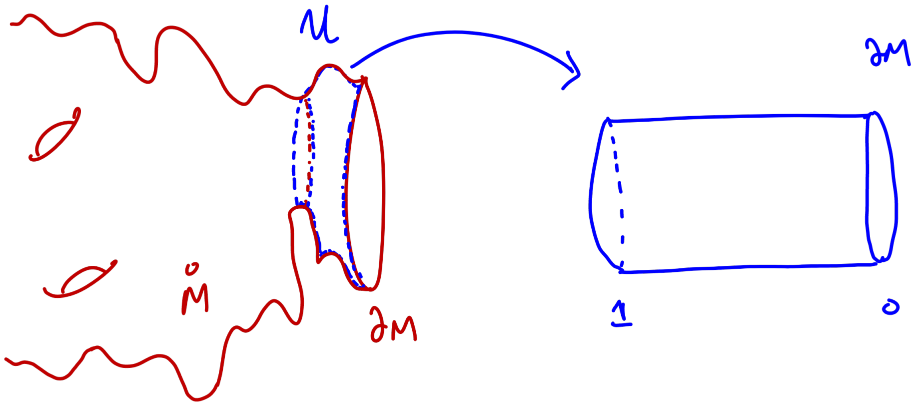
What we have really achieved is a concrete description of $\check{\mathrm{H}}(D)$ since it sits inside the negative Sobolev scales which we constructed concretely. Therefore, $\check{\mathrm{H}}(D)$ is no longer an abstract space. We know where it lives (in a finite sum of negative order Sobolev spaces *on the boundary*) and how it lives (it is densely embedded into this sum).

3.11.1 Noncompact manifolds with compact boundary

It is not a focus here to give an explicit description of what should happen in the situation of noncompact boundary for general-order problems, although, we will consider this in the first-order setting in later parts. Nevertheless, let us make a remark about this setting. Suppose that M non-compact but with ∂M compact. These are traditionally known as *exterior domains* in the Euclidean context. In order to use the pre-existing compact result, we would need to make an assumption. The obvious natural assumption would be one we have already seen: $\mathrm{C}^\infty_{\mathrm{c}}(M,E)$ is dense in $\mathrm{dom}(D_{\max})$.

Let us also recall an important result in the setting of manifolds with boundary.

Theorem 3.70 (Collar neighbourhood theorem). *There is an open neighbourhood $U \subset M$ of the boundary $\partial M \subset U$ s.t. U is diffeomorphic to $[0,1) \times \partial M$.*



In the setting of compact boundary, we would obtain that $\left[0, \frac{3}{4}\right) \times \partial M$ is a precompact set. This would allow us to institute a cutoff $\chi \in \mathrm{C}^\infty \cap \mathrm{L}^\infty([0,1) \times \partial M, [0,1])$ s.t. $\chi = 1$ inside $\left[0, \frac{1}{2}\right) \times \partial M$ and $\chi = 0$ on $\left[\frac{3}{4}, 1\right)$. It is clear that for such a cutoff, $\operatorname{spt}(d\chi) \subset \left(\frac{1}{2}, \frac{3}{4}\right) \times \partial M$ and therefore, $\operatorname{spt}(d\chi)$ is a compact set. Naturally, we can extend χ to all of M by zero, and we identify the extension again with χ itself. Let $u \in \operatorname{dom}(D_{\max})$, $u_n \in \mathrm{C}^\infty(M, E)$, $u_n \to u$. Then $u_n = (1-\chi)u_n + \chi u_n$ where χu_n is compactly supported near ∂M. Then $\gamma((1-\chi)u_n) = 0$. The desired results can now be obtained from by reduction to the compact case. We leave it as an exercise to consider the details of this construction, and a precise formulation of Theorem 3.69 in the setting of a noncompact manifold but with compact boundary.

3.11.2 Dismantling the Czech space

Despite the risk of repetition, let us again recall what we have done. We have managed to understand the Czech space $\check{\mathrm{H}}(D)$ associated with D as a finite sum of negative order Sobolev spaces on the boundary. For us, this achieves [Req 1] and [Req 2]. That means we have brought the interior problem, i.e., ${}^{\operatorname{dom}(D_{\max})}\!/\!_{\operatorname{dom}(D_{\min})}$, to the boundary, i.e. $\check{\mathrm{H}}(D)$ is a function space sitting inside a sum of negative order Sobolev spaces *over* the boundary.

But we would still like to resolve [Req 3]. So far, $\check{\mathrm{H}}(D)$ is topologised forcibly requiring it to be Banach space isomorphic to ${}^{\operatorname{dom}(D_{\max})}\!/\!_{\operatorname{dom}(D_{\min})}$. Resolving [Req 3] would mean that we can understand the topology of $\check{\mathrm{H}}(D)$ from data intrinsic to the boundary. Another way of putting it, we would like to dismantle or find an equivalent topology for $\check{\mathrm{H}}(D)$ via data obtained from ∂M. We will show that this is possible to a certain extent. Throughout this subsection, unless otherwise stated, we assume that M is compact.

Definition 3.71. Define $\mathcal{C}_D := \gamma \ker(D_{\max})$, the Hardy space of solutions of D.

Note that it is not immediate the space $\mathcal{C}_D$ is closed in $\check{\mathrm{H}}(D)$, even though $\ker(D_{\max})$ is closed in both $\mathrm{L}^2(M', E)$ and $\mathrm{dom}(D_{\max})$. We can only invoke Proposition 3.22 to conclude a space is closed when it contains $\mathrm{dom}(D_{\min})$ and it is certainly not true that $\mathrm{dom}(D_{min}) \subset \ker(D_{\max})$. In fact, $\mathrm{dom}(D_{\min}) \cap \ker(D_{\max}) = \ker(D_{\min})$ is always a finite dimensional space.

Theorem 3.72 (Theorem in [43]). *There is a projector*

$$\mathcal{P}_{\mathcal{C}} : \bigoplus_{j=0}^{m-1} \mathrm{H}^{-\frac{1}{2}-j}(\partial M, E) \to \bigoplus_{j=0}^{m-1} \mathrm{H}^{-\frac{1}{2}-j}(\partial M, E)$$

with

$$\mathcal{C}_D = \mathcal{P}_{\mathcal{C}} \bigoplus_{j=0}^{m-1} \mathrm{H}^{-\frac{1}{2}-j}(\partial M, E)\,.$$

This projector restricts to a projector on

$$\bigoplus_{j=0}^{m-1} \mathrm{H}^{s-\frac{1}{2}-j}(\partial M, E)$$

for all $s \geq 0$. *Moreover,*

$$\check{\mathrm{H}}(D) = (I - \mathcal{P}_{\mathcal{C}}) \left[\bigoplus_{j=0}^{m-1} \mathrm{H}^{m-\frac{1}{2}-j}(\partial M, E) \right] \oplus \underbrace{\mathcal{P}_{\mathcal{C}} \left[\bigoplus_{j=0}^{m-1} \mathrm{H}^{-\frac{1}{2}-j}(\partial M, E) \right]}_{=\mathcal{C}_D}.$$

Definition 3.73. Such a projector $\mathcal{P}_{\mathcal{C}}$ is called a *Calderón projector.*

This theorem says that we can understand $\check{\mathrm{H}}(D)$ 'almost' with boundary information alone. We say 'almost' because $v \in \mathcal{C}_D$ if and only if there exists $u \in \ker(D_{\max})$. If we alter the operator D, then that alternation will be reflected in $\ker(D_{\max})$ and hence in $\mathcal{C}_D$. As it stands, our description of $\check{\mathrm{H}}(D)$ still 'feels' the interior to a certain extent.

Let us imagine the way in which we can divorce a description of $\check{\mathrm{H}}(D)$ from D. Functional analytically, what this would mean is that $\check{\mathrm{H}}(D)$ should not see what happens to the operator far away from D, but rather, only in an arbitrarily small neighbourhood of the boundary. As a first attempt, let us try to rephrase this more precisely in the language of operator theory. Suppose that $\widetilde{D}$ is another operator which agrees with D on a small neighbourhood of ∂M. By this, we mean that the coefficients of these two operators are the same in this small neighbourhood. We would like in this case to obtain that $\check{\mathrm{H}}(\widetilde{D}) = \check{\mathrm{H}}(D)$. This is indeed true.

Proposition 3.74. *Let $D, \widetilde{D} \in \mathrm{Diff}_m(E, F)$, both elliptic, and suppose that there exists an open neighbourhood U of ∂M such that $D|_U = \widetilde{D}|_U$. Then, $\check{\mathrm{H}}(D) = \check{\mathrm{H}}(\widetilde{D})$.*

Proof. We leave this as an exercise. Hint: Use the cutoffs as we discussed in subsection 3.11.1. □

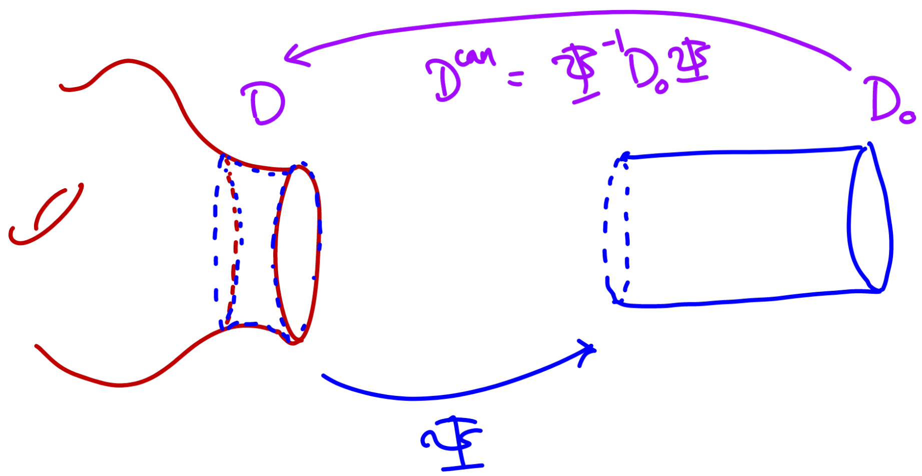
Let us now embark on a second attempt. Suppose that we fix some sort of operator on the boundary, determined only from the principal symbol information of D on the boundary. We would like this operator to determine some canonical operator D^{can}. For instance, if in $\mathbb{R}^n_+$, we have the operator

$$Du = \sum_{j=1}^{n} A^j \partial_j u + Bu\,,$$

we may want to consider the operator

$$D^{\mathrm{can}} u = \sum_{j=1}^{n} A^j \partial_j u\,.$$

Typically, such a canonical operator would have a geometric meaning. It might arise from attaching a cylinder at the boundary, or equivalently, pulling back a simpler operator on the cylinder via the collar neighbourhood theorem.



In contrast to our first attempt, the operators D and D^{can} do not agree near ∂M. Therefore, it is unclear whether we can asset that $\check{\mathrm{H}}(D^{\mathrm{can}}) = \check{\mathrm{H}}(D)$. Were it plausible to do so, then we would be able to describe the Czech space of a complicated operator such as D via a simpler 'model' operator D_0. We do not consider this problem for the general-order situation, but this will of paramount importance to our analysis in the first-order case.

3.11.3 Some types of boundary conditions

Let us recall from Definition 3.24 that we call B is a *boundary condition* if $B \subset \check{\mathrm{H}}(D)$ is a closed subspace. By D_B, we denote the induced operator with domain $\mathrm{dom}(D_B) = \{u \in \mathrm{dom}(D_{\max}) \mid \gamma u \in B\}$.

Definition 3.75 ((semi)-Elliptically regular). A boundary condition B is called *semi-elliptically regular* if $B \subset \bigoplus_{j=0}^{m-1} \mathrm{H}^{m-\frac{1}{2}-j}(\partial M, E)$. If both B and $B^\dagger$ are semi-elliptically regular, we say that B is *elliptically regular*.

Example 3.76.
1. $B = (I - \mathcal{P}_{\mathcal{C}})\left[\bigoplus_{j=0}^{m-1} \mathrm{H}^{m-\frac{1}{2}-j}(\partial M, E)\right]$ is certainly a closed subspace of $\check{\mathrm{H}}(D)$. Therefore, it is semi-elliptically regular. Note that, $\mathrm{dom}(D_B) \subset \mathrm{H}^m(\partial M, E)$. In actual fact, B is elliptically regular, but it is beyond the scope of the methods we have developed so far to demonstrate this.
2. $\mathcal{C}_D := \mathcal{P}_{\mathcal{C}}\left[\bigoplus_{j=0}^{m-1} \mathrm{H}^{-\frac{1}{2}-j}(\partial M, E)\right]$ certainly closed in $\check{\mathrm{H}}(D)$. However, $\mathrm{dom}(D_{\mathcal{C}_D}) \not\subset \mathrm{H}^m(\partial M, E)$. Therefore, it is not semi-elliptically regular or even elliptically regular. In fact $\dim(\mathcal{C}_D) = \infty \Rightarrow \dim\ker(D_{\max}) = \infty$.

Remark 3.77. Example 2. highlights an extremely important fact. It shows that, unlike in the closed manifold situation, elliptic operators may fail to satisfy elliptic regularity if the boundary condition is not chosen carefully. The regularity of the domain of the induced operator, which is of paramount importance in applications to both global analysis and PDE, require both ellipticity of the operator and a certain kind of 'ellipticity' of the boundary condition. This is what Definition 3.75 captures.

Definition 3.78. A boundary condition B is called Fredholm if D_B is a Fredholm operator. That is, $\mathrm{ran}(D_B)$ is closed and $\ker D_B$ as well as

$$\operatorname{coker} D_B := \mathrm{L}^2(M, F) \big/ \mathrm{ran}(D_B)$$

are both finite dimensional.

Proposition 3.79. *Every elliptically regular B is Fredholm.*

Remark 3.80. Again, unlike the closed case, the converse is not true. For instance, let $F \subset \mathcal{C}_D$ be a finite dimensional subspace such that

$$F \subset \oplus_{j=0}^{m-1} \mathrm{H}^{-\frac{1}{2}-j}(\partial M, E) \setminus \bigoplus_{j=0}^{m-1} \mathrm{H}^{m-\frac{1}{2}-j}(\partial M, E)\,.$$

We have already seen that $\ker \mathcal{P}_{\mathcal{C}}$ is elliptic, and therefore Fredholm. Now, consider $B = \ker \mathcal{P}_{\mathcal{C}} \oplus F$. Since F is finite dimensional, it is readily verified that

B is a boundary condition. B is just a finite dimensional alteration of $\ker \mathcal{P}_{\mathcal{C}}$, and therefore, the resulting operator D_B must be Fredholm. However, it is clear from construction that B is not semi-elliptically regular, and hence, it cannot be elliptically regular.

A very important result for boundary value problems, mirroring the higher elliptic regularity on closed manifolds, is the following regularity theorem. It asserts when we can obtain Sobolev regularity up to the boundary.

Theorem 3.81 (Higher boundary regularity). *We have that*

$$\begin{aligned}&\mathrm{dom}(D_{\max}) \cap \mathrm{H}^{s+m}(M,E)\\ &= \left\{ u \in \mathrm{dom}(D_{\max}) \;\middle|\; D_{\max} u \in \mathrm{H}^s(M,E) \text{ and } \mathcal{P}_{\mathcal{C}}\gamma u \in \bigoplus_{j=0}^{m-1} \mathrm{H}^{s+m-\frac{1}{2}-j}(\partial M, E) \right\}.\end{aligned}$$

Example 3.82. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, $\vec{n}$ the inner normal along ∂M, and $\Delta = -\sum_i \partial_i$ the usual $\mathbb{R}^n$ Laplacian. Recall the classical adjoint formula for $u, v \in \mathrm{C}^\infty(\Omega)$:

$$\int_\Omega \Delta u \overline{v} - \int_\Omega u \Delta \overline{v} = \int_{\partial\Omega} \partial_{\vec{n}} u \overline{v} - \int_{\partial\Omega} u \partial_{\vec{n}} \overline{v}\,. \tag{3.9}$$

1. The Dirichlet problem is defined as the Laplacian with domain

$$\mathrm{dom}(\Delta_{\mathrm{D}}) := \left\{ u \in \mathrm{dom}(\Delta_{\max}) \;\middle|\; u|_{\partial\Omega} = 0 \right\}.$$

On the boundary, inside the $\check{\mathrm{H}}(\Delta)$ space, this manifests as

$$B = \gamma\,\mathrm{dom}(\Delta_{\mathrm{D}}) = \left\{ \left(0, (\partial_{\vec{n}} u)|_{\partial\Omega}\right) \;\middle|\; u \in \mathrm{dom}(\Delta_{\max}) \right\} \subset \mathrm{H}^{-\frac{1}{2}}(\partial\Omega) \oplus \mathrm{H}^{-\frac{3}{2}}(\partial\Omega)\,.$$

a) *Claim:* Δ_{D} is self-adjoint.

For $v \in \mathrm{dom}(\Delta_{\mathrm{D}}^*)$ have

$$\int_\Omega \Delta_{\mathrm{D}} u \overline{v} = \int_\Omega u \Delta_{\mathrm{D}}^* \overline{v}\,.$$

From (3.9), we must have

$$\int_{\partial\Omega} (\partial_{\vec{n}} u) v = 0\,.$$

We cannot have that all $\partial_{\vec{n}} u = 0$, and by considering appropriate u such that $\partial_{\vec{n}} u \neq$, we obtain $v|_{\partial\Omega} = 0$. That is, we have $v \in \mathrm{dom}(\Delta_{\max})$ and $v|_{\partial\Omega} = 0$ so therefore, $v \in \mathrm{dom}(\Delta_{\mathrm{D}})$.

b) *Claim:* $\mathrm{dom}(\Delta_D) = \mathrm{dom}(\Delta_{\max}) \cap \mathrm{H}_0^1(\Omega)$.

For $u \in \mathrm{dom}(\Delta_{\max})$,

$$\int_\Omega \Delta_{\mathrm{D}} u \overline{u} = \int \nabla u \cdot \nabla \overline{u} = \|\nabla u\|^2$$

and s $u \in \mathrm{H}^1$. Moreover, since $u \in \mathrm{dom}(\Delta_{max})$ implies $u|_{\partial\Omega} = 0$, we conclude $u \in \mathrm{H}_0^1(\Omega)$.

c) *Claim:* $\mathrm{dom}(\Delta_{\mathrm{D}}) = \mathrm{H}^2 \cap \mathrm{H}_0^1(\Omega)$. Indeed,

$$\begin{aligned} u \in \mathrm{dom}(\Delta_{\mathrm{D}}) \quad &\Rightarrow \quad u \in \mathrm{dom}(\Delta_{\max}),\ \Delta_{\mathrm{D}} u \in \mathrm{L}^2(\Omega),\ (\partial_{\vec{n}} u)|_{\partial\Omega} \in \mathrm{H}^{\frac{1}{2}}(\partial\Omega) \\ &\Rightarrow \quad B_{\mathrm{D}} \text{ elliptically regular} \\ &\Rightarrow \quad u \in \mathrm{H}^2(\Omega)\,. \end{aligned}$$

Therefore, $u \in \mathrm{H}^2(\Omega)$.

2. The Neumann problem is given by specifying the following domain:

$$\mathrm{dom}(\Delta_{\mathrm{N}}) := \left\{ u \in \mathrm{dom}(\Delta_{\max}) \;\middle|\; (\partial_{\vec{n}} u)|_{\partial\Omega} = 0 \right\}.$$

Similar conclusions can be obtained for the Neumann Laplacian by a similar examination as for the Dirichlet case.

Remark 3.83. The Dirichlet and Neumann problems can be obtained in an alternative way, which makes it possible for greater generalisation and also for the study of fractional Sobolev scales on manifolds with boundary, without having access to any trace theorems. For that, we have to become familiar with a gadget called an energy.

On a Hilbert space, let $\mathcal{E} : \mathrm{dom}(\mathcal{E}) \times \mathrm{dom}(\mathcal{E}) \to \mathbb{C}$ be a symmetric form, $\mathrm{dom}(\mathcal{E}) \subset \mathrm{L}^2(M, E)$ is dense, and $(\mathrm{dom}(\mathcal{E}), \|\cdot\|_{\mathcal{E}})$ with $\|u\|_{\mathcal{E}}^2 := \|u\|_{\mathrm{L}^2}^2 + \sqrt{\mathcal{E}[u,u]}$ is a Banach space. Then there is a unique non-negative self-adjoint operator $\Delta_{\mathcal{E}}$, $\mathrm{dom}(\Delta_{\mathcal{E}}) \subset \mathrm{dom}(\mathcal{E})$, $\mathrm{dom}\big(\sqrt{\Delta_{\mathcal{E}}}\big) = \mathrm{dom}(\mathcal{E})$ satisfying

$$\mathcal{E}[u,v] = \left\langle \sqrt{\Delta_{\mathcal{E}}} u, \sqrt{\Delta_{\mathcal{E}}} v \right\rangle_{\mathrm{L}^2(M,E)}.$$

See representation theorems 1 add 2 in [33].

Returning back to the Dirichlet problem, we consider the energy

$$\mathcal{E}_{\mathrm{D}}[u,v] = \underbrace{\int \nabla u \cdot \nabla u}_{\text{Euclidean metric}} \, \mathrm{d}\mathscr{L} \qquad \text{with} \qquad \mathrm{dom}(\mathcal{E}_{\mathrm{D}}) = \mathrm{H}_0^1(\Omega)\,.$$

More generally, on a manifold with boundary, given a Riemannian metric g, we can consider

$$\mathcal{E}_{\mathrm{D},g}[u,v] = \int g(du, dv)\, \mathrm{d}\mu_g$$

for $u, v \in \mathrm{H}^1_0(M, g)$. That is precisely the energy with domain $\mathrm{H}^1_0(M, g)$. The operator obtained from this energy is precisely Δ_{D}, the Dirichlet Laplacian.

The Neumann problem then corresponds to the same energy but with domain $\mathrm{H}^1(M, g)$

$$\mathcal{E}_{\mathrm{N},g}[u, v] = \int g(du, dv)\ \mathrm{d}\mu_g \qquad \text{for } u, v \in \mathrm{H}^1(M, g).$$

There are numerous advantages with the energy approach. Firstly, there is no requirement to know about the boundary trace. Therefore, given any open manifold, without knowing anything about the boundary, one can extract both a Dirichlet and Neumann Laplacian. Secondly, there's no requirement that the metrics be smooth. In this situation, one will not expect the domain of the operators to be contained in $\mathrm{H}^2(M, g)$, and this could even be an operator theoretic measure of non-smoothness.

The energy methods also have a deep shared history with divergence form operators. Geometrically, such operators can be seen as changing to an auxiliary smooth background and where the original non-smoothness of the coefficients are now contained in the coefficients of an operator, in divergence form. In that case, the Laplacian might look like $\Delta_D = -a \operatorname{div} A\nabla$, where a and A contain the original metric. These operators can make sense down to the level of measurable coefficient a and A, and therefore, the smoothness requirements on the Riemannian metric g can be reduced significantly. See [8] where a useful class of measurable coefficient metrics called 'rough metrics' are defined and examined in more detail.

4 First-order elliptic operators

From this chapter onward, we will be largely concerned with first-order elliptic differential operators. General-order examples may appear to highlight salient features of general theory we discuss along the way. From here on, we assume the following:

[FO1] (M, μ) is a measured manifold.

[FO2] ∂M is compact.

[FO3] T is an inward pointing vectorfield.

[FO4] $\left(E, h^E\right), \left(F, h^F\right) \to M$ are Hermitian vector bundles.

[FO5] $D \in \mathrm{Diff}_1(E, F)$ is elliptic.

4.1 Adapted boundary operators

In subsection 3.11.2, we described the Czech space in terms of a Calderón projector. Towards the end, we discussed describing the Czech space with respect to a Calderón projector of another operator, the so-called 'model' operator. The first kind of description was when two operators agree on a neighbourhood of the boundary, and there, in Proposition 3.74, we showed this was always possible.

However, the more challenging goal was to describe $\check{\mathrm{H}}(D)$ in terms of $\check{\mathrm{H}}(D_0)$, where D_0 is built out of a part of D, but not equal to D, on a neighbourhood of the boundary. The aim of this section is to consider this goal. This is accomplished by first describing an important class of operators on the boundary called *adapted boundary operators*. These adapted boundary operators are determined from the principal symbol of D, but it is truly an object that lives on ∂M. Every adapted boundary operator then induces a 'model' operator D_0. Later, we will see that this leads to a true description of $\check{\mathrm{H}}(D)$ from boundary information alone.

Below, we demonstrate a geometric reduction lemma that upgrades the topological collar neighbourhood theorem, Theorem 3.70, to a geometric one. For this, we require the existence of an interior point covectorfield to interact with the vectorfield T.

Lemma 4.1. *There exists a unique covector field $\tau \in \mathrm{C}^\infty(\partial M, T^*M)$ s.t.*

$$T\partial M \subset \ker(\tau) \qquad \textit{and} \qquad \tau[T] = 1\,.$$

Proof. This is a routine pointwise calculation and is left as an exercise. □

The following lemma is extracted from [10], where it can be found as Lemma 2.4.

Lemma 4.2 (Geometric reduction). *There is an open neighbourhood $U \subset M$ with $\partial U \subset M$, an $R > 0$ and a $\Psi := (t, \psi) : U \to Z_R := [0, R) \times \partial M$ s.t. the following hold.*

I) $\partial M = t^{-1}(0)$.

II) $\psi|_{\partial M} = \mathrm{id}_{\partial M}$.

III) $d\Psi(T) = \partial_t$ *along* ∂M.

IV) $\tau = dt$ *along* ∂M.

V) $\Psi_*(\mu) = |dt| \otimes \nu$, *where* ν *is the induced measure on the boundary.*

Proof. a) First, extend T to $\tilde{T} \neq 0$ in some open neighbourhood U_0 of ∂M.

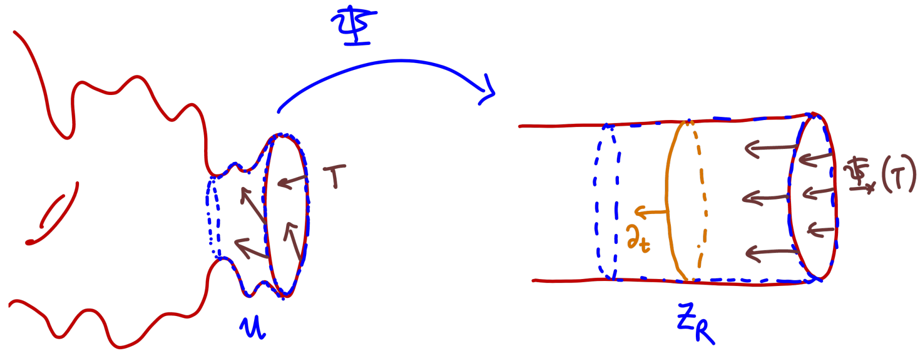
b) Then, solve for $f : U_1 \subset U_0 \to \mathbb{R}$, $\partial M \subset U_1$, where U_1 is a possibly smaller open set, satisfying the differential equation:

$$\begin{cases} 0 = \mathrm{div}_\mu\left(f\tilde{T}\right) = df\left(\tilde{T}\right) + f\,\mathrm{div}_\mu \tilde{T}\,, \\ f|_{\partial M} = 1\,. \end{cases}$$

This is, in fact, an ODE along integral curves of $\tilde{T}$ and therefore, we obtain a unique smooth f. Here, for the divergence div_μ with respect to the density μ, we have used the fact that there is a Riemannian g induced by μ and T such that $\mathrm{d}\,\mathrm{vol}_g = \mathrm{d}\mu$. Therefore, we have that $\mathrm{div}_\mu = \mathrm{div}_g$, where the latter object is obtained via the associated Levi-Civita connection.

c) Let $\tilde{\Psi}$ be the flow associated to $f\tilde{T}$. By the compactness of ∂M there is a $R > 0$ and U_2 open, again possibly smaller than U_1, with $\partial M \subset U_2$ such that $Z_R \to U_2$ given by $(t, x) \mapsto \tilde{\Psi}_t(x)$ is a diffeomorphism.

d) Set $\Psi := \tilde{\Psi}^{-1}$ and $U := U_2$. By construction, we obtain I) through to IV). Here, f was used to ensure that $\mathrm{div}_\mu\left(f\tilde{T}\right) = 0$, which yields V). □



Let us now examine D in $U \cong [0,R) \times \partial M = Z_R$. For that, first note that on M, by the ellipticity of D, there exists a unique connection ∇^D such that, given a local frame $\{e_i\}$ for TM near x we have that

$$Du(x) = \sum_{i=1}^{n} \sigma_D\big(x, e^i\big)\nabla^D_{e_i} u(x)\,,$$

where e^i is the dual frame. Now, let us focus on $x \in U$ and $e_n = \partial_t$. Then,

$$\begin{aligned}
Du(x) &= \sum_{i=1}^{n} \sigma_D\big(x, e^i\big)\nabla^D_{e_i} u(x) \\
&= \sum_{i=1}^{n-1} \sigma_D\big(x, e^i\big)\nabla^D_{e_i} u(x) + \sigma_D(x, dt)\nabla^D_{\partial_t} u(x) \\
&= \sigma_D(x, dt)\left(\sum_{i=1}^{n-1} \sigma_D(x, dt)^{-1}\sigma_D\big(x, e^i\big)\nabla^D_{e_i} u(x) + \nabla^D_{\partial_t} u(x)\right) \\
&= \sigma_D(x, dt)\left(\partial_t u(x) + \underbrace{\sum_{i=1}^{n-1} \underbrace{\underbrace{\sigma_D(x, dt)^{-1}}_{F_x \to E_x}\underbrace{\sigma_D\big(x, e^i\big)}_{E_x \to F_x}}_{E_x \to E_x}\nabla^D_{e_i} u(x)}_{\in \mathrm{Diff}_1\left(E|_{\partial M}\right)\ \text{elliptic}} + \underbrace{\big(\nabla^D_{\partial t} - \partial_t\big)}_{\in \mathrm{Diff}_0} u(x)\right).
\end{aligned} \tag{4.1}$$

From this, it is clear that the interesting operator of order one on the boundary is precisely

$$\sigma_D(x, dt)^{-1}\sigma_D\big(x, e^i\big)\nabla^D_{e_i} u(x)\,.$$

On making the required identifications, it is easy to see that this is an operator acting on the bundle $E|_{\{t\}\times\partial M} \to \{t\} \times \partial M$.

Definition 4.3. An operator $A \in \mathrm{Diff}_1\Big(E|_{\partial M}\Big)$ is called an *adapted boundary operator to* D if its principal symbol satisfies

$$\sigma_A(x, \xi) = \sigma_D(x, \tau)^{-1} \circ \sigma_D(x, \xi)$$

for all $\xi \in T^*_x \partial M$.

Remark 4.4. Clearly, an adapted operator on the boundary A is elliptic.

Remark 4.5. Since ∂M has no boundary and A is elliptic, we have that $\bar{A} = A_{\max} = A_{\min}$. Consequently, with a slight abuse of notation, we will denote by A the operator $A \in \mathrm{Diff}_1(E|_{\partial M})$ as well as $\bar{A} : \mathrm{H}^1(\partial M, E) \subset \mathrm{L}^2(\partial M, E) \to \mathrm{L}^2(\partial M, E)$.

Given that we have highlighted this class of operators, the goal is to be able to control $\check{\mathrm{H}}(D)$ in terms of A, an operator purely on the boundary. The point here to note is that the relationship between A and D is only that A is built out of the principal symbol of D restricted to the boundary. Therefore, A is really an operator which only sees the boundary and nothing else. However, as we expressed in subsection 3.11.2, it is not immediate how we can control the operator D knowing an operator A only defined on the boundary. For that, we have the following lemma which gives a hint of how to perform an operator theoretic reduction. This lemma appears as Lemma 4.1 in [10], albeit there, it is assumed that A is symmetric.

Lemma 4.6 (Operator reduction). *Let $A, \tilde{A}$ be adapted boundary operators to D and $D^\dagger$ respectively. Then there are differential operators $R_t \in \mathrm{Diff}_1\left(E|_{\partial M}\right)$ and $\tilde{R}_t \in \mathrm{Diff}_1\left(F|_{\partial M}\right)$ (i.e. at most order 1) on Z_R varying smoothly in $t \in [0, R)$ s.t. inside $U \cong Z_R$,*

$$D = \sigma_t(\partial_t + A + R_t) \qquad \textit{and} \qquad D^\dagger = -\sigma_t^*\left(\partial_t + \tilde{A} + \tilde{R}_t\right),$$

where $\sigma_t := \sigma_D(x, dt)$. Moreover, given $R' < R$, there exists a constant $C' < \infty$ s.t.

$$\|R_t u\|_{\mathrm{L}^2(\partial M)} \le C'\left(t\|Au\|_{\mathrm{L}^2(\partial M)} + \|u\|_{\mathrm{L}^2(\partial M)}\right),$$
$$\left\|\tilde{R}_t u\right\|_{\mathrm{L}^2(\partial M)} \le C'\left(t\left\|\tilde{A}u\right\|_{\mathrm{L}^2(\partial M)} + \|u\|_{\mathrm{L}^2(\partial M)}\right)$$

in $Z_{R'}$.

Proof. We only prove this statement for the operator D, given a boundary adapted operator A, since the proof for $D^\dagger$ given $\tilde{A}$ is obtained by the exact same argument.

As we have already seen in (4.1),

$$D = \sigma_t(\partial_t + X_t),$$

where $\sigma_t := \sigma_D(x, dt)$ and $X_t \in \mathrm{Diff}_1(E|_{\partial M})$ elliptic. Restricting to the boundary, i.e. $t = 0$, it is clear that $\sigma_{X_0}(x, \xi) = \sigma_A(x, \xi)$.

Define:

$$R_t := X_t - A.$$

Note that R_0 is of order 0 since the principal symbol of X_0 and A are equal. Therefore, we have that $R_t \in \mathrm{Diff}_1\big(E|_{\partial M}\big)$ and locally, inside a coordinate chart, we can write

$$R_t u = \sum_i R_t^i \partial_i u + S_t u\,,$$
$$(R_t - R_0)u = \sum_i \big(R_t^i - R_0^i\big)\partial_i u + (S_t - S_0)u\,.$$

Since $t \mapsto R_t^i$ and $t \mapsto S_t$ are smooth, which means precisely that their coefficients are smooth, in particular implies that these coefficients are Lipschitz. Given $R' < R$, setting $R'' := R' + \frac{R-R'}{2}$, on $\overline{Z_{R''}}$, we obtain the estimates

$$\big|\big(R_t^i - R_0^i\big)v\big| \lesssim t|v| \qquad \text{and} \qquad |(S_t - S_0)v| \lesssim t|v|\,.$$

Here, the implicit Lipschitz constant obviously depends on R''. Therefore, for $t \in [0, R']$, we have

$$|(R_t - R_0)u|^2 \lesssim t^2 \left(\sum_{i=1}^n |\partial_i u|^2 + |u|^2\right).$$

Instituting a finite partition of unity using the compactness of ∂M, patching, and integrating the resulting expression, we obtain

$$\|(R_t - R_0)u\|_{\mathrm{L}^2(\partial M)} \lesssim t\|u\|_{\mathrm{H}^1(\partial M)}\,.$$

Therefore,

$$\begin{aligned}
\|R_t u\|_{\mathrm{L}^2(\partial M)} &\lesssim t\|u\|_{\mathrm{H}^1(\partial M)} + \|R_0 u\|_{\mathrm{L}^2(\partial M)} \\
&\lesssim t\|u\|_{\mathrm{H}^1(\partial M)} + \|u\|_{\mathrm{L}^2(\partial M)} \\
&\lesssim t\Big(\|Au\|_{\mathrm{L}^2(\partial M)} + \|u\|_{\mathrm{L}^2(\partial M)}\Big) + \|u\|_{\mathrm{L}^2(\partial M)} \\
&\lesssim t\|Au\|_{\mathrm{L}^2(\partial M)} + \|u\|_{\mathrm{L}^2(\partial M)}\,,
\end{aligned}$$

where in the first inequality we used the reverse triangle inequality, in the second the fact that R_0 is a bounded operator, and the penultimate inequality we used the ellipticity of A. □

Definition 4.7 (Model operator on Z_R given adapted A). Given an adapted boundary operator A to D, the operator

$$D_0 := \sigma_0(\partial_t + A)\,,$$

where $\sigma_0 = \sigma_D(x, \tau) : E_x \to F_x$, is called the *model operator.*

Remark 4.8. Note that D_0 induces an induces an adapted boundary operator

$$\tilde{A} := -\big(\sigma_0^{-1}\big)^* A^* \sigma_0^*$$

for $D^\dagger$. This is obtained by simply computing the formal adjoint of D_0 in Z_R:

$$D_0^\dagger = (-\partial_t + A^*)\sigma_0^* = -\sigma_0^*\big(\partial_t - \big(\sigma_0^{-1}\big)^* A^* \sigma_0^*\big)\,.$$

Let us again recall our discussion earlier, as well as in section 3.11.2, where we alluded to the fact that we wish to control D in terms of boundary information alone. So far, we have highlighted a class of operators A, the adapted boundary operators to D, as candidates which will allow us to accomplish this goal. The model operator simply 'pushes' A to the interior and it can be thought of as an interior realisation of A. What we mean precisely by this is that we wish to obtain $\check{\mathrm{H}}(D_0) = \check{\mathrm{H}}(D)$, and then have $\check{\mathrm{H}}(D_0)$ being described by the operator A.

In order to accomplish such a goal, an important necessary condition is to know that the maximal domain of D_0 and D are equal on a neighbourhood of ∂M. Given the geometric reduction as well as operator theoretic reduction in Lemmas 4.2 and 4.6, we have candidate neighbourhoods where we can anticipate this equality to hold. In the earlier reduction which we presented, i.e., in Proposition 3.74, this was immediate since $\widetilde{D}$ and D were equal on a neighbourhood of ∂M. At this stage, we will not embark on a rigorous proof of the reduction of D to D_0. However, consider the following calculation:

$$\begin{aligned} D - D_0 &= \sigma_t(\partial_t + A + R_t) - \sigma_0(\partial_t + A) \\ &= (\sigma_t - \sigma_0)\sigma_0^{-1} D_0 + \sigma_t R_t \,. \end{aligned} \tag{4.2}$$

For an $R' < R$, it is clear that the term $(\sigma_t - \sigma_0)$ is uniformly bounded inside $Z_{R'}$ due to smoothness of σ_t and compactness of ∂M. Moreover, by Lemma 4.6, the remainder term $\sigma_t R_t$ is controlled by A.

Another noteworthy result is the following.

Proposition 4.9. *For all $u \in \mathrm{C}_c^\infty(M, E)$ and all $v \in \mathrm{C}_c^\infty(M, F)$ we have*

$$\langle Du, v\rangle_{\mathrm{L}^2(M,F)} - \left\langle u, D^\dagger v\right\rangle_{\mathrm{L}^2(M,E)} = \left\langle \sigma_0 u|_{\partial M}, v|_{\partial M}\right\rangle_{\mathrm{L}^2(\partial M,F)} .$$

We see here that the key player that appears in the boundary term is in fact σ_0.

Therefore, combining equation (4.2) along with Proposition 4.9, we have a strong indication of the plausibility to reduce questions regarding D to D_0. At this stage, we will leave it as impressionistic and suggestive, and return to a rigorous analysis later. However, we see from this that understanding the model operator on the cylinder is of paramount importance in order for this reduction to work. In particular, this means having to study the operator theory of A in greater detail.

Let us make one further remark to conclude the section. Eventually, we wish to extend Proposition 4.9 to $D_{\max}$ and $D^\dagger_{\max}$. But we are constrained by their values on $\mathrm{C}_c^\infty(M, E) \subset \mathrm{dom}(D_{\max})$ and $\mathrm{C}_c^\infty(M, F) \subset \mathrm{dom}(D_{\min})$. The pairing at the boundary, i.e. the term on the right is

$$\left\langle \sigma_0 u|_{\partial M}, v|_{\partial M}\right\rangle_{\mathrm{L}^2(\partial M,F)} .$$

This is the L^2-inner product on the boundary. Therefore, the extension to the maximal domains of the respective operators must respect this constraint. This is precisely what forces us away from the Hilbert space inner product of spaces where these boundary conditions lie and instead prompts us to consider perfect pairings between different Hilbert spaces.

4.2 Spectral Theory and H^∞-functional calculus

As aforementioned, we wish to understand adapted boundary operators in a more precise manner. That requires us to introduce one of the most significant aspects of operator theory - the spectrum of an operator. We give an account of this in full generality.

Let $T : \mathcal{B} \to \mathcal{B}$ be an operator on a Banach space.

Definition 4.10 (Resolvent and resolvent set). For $\zeta \in \mathbb{C}$ we say $\zeta \in \mathrm{res}(T)$ if $(\zeta - T)$ is invertible. That is, $(\zeta - T)$ has dense range and there is a $c \in (0, \infty)$ s.t. $\|(\zeta - T)u\| \geq c\|u\|$ for all $u \in \mathrm{dom}(T)$.

The set $\mathrm{res}(T)$ is called the *resolvent set* and for $\zeta \in \mathrm{res}(T)$, the operator $(\zeta - T)^{-1} \in \mathscr{B}(\mathcal{B})$ is a *resolvent*.

Remark 4.11. Note that when $\zeta \in \mathrm{res}(T)$, from the estimate in the definition, we automatically have that $(\zeta - T)$ is injective. Therefore, we have $(\zeta - T)^{-1} : \mathrm{ran}(\zeta - T) \to \mathcal{B}$, with the estimate

$$\left\|(\zeta - T)^{-1} u\right\| \leq c\|u\|,$$

for all $u \in \mathrm{ran}(\zeta - T)$. However, since $\mathrm{ran}(\zeta - T)$ is dense, the operator $(\zeta - T)^{-1}$ extends uniquely to a bounded operator on all of $\mathcal{B}$. It is for this reason we can consider the resolvent $(\zeta - T)^{-1} \in \mathscr{B}(\mathcal{B})$.

Proposition 4.12 (First resolvent equation). *For $\zeta, \xi \in \mathrm{res}(T)$, we have the first resolvent equation:*

$$(\zeta - T)^{-1} - (\xi - T)^{-1} = (\zeta - \xi)(\zeta - T)^{-1}(\xi - T)^{-1}.$$

In particular, resolvents commute.

Definition 4.13 (Spectrum). The spectrum of T is

$$\mathrm{spec}(T) := \mathbb{C} \setminus \mathrm{res}(T).$$

We say:

- $\lambda \in \mathrm{spec}_{\mathrm{p}}(T)$ if $(\lambda - T)$ is not injective,

- $\lambda \in \mathrm{spec_c}(T)$ if $(\lambda - T)$ is injective with dense-range but $(\lambda - T)^{-1}$ is unbounded,
- $\lambda \in \mathrm{spec_r}(T)$ if there is a $c \in (0, \infty)$ s.t. $\|(\lambda - T)u\| \geq c\|u\|$ but $(\lambda - T)$ does not have dense range.

The set $\mathrm{spec_p}(T)$ is called the *point spectrum*, $\mathrm{spec_c}(T)$ the *continuous spectrum* and $\mathrm{spec_r}(T)$ the *residue spectrum.*

As the following proposition highlights, only closed operators have the chance to enjoy a non-trivial spectral theory.

Proposition 4.14. *If $T : \mathcal{B} \to \mathcal{B}$ is not closed, then $\mathrm{spec}(T) = \mathbb{C}$.*

Proof. We prove the contrapositive. Suppose that $\mathrm{spec}(T) \subsetneq \mathbb{C}$. Therefore $\mathrm{res}(T) \neq \varnothing$ so fix $\zeta \in \mathrm{res}(T)$. Now, we have that by definition, $(\zeta - T)^{-1} \in \mathscr{B}(\mathcal{B})$. By Exercise 2.28, we then have that $((\zeta - T)^{-1})^{-1} = (\zeta - T)$ is closed. Since $v \mapsto \zeta v$ is a bounded operator, it follows that $T = (\zeta - T) + \zeta$ is also closed. Hence, T is a closed operator. □

As a consequence of this proposition, throughout, we will only consider closed operators for spectral theoretic purposes.

Proposition 4.15. *If $T \in \mathscr{C}(\mathcal{B})$ the following hold.*

I) $\mathrm{spec}(T)$ *is closed.*

II) $\mathrm{res}(T) \neq \varnothing \quad \Rightarrow \quad \zeta \mapsto (\zeta - T)^{-1}$ *is holomorphic.*

III) If $U : \mathcal{B} \to \mathcal{B}'$ is an isomorphism of Banach spaces, then

$$\mathrm{res}\big(U^{-1}TU\big) = \mathrm{res}(T) \qquad \textit{and} \qquad \mathrm{spec}\big(U^{-1}TU\big) = \mathrm{spec}(T)\,.$$

Moreover, whenever $\zeta \in \mathrm{res}(T)$, we have $(\zeta - U^{-1}TU)^{-1} = U(\zeta - T)^{-1}U^{-1}$.

In later parts, it is essential for us to also know how to relate the spectrum of an adjoint operator to the spectrum of the original operator. For this, let us first highlight the following important fact about adjoints of reflexive pairings.

Proposition 4.16. *Let $\langle \mathcal{B}_1, \mathcal{B}_2 \rangle$ be reflexive and $T : \mathcal{B}_2 \to \mathcal{B}$ densely-defined and closable. Then:*

I) The adjoint $T^* : \mathcal{B}_1 \to \mathcal{B}_1$ is densely-defined and closed.

II) $\ker(T^*) = \mathrm{ran}(T)^{\perp, \langle \mathcal{B}_1, \mathcal{B}_2 \rangle}$.

III) If $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{H}$ is a Hilbert space, then

$$\mathcal{H} = \ker(T^*) \oplus \overline{\operatorname{ran}(T)} = \ker\big(\overline{T}\big) \oplus \overline{\operatorname{ran}(T^*)}\,.$$

Proof. a) Ad I). The conclusion for the canonical adjoint $T^{*,\mathrm{can}}$ w.r.t. $\langle \mathcal{B}_2^*, \mathcal{B}_2\rangle$ is obtained by examining $\operatorname{inv}\operatorname{graph}(T^{*,\mathrm{can}}) = \operatorname{graph}(T)^\perp$. Since we assume $\langle \mathcal{B}_1, \mathcal{B}_2\rangle$ is reflexive, we obtain the stated conclusion for T^* through Proposition 2.48. We leave the details as an exercise.

b) Ad II). We first prove $\ker(T^*) \subset \operatorname{ran}(T)^{\perp,\langle \mathcal{B}_1,\mathcal{B}_2\rangle}$. For that, fix $u \in \ker(T^*)$. Then $\forall u \in \operatorname{dom}(T^*), w \in \operatorname{dom}(T)$ have

$$0 = \langle T^* u, w\rangle = \langle u, Tw\rangle\,.$$

That is, $u \in \operatorname{ran}(T)^{\perp,\langle \mathcal{B}_1,\mathcal{B}_2\rangle}$.

Next, we show that $\ker(T^*) \supset \operatorname{ran}(T)^\perp$. If $u \in \operatorname{dom}(T)^\perp$ we have $\langle u, Tw\rangle = 0$ for all $w \in \operatorname{dom}(T)$. Therefore,

$$0 = \langle u, Tw\rangle = \langle T^* u, w\rangle = 0$$

for all $w \in \operatorname{dom}(T)$. Since $\operatorname{dom}(T)$ is dense in $\mathcal{B}_2$ and we have a perfect pairing, we have that $T^* u = 0$. That is, $u \in \ker(T^*)$.

c) Ad III). This follows from II), and it is left as an exercise. □

As always in spectral theory, we study the spectrum through understanding resolvents (when they exist). In order to relate the resolvent of the adjoint to the adjoint of the resolvent, we first note the following commutativity of adjoints and inversion when the inverse is a bounded operator.

Lemma 4.17. *Let $\langle \mathcal{B}_1, \mathcal{B}_2\rangle$ be reflexive and $T \in \mathscr{C}(\mathcal{B}_2)$ densely-defined. Then*

$$T^{-1} \in \mathscr{B}(\mathcal{B}_2) \quad \Leftrightarrow \quad \big(T^{-1}\big)^* \in \mathscr{B}(\mathcal{B}_1)$$

and $(T^*)^{-1} = (T^{-1})^*$.

Proof. This is a routine calculation, which we leave as an exercise. □

Notation 4.18. For $S \subset \mathbb{C}$ we write $S^{\mathrm{conj}} := \{\bar{z} \mid z \in S\}$ as the complex conjugation of the set S.

Proposition 4.19. *Let $\langle \mathcal{B}_1, \mathcal{B}_2\rangle$ reflexive and $T \in \mathscr{C}(\mathcal{B}_2)$ densely-defined. Then*

$$\operatorname{spec}(T^*) = \operatorname{spec}(T)^{\mathrm{conj}} \qquad \text{and} \qquad \operatorname{res}(T^*) = \operatorname{res}(T)^{\mathrm{conj}}\,.$$

Proof. It suffices to show $\zeta \in \mathrm{res}(T) \Rightarrow \bar{\zeta} \in \mathrm{res}(T^*)$. By the conjugate linearity of the inner product in the second slot,

$$\left\langle \left(\bar{\zeta} - T^*\right)u, v\right\rangle = \langle u, (\zeta - T)v\rangle\,,$$

which yields $(\zeta - T)^* = \left(\bar{\zeta} - T^*\right)$. By the previous Lemma 4.17 we have that $(\zeta - T)^* = \left(\bar{\zeta} - T^*\right)$ is invertible and therefore $\bar{\zeta} \in \mathrm{res}(T^*)$. □

Example 4.20. 1. Let $B \in \mathbb{C}^{n\times n}$ be a matrix. Then $\mathrm{spec}(B) = \mathrm{spec}_{\mathrm{p}}(B)$. These are precisely the eigenvalues. However, the corresponding eigenspaces may not span the whole space, and therefore, generalised eigenspaces need to be considered. This is best seen through the matrix

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

2. Consider $T : \mathcal{H} \to \mathcal{H}$ self-adjoint. I.e. T is a densely-defined symmetric operator with $T^* = T$. Note this implies T is automatically closed. Then,

   I) $\mathrm{spec}(T) \subset \mathbb{R}$, and

   II) we have 'resolvent estimates':

   $$\left\|(\zeta - T)^{-1}u\right\| \leq \frac{1}{\|\mathrm{Im}(\zeta)\|}\|u\|\,.$$

   Proof. a) Ad I). For any operator, we always have that $\langle Tu, u\rangle = \overline{\langle u, Tu\rangle}$. However, due to self-adjointness of T, we have that $\langle Tu, u\rangle = \langle u, Tu\rangle$. Therefore, $\langle Tu, u\rangle \in \mathbb{R}$.

   Now, fix $\zeta \in \mathbb{C}\setminus\mathbb{R}$. Then, we have that $\langle(\zeta - T)u, u\rangle = \zeta\|u\|^2 - \langle Tu, u\rangle$ from which we conclude

   $$\mathrm{Im}\langle(\zeta - T)u, u\rangle = (\mathrm{Im}\,\zeta)\|u\|^2\,.$$

   Therefore,

   $$\begin{aligned}\|u\|^2 &\leq \frac{1}{|\mathrm{Im}(\zeta)|}\,\mathrm{Im}\langle(\zeta - T)u, u\rangle \\ &\leq \frac{1}{|\mathrm{Im}(\zeta)|}|\langle(\zeta - Tu), u\rangle| \\ &\leq \frac{1}{|\mathrm{Im}(\zeta)|}\|(\zeta - T)u\|\|u\|\,.\end{aligned}$$

   Hence,

   $$\|u\| \leq \frac{1}{|\mathrm{Im}(\zeta)|}\|(\zeta - T)u\|\,, \tag{4.3}$$

   so in particular, $(\zeta - T)$ is injective.

   We only assumed that $\zeta \in \mathbb{C}\setminus\mathbb{R}$, so the same argument yields that $(\bar{\zeta} - T)$ is injective. By Proposition 4.16 II) we have that $\mathrm{ran}(\zeta - T)^{\perp} = \ker((\zeta - T)^*)$ and by Proposition 4.16 III),

   $$\mathcal{H} = \ker((\zeta - T)^*) \oplus \overline{\mathrm{ran}(\zeta - T)} = \overline{\mathrm{ran}(\zeta - T)}\,.$$

Here, the second equality follows from the fact that

$$(\zeta - T)^* = (\overline{\zeta} - T^*) = (\overline{\zeta} - T)$$

using the self-adjointness of T and the injectivity of $(\overline{\zeta} - T)$ which we alluded to earlier. This shows that $(\zeta - T)$ has dense range and on combining this with the estimate (4.3) implies that $\zeta \in \operatorname{res}(T)$.

b) Ad II). This is immediate from the estimate (4.3) since we have already established that $\mathbb{C} \setminus \mathbb{R} \subset \operatorname{res}(T)$. □

3. If T is non-negative self-adjoint, i.e. $\langle Tu, u\rangle \geq 0$, then $\operatorname{spec}(T) \subset [0, \infty)$.

   For example, the Euclidean Laplacian $\Delta = -\sum_{i=1}^n \partial_i^2$ on $\mathrm{L}^2(\mathbb{R}^n)$ is a non-negative self-adjoint operator and there, we have $\operatorname{spec}(\Delta) = [0, \infty)$.

4.3 Pure point spectrum

We have seen that the spectrum of an operator can be classified into three distinct parts. Later on, it will be important for us to know that the operators we consider have pure point spectrum. However, we have seen that even classical examples, like the Euclidean Laplacian, fails this criterion. To understand when an operator has point spectrum, we need to understand a certain subclass of operators.

Definition 4.21. An operator $T \in \mathscr{B}(\mathcal{B})$ is said to be *compact* if for all $B_R(0) = \{x \in \mathcal{B} \mid \|x\| < R\} \subset \mathcal{B}$, we have $T(B_R(0))$ is a precompact set (i.e. $\overline{T(B_R(0))}$ is compact).

An extremely important fact regarding compact operators is the following. It allows one to conclude when a bounded operator is compact through factoring it via a compact operator.

Proposition 4.22. *Let $S, T \in \mathscr{B}(\mathcal{B})$ and T compact. Then TS and ST are compact.*

The following then gives criteria for pure point spectrum for compact operators and for closed operators via their resolvents.

Proposition 4.23.

I) If $T \in \mathscr{B}(\mathcal{B})$ is compact, then $\operatorname{spec}(T) = \operatorname{spec}_{\mathrm{p}}(T)$ consists of isolated points with 0 being the only point of accumulation. The generalised eigenspaces are finite dimensional.

II) If $T \in \mathscr{C}(\mathcal{B})$, $\zeta \in \operatorname{res}(T)$ and $(\zeta - T)^{-1}$ is compact, then $\operatorname{spec}(T) = \{\lambda_i \mid i \in \mathbb{N}\}$ is an isolated set of points, the generalised eigenspaces are finite dimensional and the only accumulation point is at ∞.

Moreover $\zeta' \in \operatorname{res}(T) \Rightarrow (\zeta' - T)^{-1}$ *compact.*

The proofs of these propositions, as well as a wider discussion surrounding compact operators, can be found in Chapter III, Sections 4, 7 and 8 in [33]. These propositions have an important consequence for elliptic differential operators on compact manifolds with boundary, as captured in the following.

Corollary 4.24. *Let $D \in \operatorname{Diff}_m(E)$ be elliptic, $E \to M'$, M' compact with $\partial M = \varnothing$. If $\operatorname{res}(\bar{D}) \neq \varnothing$, then $\operatorname{spec}(\bar{D})$ is discrete and the generalised eigenspaces are finite dimensional and smooth (i.e. they consist of smooth sections).*

Proof. Recall that by Theorem 3.38, we have that $\bar{D} = D_{\max} = D_{\min}$ since M' is compact and $\partial M' = \varnothing$. The same theorem guarantees elliptic regularity and we have $\operatorname{dom}(\bar{D}) = \mathrm{H}^m(M', E)$. Now,

$$\mathrm{H}^m(M', E) \subset \mathrm{H}^1(M', E) \overset{\text{compact}}{\hookrightarrow} \mathrm{L}^2(M', E)\,.$$

Since M' is compact we have that $\zeta \in \operatorname{res}(\bar{D})$ and we can factor the map $(\zeta - \bar{D})^{-1} : \mathrm{L}^2(M', E) \to \mathrm{L}^2(M', E)$ as

$$\left(\zeta - \bar{D}\right)^{-1} : \mathrm{L}^2(M', E) \to \mathrm{H}^m(M', E) \overset{\text{compact}}{\hookrightarrow} \mathrm{L}^2(M', E)\,.$$

By Proposition 4.22, we have that $\left(\zeta - \bar{D}\right)^{-1}$ is a compact operator.

Proposition 4.23 guarantees discrete spectrum with finite dimensional generalised eigenspaces for $\bar{D}$. Moreover, if φ is a generalised eigenvector of $\bar{D}$, then $\varphi \in \bigcap_{\ell \in \mathbb{N}} \mathrm{H}^\ell(M', E) = \mathrm{C}^\infty(M', E)$. □

4.4 Banach valued integrals

In what is to follow, we will need to integrate Banach valued functions. Measure theory in the Banach-valued setting has been widely studied and there is a broad variety of deep and sophisticated topics. Much of this surrounds questions of measurability for Banach-valued maps. For our purposes, in the worst case scenario, we shall only need to integrate piecewise continuous functions. Therefore, we require the simplest kind of integration in the Banach-valued setting - the generalisation of the Riemann integral.

Definition 4.25 (Banach-valued definite Riemann integral). For a continuous function $f : [a, b] \to \mathcal{B}$, define the *definite Riemann integral*

$$\int_a^b f(t)\ \mathrm{d}t := \lim_{n \to \infty} \sum_{j=0}^{n-1} \frac{b-a}{n} f(a + j(fracb - an))\,.$$

Remark 4.26. The limit certainly exists if f is continuous since the sequence

$$s_n := \sum_{j=0}^{n-1} \frac{b-a}{n} f\left(a + j\left(\frac{b-a}{n}\right)\right)$$

is Cauchy in $\mathcal{B}$.

It is also useful to obtain an indefinite Banach valued integral.

Definition 4.27 (Banach-valued indefinite Riemann integral). If $f : (a, b] \to \mathcal{B}$ is continuous, define

$$\int_a^b f(t)\ \mathrm{d}t := \lim_{a' \to a} \int_{a'}^b f(t)\ \mathrm{d}t$$

whenever the limit exists.

Proposition 4.28. *For a continuous function $f : (a, b] \to \mathcal{B}$, we have*

$$\int_a^b \|f(t)\|\ \mathrm{d}t < \infty \quad \Rightarrow \quad \left\|\int_a^b f(t)\ \mathrm{d}t\right\| \leq \int_a^b \|f(t)\|\ \mathrm{d}t\,.$$

Note that each of these definitions are easily extended to the case where continuity is replaced with piecewise continuity. With that in mind, we also consider a contour integral.

Definition 4.29 (Banach-valued contour integral). If $\gamma : [a, b] \to \Omega \subset \mathbb{C}$ is piecewise continuously differentiable and $f : \Omega \to \mathcal{B}$ continuous, then define

$$\oint_\gamma f(\zeta)\ \mathrm{d}\zeta := \int_a^b (f \circ \gamma)(t)\dot{\gamma}(t)\ \mathrm{d}t\,.$$

Here we used an integral symbol which suggests integration over a closed curve. That's because we want to mainly consider curves 'enveloping' things in the following sense.

Definition 4.30. Let $\Lambda \subset \Omega \subset \mathbb{C}$. We say that $\gamma : [a, b] \to \Omega$ *envelops* Λ in Ω, if γ is a continuous and piecewise continuously differentiable closed curve in $\Omega \setminus \Lambda$ that has winding number 1 w.r.t. each point $z \in \Lambda$, i.e.

$$\frac{1}{2\pi\imath} \oint_\gamma \frac{1}{\zeta - z}\ \mathrm{d}\zeta = 1\,.$$

If $\Lambda = \{\omega\}$ we also say that γ envelops ω in Ω.

Remark 4.31.

1. Visually, a curve having winding number 1 w.r.t. a point means that it runs around this point in a counter-clockwise manner exactly once.

2. With this notion of integral, many results of $\mathbb{C}$-valued holomorphic functions hold for $\mathcal{B}$-valued holomorphic functions. I.e., $f : \Omega \to \mathcal{B}$ holomorphic, then

$$f(\omega) = \frac{1}{2\pi\imath} \oint_\gamma \frac{f(z)}{z-\omega} dz$$

for any curve γ enveloping ω.

The Cauchy integral formula in the Banach space setting will be at the heart of our considerations in this section. We will use this to define functions of operators. But first, let us motivate this by considering the classical situation when $\mathcal{B} = \mathbb{C}$.

Typically, in the Cauchy integral formula, a holomorphic function is fixed and we write the expression to recover values of that function via the integrand. Let us now turn this around. Fix a point ω, an open set Ω with $\omega \in \Omega$ and consider holomorphic functions $f : \Omega \to \mathbb{C}$. Then, the Cauchy integral formula allows us to recover $f(\omega)$ through a contour integration. Again, let us emphasise, ω is fixed and we are considering different holomorphic functions $f : \Omega \to \mathbb{C}$.

It is natural to consider fixing such a point ω, as it can be related the the spectrum of an operator. For $\omega \in \mathbb{C}$ fixed, consider the multiplication operator

$$M_\omega : \mathbb{C} \to \mathbb{C}, z \mapsto \omega z\,.$$

Then

$$\lambda \in \operatorname{spec}(M_\omega) \quad\Leftrightarrow\quad \exists z \neq 0\colon \lambda z = M_\omega z = \omega z \quad\Leftrightarrow\quad \omega = \lambda\,,$$

so $\operatorname{spec}(M_\omega) = \{\omega\}$.

As before, let Ω be an open neighbourhood of $\{\omega\} = \operatorname{spec}(M_\omega)$ and $f : \Omega \to \mathbb{C}$ a holomorphic function. Now, suppose that γ envelops $\operatorname{spec}(M_\omega)$ inside Ω. By the Cauchy integral formula,

$$f(\omega) = \frac{1}{2\pi\imath} \oint_\gamma \frac{f(\zeta)}{\zeta-\omega}\,\mathrm{d}\zeta = \frac{1}{2\pi\imath} \oint_\gamma f(\zeta)(\zeta-\omega)^{-1}\,\mathrm{d}\zeta\,.$$

Let us now multiply each side by $z \in \mathbb{C}$:

$$f(\omega)z = \frac{1}{2\pi\imath} \oint_\gamma f(\zeta)(\zeta-\omega)^{-1}z\,\mathrm{d}\zeta\,.$$

Now, let us understand $(\zeta-\omega)^{-1}$. Fix $z \in \mathbb{C}$ and note that

$$z = (\zeta-\omega)^{-1}(\zeta-\omega)z = (\zeta-\omega)^{-1}(\zeta-M_\omega)z\,, \qquad \text{and}$$
$$z = (\zeta-\omega)(\zeta-\omega)^{-1}z = (\zeta-M_\omega)(\zeta-\omega)^{-1}z\,.$$

That is, $(\zeta-\omega)^{-1} = (\zeta - M_\omega)^{-1}$ is a resolvent of M_ω. Therefore, we can rewrite the expression above in terms of this resolvent to obtain

$$f(\omega)z = \frac{1}{2\pi\imath}\oint_\gamma f(\zeta)(\zeta - M_\omega)^{-1} z \, \mathrm{d}\zeta\,,$$

which justifies the definition

$$f(M_\omega)z := \frac{1}{2\pi\imath}\oint_\gamma f(\zeta)(\zeta - M_\omega)^{-1} z \, \mathrm{d}\zeta\,. \tag{4.4}$$

It is easy to see that if we choose $f = \mathrm{id}$, we obtain $f(M_\omega) = M_\omega$ and if we choose $f = 1$, we obtain $f(M_\omega) = \mathrm{id}$.

Since (4.4) is expressed in terms terms of the resolvent of M_ω, integrated along a curve that envelops the spectrum of M_ω, we have a clear indication of how we can obtain functions of arbitrary operators. We first consider this for bounded operators on a Banach space. Since we need the integrand to converge, we need to understand the spectrum of such an operator. The following important proposition furnishes us with that information.

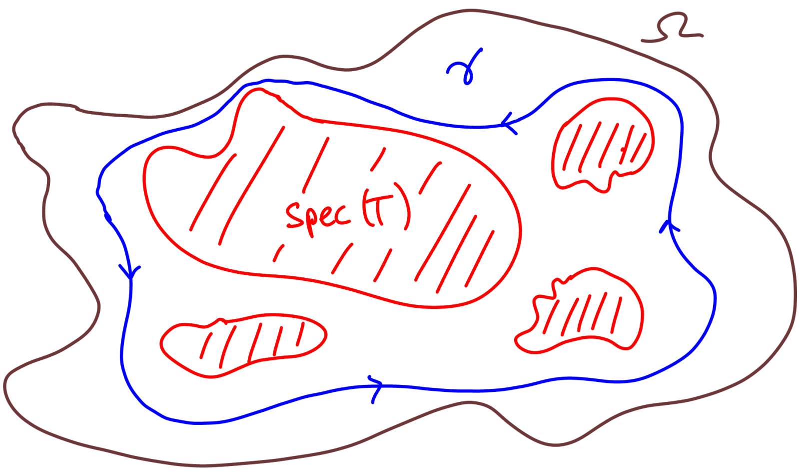
Proposition 4.32. *Let $T \in \mathscr{B}(\mathcal{B})$. Then* $\mathrm{spec}(T)$ *is compact.*

Proof. This is a direct consequence of Runge's theorem. □

With this, we can define functions of bounded operators for a sufficiently large class of functions.

Definition 4.33 (Riesz-Dunford functional calculus for bounded operators). Let $\Omega \subset \mathbb{C}$ be an open set and $f : \Omega \to \mathbb{C}$ holomorphic. Let γ be a curve enveloping $\mathrm{spec}(T)$ inside Ω. Define

$$f(T)u := \frac{1}{2\pi i}\oint_\gamma f(\zeta)(\zeta - T)^{-1} u \, \mathrm{d}\zeta\,.$$



Proposition 4.34. *The operator* $f(T) \in \mathscr{B}(\mathcal{B})$. *Moreover, we have that*

$$
\begin{aligned}
f = \mathrm{id} \Rightarrow f(T) &= T\,, \\
f = 1 \Rightarrow f(T) &= \mathrm{id}\,, \\
(\alpha f_1 + \beta f_2)(T) &= \alpha f_1(T) + \beta f_2(T)\,, \\
(f_1 f_2)(T) &= f_1(T) f_2(T)\,, \\
(f_1 \circ f_2)(T) &= f_1(f_2(T))\,.
\end{aligned}
$$

Proof. By definition,

$$\|f(T)u\| \leq \mathrm{len}(\gamma)\left(\sup_{\zeta \in \mathrm{ran}(\gamma)} |f(\zeta)|\right)\left(\sup_{\zeta \in \mathrm{ran}(\gamma)} \left\|(\zeta - T)^{-1}\right\|\right)\|u\| \lesssim \|u\|\,,$$

since $f : \Omega \to \mathbb{C}$ is holomorphic as is $\zeta \mapsto (\zeta - T)^{-1} : \Omega \to \mathcal{B}$. Therefore, $f(T) \in \mathscr{B}(\mathcal{B})$. The remaining facts are readily verified, although $(f_1 \circ f_2)(T) = f_1(f_2(T))$ requires a slightly more tedious calculation. □

Remark 4.35. These properties mean that $f \mapsto f(T) : \mathrm{Hol}(\Omega) \to \mathscr{B}(\mathcal{B})$ is a homomorphism into a commutative subalgebra of $\mathscr{B}(\mathcal{B})$.

Example 4.36. Suppose that we can write $\mathrm{spec}(T) = \Lambda_1 \sqcup \ldots \sqcup \Lambda_K$ as a disjoint union such that there are mutually disjoint open sets Ω_i with $\Lambda_i \subset \Omega_i$. Suppose further that we obtain curves γ_i, such that γ_i envelops Λ_i in Ω_i. Let $\Omega := \bigcup_{i=1}^K \Omega_i$, and define

$$\chi_i(\zeta) := \begin{cases} 1 & \text{if } \zeta \in \Omega_i\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

By the mutual disjointedness of Ω_i, the maps $\chi_i : \Omega \to \mathbb{C}$ are holomorphic, $\chi_i \chi_j = 0$ when $i \neq j$, and $\chi_i^2 = \chi_i$. Since the functional calculus is a homomorphism, we have that $\chi_i(T)$ is a projector. Therefore,

$$\mathcal{B} = \bigoplus_{i=1}^K \chi_i(T)\mathcal{B}\,.$$

Moreover T commutes with $\chi_i(T)$, and therefore, $T|_{\chi_i(T)\mathcal{B}} : \chi_i(T)\mathcal{B} \to \chi_i(T)\mathcal{B}$. In fact, $\mathrm{spec}(T|_{\chi_i(T)\mathcal{B}}) = \Lambda_i$ and

$$Tu = \sum_{i=1}^K T|_{\chi_i(T)\mathcal{B}} \chi_i(T) u\,,$$

which can be seen as a decomposition of the underlying space $\mathcal{B}$ with respect to T, and obtaining T via restricted operators on subspaces of $\mathcal{B}$.

If $\mathcal{B} = \mathbb{C}^K$, then T is a matrix and $\mathrm{spec}(T) = \{\lambda_1, \ldots, \lambda_K\}$. In this situation, the space $\chi_i(T)\mathbb{C}^K$ is precisely the generalised eigenspace of λ_i and the operator $T|_{\chi_i(T)\mathbb{C}^K}$ is precisely the Jordan block corresponding to λ_i.

4.5 Functions of closed operators

We were able to define operators $f(T)$ in the previous section by utilising the fact that $\operatorname{spec}(T)$ is compact when T is bounded to ensure the defining integral converges. Our desire is to extend this to the case of closed operators. We first start with demonstrating that the underlying space can be separated via the spectrum under certain conditions, even when the operator may be unbounded. This generalises Example 4.36.

Proposition 4.37. *Let $T \in \mathscr{C}(\mathcal{B})$ and suppose that $\operatorname{spec}(T) = \Lambda_1 \sqcup \Lambda_2$ where Λ_1 is enveloped by a curve $\gamma : [a, b] \to \mathbb{C}$. Then*

$$Pu := \frac{1}{2\pi i} \oint_\gamma (\zeta - T)^{-1} u \,\mathrm{d}\zeta$$

defines a projection $P \in \mathscr{B}(\mathcal{B})$ and $\mathcal{B} = P\mathcal{B} \oplus (I - P)\mathcal{B}$. Moreover, $T|_{P\mathcal{B}} : P\mathcal{B} \to P\mathcal{B}$ and $T|_{(I-P)\mathcal{B}} : (I - P)\mathcal{B} \to (I - P)\mathcal{B}$, i.e. the restriction of the operators respect the subspaces. Also, $T|_{P\mathcal{B}} \in \mathscr{B}(P\mathcal{B})$ and $\operatorname{spec}\Big(T|_{P\mathcal{B}}\Big) = \Lambda_1$, $\operatorname{spec}\Big(T|_{(I-P)\mathcal{B}}\Big) = \Lambda_2$.

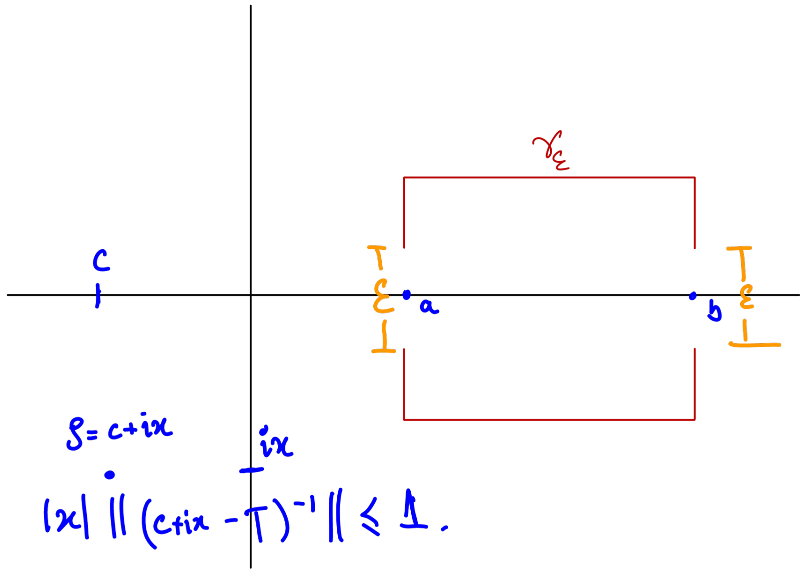
Remark 4.38. Note that this projector can be interpreted as $\chi(T)$, where $\chi : \Omega \to \mathbb{C}$ is constructed by fixing disjoint open neighbourhoods $\Omega_i \supset \Lambda_i$, setting $\Omega := \Omega_1 \cup \Omega_2$ and defining

$$\chi(\zeta) := \begin{cases} 1 & \text{if } z \in \Omega_1\,, \\ 0 & \text{if } z \in \Omega_2\,. \end{cases}$$

Let us now consider forming functions of closed operators. Using the Euclidean Laplacian as a guiding example, we saw that the spectrum may be an unbounded set. However, we also saw in Example 4.20 2. II) that a self-adjoint operator enjoys certain 'resolvent bounds', which dictate the way in which the norm of resolvents can 'blow up' when approaching the spectrum. As we discussed in Subsection 3.8, functions of self-adjoint operators can be constructed for continuous or even Borel functions on the real line. In fact, the resolvent bounds are at the analytical heart of the construction of the the spectral measure which facilities this construction. Let us consider this functional calculus from our point of view that we have developed here.

Recall that T self-adjoint implies $\operatorname{spec}(T) \subset \imath\mathbb{R}$. Given $a < b$ and $\varepsilon > 0$, define curve γ to be a rectangular region enveloping points in (a, b) with the two sides through a and b. Let γ_ε be this curve but with a ball of ε removed from γ around a and b. Define:

$$P_\varepsilon^{[a,b]} u := \frac{1}{2\pi\imath} \oint_{\gamma_\varepsilon} (\zeta - T)^{-1} u \,\mathrm{d}\zeta\,.$$



The resolvent bounds in Example 4.20 2. II) are precisely $|\mathrm{Im}(\zeta)|\left\|(\zeta - T)^{-1}\right\| \leq 1$ for all $\zeta \in \mathbb{C} \setminus \mathbb{R}$. By design of the curve γ_ε, it is easy to see that Then $\|P_\varepsilon\| \leq c$ for $c < \infty$ independent of ε. Therefore, let

$$P^{[a,b]}u := \lim_{\varepsilon \to 0} P_\varepsilon^{[a,b]} ,$$

which is readily verified to be a projector. In fact, $P^{[a,b]} = \mathrm{d}E_T|_{[a,b]}$, where $\int E_T$ is the spectral measure associated to a self-adjoint operator which we alluded to in Section 3.8. With the spectral measure, continuous (and also Borel) functions on the real line can be integrated to form operators $f(T)$, as we also saw in Section 3.8.

This illustrates in the case of self-adjoint operators how resolvent estimates give us a method of access to functions of operators. However, as we will see, the analysis of adapted boundary operators necessitates that we go beyond the self-adjoint realm. Nevertheless, the analytical features surrounding the self-adjoint case serve as a template which will allow us to consider a more general class of operators which will contain the self-adjoint situation as a special case.

Notation 4.39. For a complex number $z \in \mathbb{C} \setminus \{0\}$ let $\arg(z) \in (-\pi, \pi]$ s.t. $z = |z| e^{i \arg(z)}$.

Definition 4.40 (Open/closed bisectors/sectors). Define the closed and open bisectors of angle $\mu < \frac{\pi}{2}$ by:

$$\mathrm{S}_\mu := \{\zeta \in \mathbb{C} \mid |\arg(\zeta)| \leq \mu \text{ or } |\arg(-\zeta)| \leq \mu\} \cup \{0\} ,$$
$$\mathrm{S}^\circ_\mu := \mathring{\mathrm{S}}_\mu .$$

The closed and open sectors of angle $\mu < \pi$ are then

$$\mathrm{S}_{\mu\pm} := \{\zeta \in \mathbb{C} \mid |\arg(\pm\zeta)| \le \mu\} \cup \{0\}\,,$$
$$\mathrm{S}^\circ_{\mu\pm} := \mathring{\mathrm{S}}_{\mu\pm}\,.$$

Remark 4.41. Note that S°_μ and $\mathrm{S}^\circ_{\mu\pm}$ are open sets and both *exclude* the point 0.

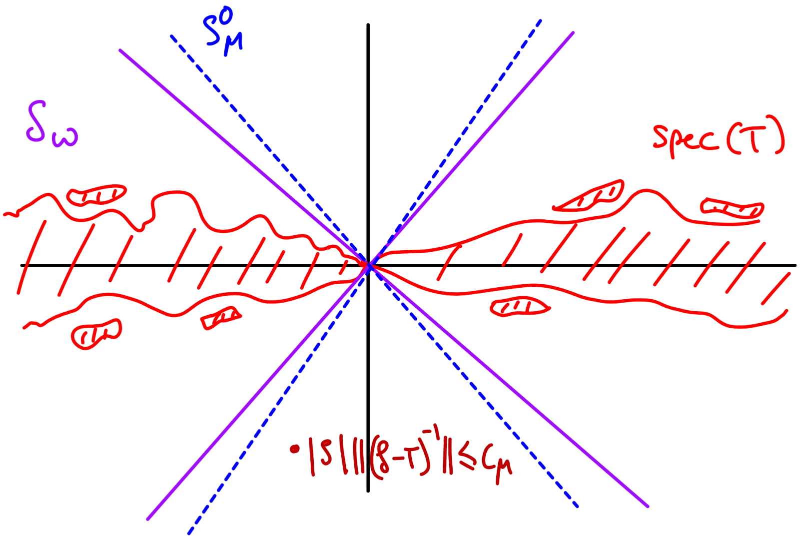
Definition 4.42 (Bisectorial/Sectorial operator). Let $T \in \mathscr{C}(\mathcal{B})$ be a densely defined closed operator on a Banach space and $\omega < \frac{\pi}{2}$. Suppose that

(I) $\operatorname{spec}(T) \subset \mathrm{S}_\omega$, and

(II) $\forall \mu \in \left(\omega, \frac{\pi}{2}\right) \exists C_\mu < \infty \, \forall \zeta \notin \mathrm{S}_\mu$:

$$|\zeta| \left\| (\zeta - T)^{-1} \right\| \le C_\mu\,.$$

Such an operator is called *bisectorial* or *ω-bisectorial.* If $\operatorname{spec}(T) \subset \mathrm{S}_{\omega+}$, then we say it is *sectorial* or *ω-sectorial.*



Remark 4.43. 1. The estimates in (II) are precisely the resolvent estimates which generalise the estimates we saw for self-adjoint operators.

2. Sectoriality can also be defined for $\omega \in \left[\frac{\pi}{2}, \pi\right)$.

Example 4.44. 1. Every self-adjoint operator is 0-bisectorial.

2. Every non-negative self-adjoint operator is 0-sectorial.

Proposition 4.45. *$\langle \mathcal{B}_1, \mathcal{B}_2 \rangle$ reflexive, $T \in \mathscr{C}(\mathcal{B}_2)$ be ω-(bi)sectorial. Then $T^* \in \mathscr{C}(\mathcal{B}_1)$ is also ω-(bi)sectorial.*

Definition 4.46. We define two function spaces on the open bisector. The bounded holomorphic functions are

$$\mathrm{Hol}^\infty(\mathrm{S}_\mu^\circ) := \left\{ f : \mathrm{S}_\mu^\circ \to \mathbb{C} \;\middle|\; f \text{ is holomorphic and bounded} \right\},$$

while the polynomially decreasing holomorphic functions, called *psi-class functions*, are

$$\Psi(\mathrm{S}_\mu^\circ) := \left\{ \psi \in \mathrm{Hol}^\infty(\mathrm{S}_\mu^\circ) \;\middle|\; \exists \alpha > 0, c < \infty \, \forall \zeta \in \mathrm{S}_\mu^\circ \colon |\psi(\zeta)| \leq c \min\left\{ |\zeta|^\alpha, |\zeta|^{-\alpha} \right\} \right\}.$$

Example 4.47. The functions

$$\zeta \mapsto \frac{\zeta}{1+\zeta^2} \qquad \text{and} \qquad \zeta \mapsto \zeta^\beta e^{-\zeta}$$

for $\beta > 0$ are examples of psi-class functions.

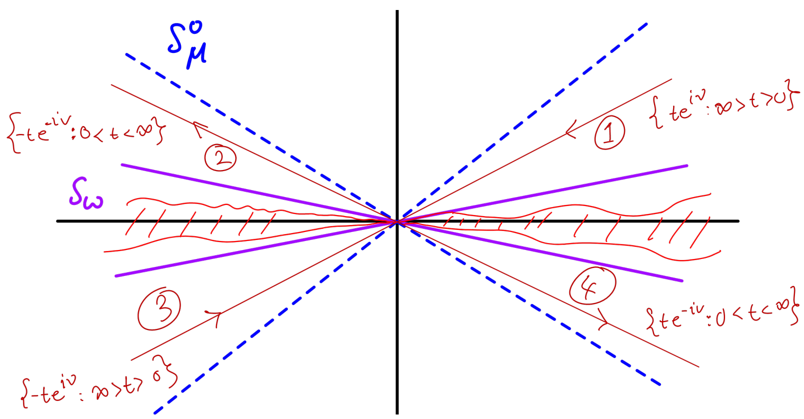
Definition 4.48. For $\psi \in \Psi(S_\mu^0)$ and $u \in \mathcal{B}$, define

$$\psi(T)u := \frac{1}{2\pi i} \int_\gamma \psi(\zeta)(\zeta - T)^{-1} u \, \mathrm{d}\zeta,$$

where γ is a curve parametrised as

$$\begin{aligned} \gamma = &\left\{ te^{i\nu} \;\middle|\; \infty > t > 0 \right\} \cup \left\{ -te^{-i\nu} \;\middle|\; 0 < t < \infty \right\} \\ &\cup \left\{ -te^{i\nu} \;\middle|\; -\infty > t > 0 \right\} \cup \left\{ te^{-i\nu} \;\middle|\; 0 < t < \infty \right\} \end{aligned}$$

for $\nu \in (\omega, \mu)$.



Remark 4.49. 1. The integral here is made sense of by using an approximating procedure. More precisely, we truncate the curve at radii ε and ε^{-1}, and then take the limit as $\varepsilon \to 0$. Since this is for each fixed $u \in \mathcal{B}$, the convergence is in the strong operator topology.

2. Using the Cauchy integral theorem, it is easy to show that the definition is independent of the specific $\nu \in (\omega, \mu)$.

3. This is an absolutely convergent integral due to decay of ψ and resolvent estimates for bisectorial T. So,

$$\|\psi(T)u\|_{\mathcal{B}} \leq C_{\psi,T}\|u\|_{\mathcal{B}}\,.$$

The following are important properties that this *pre*-functional calculus enjoys. We say 'pre' here since neither the function 1, nor the resolvent $z \mapsto (\zeta - z)$ for $\zeta \in \operatorname{res}(T)$, is a psi-class function.

Proposition 4.50. *I) $\psi \mapsto \psi(T) : \Psi(\mathrm{S}^{\circ}_{\mu}) \to \mathscr{B}(\mathcal{B})$ is a homomorphism of vector spaces.*

II) If S commutes with $(\zeta - T)^{-1}$ for some $\zeta \in \operatorname{res}(T)$ (i.e. $S(\zeta - T)^{-1} \subset (\zeta - T)^{-1}S$), then S commutes with $\psi(T)$.

III) $\operatorname{spec}(\psi(T)) = \psi(\operatorname{spec}(T))$.

IV) If $\mathcal{B}$ is reflexive, then $\mathcal{B} = \ker(T) \oplus \overline{\operatorname{ran}(T)}$.

Proof. a) Parts I) through III) are readily verified.

b) Ad IV). Let

$$\mathcal{B}_0 := \left\{ u \in \mathcal{B} \;\middle|\; \lim_{n\to\infty} (1 - \imath nT)^{-1} \text{ exists} \right\}$$

and define $P : \mathcal{B}_0 \to \mathcal{B}_0$ by

$$Pu = \lim_{n\to\infty} \frac{1}{\imath n}\left(\frac{1}{\imath n} - T\right)^{-1} u = \lim_{n\to\infty} (1 - \imath nT)^{-1} u\,.$$

It is easily verified that $P \in \mathscr{B}(\mathcal{B}_0)$ is a projector. Moreover, a routine calculation will yield that

$$P\mathcal{B}_0 = \ker(T) \qquad \text{and} \qquad (I - P)\mathcal{B}_0 = \operatorname{ran}(T)\,.$$

The reflexivity assumption on $\mathcal{B}$ is required in proving $\mathcal{B}_0 = \mathcal{B}$. □

4.6 Fractional Powers revisited

In Section 3.8, we demonstrated how the spectral theorem for self-adjoint operators can be used to form fractional powers of non-negative self-adjoint operators. There, we took the spectral theorem as a given in this construction. In this section, we will consider the construction of fractional powers of operators for ω-sectorial operators when $\omega < \pi$ via psi-functions of such operators.

Definition 4.51. Let T be ω-sectorial for $\omega \in [0, \pi)$ and let $\alpha > 0$. Moreover, let

$$f_\alpha(\zeta) := \frac{\zeta^\alpha}{1 + \zeta^{2\lceil\alpha\rceil}} ,$$

and it is clear that $f_\alpha \in \Psi\big(\mathrm{S}^\circ_\mu\big)$. Define:

$$\begin{aligned} \mathrm{dom}(T^\alpha) &:= \big\{u \in \mathcal{B} : f_\alpha(T)u \in \mathrm{dom}(T^{2\lceil\alpha\rceil})\big\}, \\ T^\alpha &:= \big(1 + T^{2\lceil\alpha\rceil}\big) f_\alpha(T) . \end{aligned}$$

Remark 4.52. The exponent $2\lceil\alpha\rceil$ can be replaced by any integer $> \alpha$. Note that in doing so, the function f needs to be also altered. Explicitly, fix some positive integer $N > \alpha$. Necessarily, we will have that $\lceil\alpha\rceil \leq N$. Letting

$$f^N_\alpha(\zeta) := \frac{\zeta^\alpha}{1 + \zeta^{2N}} ,$$

we have that

$$T^\alpha = (1 + T^{2N}) f^N_\alpha(T) .$$

The domain of the operator via this construction is the same as before, since we required $f^N_\alpha(T)u \in \mathrm{dom}(T^{2N})$. As N changes, so does f^N_α as well $(1 + T^{2N})$. This is seen from the following impressionistic calculation, which can be made rigorous (exercise):

$$\begin{aligned} T^\alpha &= (1 + T^{2\lceil\alpha\rceil}) f_\alpha(T) \\ &= (1 + T^{2\lceil\alpha\rceil})(1 + T^{2N})(1 + T^{\lceil 2N\rceil})^{-1} f_\alpha(T) \\ &= (1 + T^{2N})(1 + T^{\lceil 2N\rceil})^{-1}(1 + T^{2\lceil\alpha\rceil}) f_\alpha(T) \\ &= (1 + T^{2N}) f^N_\alpha(T) . \end{aligned}$$

For example, letting $\alpha := 2$ and $N = 10^6$, we see that

$$f^N_\alpha(\zeta) := \frac{\zeta^2}{1 + \zeta^{10^6}} ,$$

and

$$\Big(1 + T^{10^6}\Big) \tilde{f}_\alpha(T) = \Big(1 + T^{10^6}\Big) \frac{T^2}{1 + T^{10^6}} = T^2 .$$

Proposition 4.53. *I) T^α is $\alpha\omega$-sectorial if $\alpha \in (0,1)$.*

II) For $\alpha < \beta$ we have $\mathrm{dom}(T^\beta) \subset \mathrm{dom}(T^\alpha)$.

III) $T^{\alpha+\beta} = T^\alpha \circ T^\beta$.

IV) $\mathrm{spec}(T^\alpha) = \mathrm{spec}(T)^\alpha$.

V) If $S \in \mathscr{B}(\mathcal{B})$ commutes with a resolvent $(\zeta - T)^{-1}$, then S also commutes with T^α.

Example 4.54. Let T be non-negative and self-adjoint. We have already seen that such an operator is 0-sectorial. The fractional power of T constructed here is, as it is to be expected, consistent with the construction of fractional powers for T using the spectral theorem as we saw in Section 3.8. One way to see this is to access the spectral measure via resolvent estimates as we discussed at the start of Section 4.5.

We also highlight the following important proposition, which connects fractional powers of sectorial operators with interpolation scales.

Proposition 4.55. *If T is ω-sectorial, then $\mathrm{dom}(T^\alpha) = [\mathrm{dom}(T), \mathcal{H}]_\alpha$ for $0 < \alpha < 1$.*

4.7 Heat equation and semigroups

Semigroups play a central role in the analysis of elliptic operators in the presence of boundary. This can be seen easily through the model operator $D_0 = \sigma_0(\partial_t + A)$ from Section 4.1. On interpreting the transversal direction as 'time', the study of the operator $\partial_t + A$ can be interpreted as a 'heat equation' with respect to the operator A. In order to consider heat equations and their solutions, we begin with the following fundamental technical result. Throughout, we assume that T is ω-sectorial.

Theorem 4.56 (McIntosh convergence lemma). *Let T be ω-sectorial on the reflexive Banach space $\mathcal{B}$. Suppose that:*

(I) $\psi_n \in \Psi(\mathrm{S}^\circ_{\mu+})$ s.t. there is a $c < \infty$ with $\|\psi_n\|_\infty < c$ uniformly in n, and

(II) $\psi_n \to \psi$ where $\psi \in \Psi(\mathrm{S}^\circ_{\mu+})$ uniformly on compact subsets of $\mathrm{S}^\circ_{\mu+}$.

Then, for each $u \in \mathcal{B}$, $\psi_n(T)u \to \psi(T)u$.

Remark 4.57. Note that the convergence here is in the strong operator topology, not in the operator norm. I.e., given $u \in \mathcal{B}$ and an $\varepsilon > 0$, the assertion is the existence of $N > 0$ such that for $n \geq N$,

$$\|\psi_n(T)u - \psi(T)u\| < \varepsilon\,.$$

In contrast, to converge in the uniform operator topology, we would require

$$\|\psi_n(T)u - \psi(T)u\| < \varepsilon\|u\|$$

for ε, N, n independent of u, making this a stronger condition.

The convergence lemma is an enormously useful technical device. It allows us to immediately obtain properties for functions of operators that are valid for functions. One such example is the derivative of a t-varying family of functions.

Let us now consider the heat equation with respect to T. That is, given some $u_0 \in \mathcal{B}$ as the initial condition, we want $u \in \mathrm{C}^1((0,1), \mathcal{B})$ satisfying

$$(\partial_t + T)u = 0 \qquad \text{with} \qquad \lim_{t\to 0} u(t) = u_0\,. \tag{4.5}$$

To generate solutions, we construct a 'semigroup' $\mathrm{e}^{-tT}u_0 =: u(t)$. This requires us to construct $f(tT)$ where $f(\zeta) := \mathrm{e}^{-t\zeta}$. The problem is that, although $f(\zeta) \to 0$ as $|\zeta| \to \infty$, we have that $f(\zeta) \to 1$ as $|\zeta| \to 1$. That is, $f \notin \Psi(\mathrm{S}^{\circ}_{\mu+})$. To rectify this problem, we consider instead

$$\psi(\zeta) := \mathrm{e}^{-\zeta} - \frac{1}{1+\zeta}\,.$$

It is clear that $\psi \in \Psi\big(\mathrm{S}^{\circ}_{\mu+}\big)$.

Definition 4.58 (Exponential semigroup). Define

$$\mathrm{e}^{-tT} := (1 + tT)^{-1} + \psi(tT)\,.$$

The exponential semigroup of T satisfies the following key properties.

Proposition 4.59. *Let $\mathcal{B}$ be a reflexive Banach space and $T : \mathcal{B} \to \mathcal{B}$ ω-sectorial. Then:*

I) $\exists c < \infty\, \forall t \in (0,\infty)\colon \left\|\mathrm{e}^{-tT}\right\| \leq c$. *In particular,* $\mathrm{e}^{-tT} \in \mathscr{B}(\mathcal{B})$ *for all* $t > 0$.

II) $\mathrm{e}^{-tT}\mathrm{e}^{-sT} = \mathrm{e}^{-(t+s)T}$.

III) $t \mapsto \mathrm{e}^{-tT} : (0,\infty) \to \mathscr{B}(\mathcal{B})$ *is continuous.*

IV) $\frac{d}{dt}\mathrm{e}^{-tT} = -T\mathrm{e}^{-tT} \supset \mathrm{e}^{-tT}T$.

V) If $u \in \ker(T)$, *then* $\mathrm{e}^{-tT}u = u$.

VI) For all $u \in \mathcal{B}$ *we have*

$$\mathrm{e}^{-tT}u \xrightarrow[t\to 0]{} u \qquad \textit{and} \qquad \mathrm{e}^{-tT}u \xrightarrow[t\to\infty]{} 0\,.$$

VII) We have $u \in \mathrm{dom}(T)$ iff $\lim_{t\to 0} \frac{1}{t}\big(\mathrm{e}^{-tT}u - u\big)$ exists and $-Tu = \lim_{t\to 0} \frac{1}{t}\big(\mathrm{e}^{-tT}u - u\big)$.

VIII) If $S \in \mathscr{C}(\mathcal{B})$ commutes with a resolvent $(\zeta - T)^{-1}$, then S commutes with e^{-tT}.

Remark 4.60. This is proved by using Theorem 4.56. The derivative in IV) is just the usual derivative, $\mathscr{B}(\mathcal{B})$ is a Banach space.

Let us now return back to the heat equation (4.5). For a given initial datum $u_0 \in \mathcal{B}$, define:

$$u(t) := \mathrm{e}^{-tT}u_0\,.$$

It is readily verified that $u \in \mathrm{C}^1((0,\infty),\mathcal{B})$ and that it solves (4.5).

Proposition 4.61. *Let T be an ω-sectorial operator on a reflexive Banach space $\mathcal{B}$. Then*

$$\mathcal{B}_T^\infty := \bigcap_{j=1}^\infty \mathrm{dom}\big(T^j\big) \subset \mathcal{B}$$

is a dense subspace of $\mathcal{B}$. In particular, if $u \in \mathcal{B}$, then $\mathrm{e}^{-tT}u \in \mathcal{B}_T^\infty$ and $\mathrm{e}^{-tT}u \to u$ as $t \to 0$.

Proof. Let $f_k := \big(\zeta \mapsto \zeta^k \mathrm{e}^{-\zeta}\big) \in \Psi\big(\mathrm{S}^\circ_{\mu+}\big)$, then $f_k(tT) \in \mathscr{B}(\mathcal{B})$. Clearly, $f_k(tT) = (tT)^k\mathrm{e}^{-tT}$ and by explicitly examining $\mathrm{dom}(f_k(tT)) = \mathcal{B}$, we find that

$$\mathcal{B} = \mathrm{dom}(f_k(tT)) = \left\{u \in \mathcal{B} \;\middle|\; \mathrm{e}^{-tT}u \in \mathrm{dom}\Big((tT)^k\Big)\right\},$$

and therefore,

$$\mathrm{e}^{-tT}u \in \mathrm{dom}\Big((tT)^k\Big) = \mathrm{dom}\big(T^k\big)\,.$$

Since k is arbitrary, we obtain $\mathrm{e}^{-tT}u \in \mathcal{B}_T^\infty$. The density is due to Proposition 4.59 VI) which yields $\mathrm{e}^{-tT}u \to u$ as $t \to 0$. □

Example 4.62. Let M' be compact with $\partial M' = \varnothing$. Let $\big(E, h^E\big) \to M'$ be a Hermitian vector bundle and ∇ a connection on it. Consider $\Delta := \nabla^*\bar{\nabla}$.

Let $u \in \mathrm{L}^2(M',E)$ and note that by Theorem 3.38 and the Sobolev embedding theorem,

$$\mathrm{e}^{-t\Delta}u \in \bigcap_{j=1}^\infty \mathrm{dom}\big(\Delta^j\big) = \bigcap_{j=1}^\infty \mathrm{H}^{2j}(M',E) \subset \mathrm{C}^\infty(M,E)\,.$$

4.8 The H^∞-functional calculus

In this section, we illustrate the salient points in the construction of the H^∞-functional calculus. Unlike the ability to take functions of operators by psi-class functions, this is a full fledged functional calculus, meaning that functions $\zeta \mapsto 1$ and $\zeta \mapsto (z-\zeta)^{-1}$ when $z \in \mathrm{res}(T)$ are within reach of the functional calculus.

Although the functional calculus can be defined on general Banach spaces, we limit our attention to the Hilbert space case. Our applications only demand this setting and in addition, it avoid some unnecessary technical hurdles specialising to Hilbert spaces.

Before we embark on constructing the functional calculus, let us mention its resemblance to the Fourier transform. We have already seen in earlier parts that the Fourier transform can be seen as a functional calculus for the Laplacian on Euclidean space. To define the Fourier transform on $\mathrm{L}^2(\mathbb{R}^n)$, we first define it in integral form on Schwarz-class functions, which are a dense subspace of $\mathrm{L}^2(\mathbb{R}^n)$. Then, by proving the relevant estimates, namely that for Schwarz-class functions the Fourier transform is bounded in $\mathrm{L}^2(\mathbb{R}^n)$, it is extended by density to all of $\mathrm{L}^2(\mathbb{R}^n)$. The construction of the H^∞ functional calculus is reminiscent of the construction of the Fourier transform, where the psi-class functions now play the role of the Schwarz-class functions. The estimate required on psi-class functions are captured in the following definition.

Definition 4.63 (H^∞-functional calculus). If T is ω-bisectorial and there is a $c < \infty$ s.t. for all $\psi \in \Psi(\mathrm{S}^\circ_\mu)$ we have that

$$\|\psi(T)\| \leq c\|\psi\|_\infty\,,$$

then we say that T has an $\mathrm{H}^\infty(\mathrm{S}^\circ_\mu)$-functional calculus. The smallest such c we denote by $C_{\mathrm{H}^\infty}(T)$.

Remark 4.64.

1. Note that we always have $\|\psi(T)\| \leq c_{\psi,T}$. Here we are saying that $c_{\psi,T}$ has an additional structure, namely $c_{\psi,T} \leq c\|\psi\|_\infty$.
2. We are jumping the gun by using the terminology 'H^∞-functional calculus' for an estimate for psi-class functions. However, the reasons for this choice of nomenclature will become apparent shortly.

Definition 4.65. Define:

$$\mathrm{H}^\infty(\mathrm{S}^\circ_\mu) := \left\{ f : \mathrm{S}^\circ_\mu \cup \{0\} \to \mathbb{C} \;\middle|\; f \text{ is bounded and } f|_{\mathrm{S}^\circ_\mu} \in \mathrm{Hol}^\infty(\mathrm{S}^\circ_\mu) \right\}.$$

Remark 4.66. It is essential that we allow for discontinuities across zero. The quintessential application we consider, namely spectral projectors to the left or right halves of the complex plane determined by negative or positive real parts, are such functions which are discontinuous across zero.

Lemma 4.67 (Second McIntosh convergence lemma). *Suppose T has an $\mathrm{H}^\infty\big(\mathrm{S}^\circ_\mu\big)$-functional calculus. Then for all $f \in \mathrm{H}^\infty\big(\mathrm{S}^\circ_\mu\big)$ there is a sequence $f_n \in \Psi\big(\mathrm{S}^\circ_\mu\big)$ such that:*

1. *$f_n \to f$ uniformly on compact subsets of S°_μ, and*
2. *whenever $u \in \mathcal{H}$, the sequence $(f_n(T)u)$ is Cauchy.*

This is the crucial lemma that allows us to formulate the H$^\infty$-functional calculus.

Definition 4.68. If T has an $\mathrm{H}^\infty\big(\mathrm{S}^\circ_\mu\big)$-functional calculus, then for $f \in \mathrm{Hol}^\infty\big(\mathrm{S}^\circ_\mu\big)$ define

$$f(T)u := f(0)P_{\ker(T),\overline{\operatorname{ran}(T)}}u + \lim_{n\to\infty} f_n(T)u\,,$$

for $u \in \mathcal{H}$.

If f_n and $\tilde{f}_n$ are two different sequences, converging uniformly to f and for which $f_n(T)u$ and $\tilde{f}_n(T)u$ are Cauchy, then

$$\lim_{n\to\infty} f_n(T)u = \lim_{n\to\infty} \tilde{f}_n(T)u\,.$$

Therefore, this is well-defined.

Remark 4.69. Note here that it is essential that the kernel of the operator is handled separately. The reason here is that $\ker(T) \subset \ker(\psi(T))$ for $\psi \in \mathrm{S}^\circ(S_\mu)$. In a sense, this is also why we are able to account for discontinuities across zero.

There is no great mystery here that the kernel plays a special role. By the very bisectoriality of T, the points at 0 and ∞ are special as we have desirable resolvent bounds as we estimate these points.

Proposition 4.70 (Properties of the H$^\infty$-functional calculus). *For T an ω-bisectorial operator with an H$^\infty$-functional calculus on a Hilbert space $\mathcal{H}$, the following hold.*

I) *$f \mapsto f(T) : \mathrm{H}^\infty\big(\mathrm{S}^\circ_\mu\big) \to \mathscr{B}(\mathcal{B})$.*

II) *There is a $c < \infty$ s.t. $\|f(T)\| \le c\|f\|_\infty$.*

III) *$f \mapsto f(T)$ is an algebra homomorphism, i.e.*

$$\begin{aligned}(\alpha f + \beta g)(T) &= \alpha f(T) + \beta g(T)\,,\\ (fg)(T) &= f(T)g(T)\,,\\ 1(T) &= \mathrm{id}\;.\end{aligned}$$

Moreover,

$$\left(z \mapsto \frac{1}{\zeta - z}\right)(T) = (\zeta - T)^{-1} \qquad \text{for all } \zeta \notin \mathrm{S}_\mu.$$

IV) If $S \in \mathscr{C}(\mathcal{H})$ commutes with a resolvent $(\zeta - T)^{-1}$, then S commutes with $f(T)$.

Remark 4.71. If an operator $S \in \mathscr{C}(\mathcal{H})$ commutes with one resolvent $(z-T)^{-1}$, then, since $(\zeta \mapsto (z' - \zeta)^{-1})(T) = (z' - T)^{-1}$ for any other $z' \in \operatorname{res}(T)$, we have that S also commutes with any other resolvent.

While the notion of an H^∞-functional calculus for T as defined in Definition 4.63, it is not always easy to prove that such estimates hold. In the following theorem, originally proved by McIntosh in [39], is an enormously important equivalent criterion for detecting whether an operator T has an H^∞-functional calculus.

Theorem 4.72 (McIntosh). *The following are equivalent:*

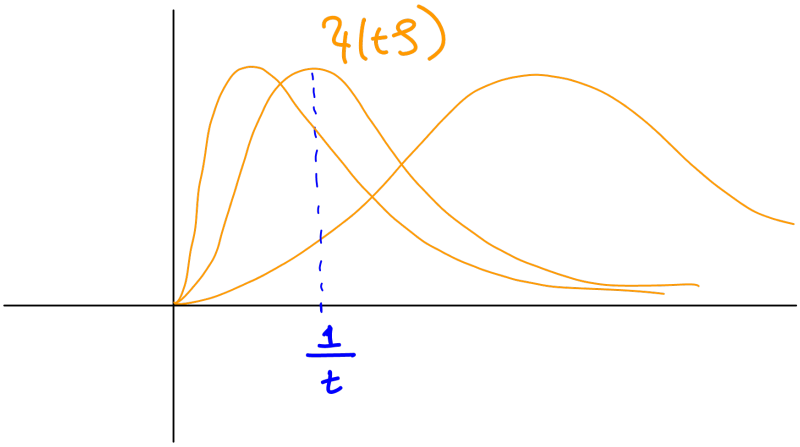
(I) T has an H^∞-functional calculus.

(II) There is a $\psi \in \Psi\big(\mathrm{S}^\circ_\mu\big)$ not identically 0 on either sector $\mathrm{S}^\circ_{\mu\pm}$ satisfying the 'quadratic estimate'

$$\int_0^\infty \|\psi(tT)u\|^2 \, \frac{\mathrm{d}t}{t} \simeq \|u\|^2 \qquad \text{for all } u \in \overline{\operatorname{ran}(T)}. \tag{4.6}$$

(III) For all $\psi \in \Psi\big(\mathrm{S}^\circ_\mu\big)$ not identically 0 on $\mathrm{S}^\circ_{\mu\pm}$, the estimate (4.6) is satisfied.

This theorem is what connects the H^∞-functional calculus with real-variable harmonic analysis methods. Let us get an intuitive sense for these quadratic estimates. The functions $\psi \in \Psi(\mathrm{S}^\circ_\mu)$ decay polynomially at 0 and ∞, and therefore, they can be seen as complex 'bump' functions. They can be thought of as smoothed out indicator functions on the complex plane. With this, and considering $u \in \overline{\operatorname{ran}(T)}$ as a 'signal', we can interpret $\psi(tT)$ as a 'band-pass filter' and $\psi(tT)u$ as a band-pass filter applied to u. What the quadratic estimate in (4.6) then says is that the norm of u can be reconstructed, up to a constant, through summing over band-pass filters.



On a manifold, quadratic estimates (4.6) are typically proved through trivialisations. In the noncompact setting, these trivialisations may need to be chosen carefully, exploiting geometric features like potential curvature bounds. The advantage of the quadratic estimates is that, despite localisations, the analysis is actually performed on an infinite family of operators. Namely, we need to consider $t \mapsto \psi(tT)$ as t ranges from 0 to ∞. Seeing 0 and ∞ as the high and low frequencies of T respectively, this amounts to being able to 'see' global information regarding the operator despite localisation. In analysis, particularly in the noncompact setting, this effectively allows for desired estimates to be obtained under weaker assumptions than trying to localise the operator T itself.

Example 4.73. Consider

$$\chi^{\pm}(\zeta) := \begin{cases} 1 & \text{if } \pm \operatorname{Re}(\zeta) > 0\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

then $\chi^{\pm} \in \mathrm{H}^\infty\big(\mathrm{S}^\circ_\mu\big)$. Let

$$\operatorname{sgn}(\zeta) := \chi^+(\zeta) - \chi^-(\zeta)\,.$$

Then $\chi^{\pm}(T), \operatorname{sgn}(T) \in \mathscr{B}(\mathcal{H})$, and $T|_{\chi^{\pm}(T)\mathcal{H}} : \chi^{\pm}(T)\mathcal{H} \to \chi^{\pm}(T)\mathcal{H}$ since T trivially commutes with all $(\zeta - T)^{-1}$.

Then $|T| := T \operatorname{sgn}(T)$ is an ω-sectorial operator with $\operatorname{dom}(|T|) = \operatorname{dom}(T)$ and is called the *McIntosh modulus* of T.

In the applications which we are interested in, we actually require the quadratic estimates (4.6) and we access them through an alternative means of establishing an H$^\infty$-functional calculus. More precisely, in applications to differential operators on manifolds without boundary, elliptic regularity allows us to identify domains of operators. This identification turns out to be of vital importance to the existence of an H$^\infty$-functional calculus. At the heart of this lies the fact that the H$^\infty$-functional calculus connects in an intimate way to interpolation scales. The precise theorem that yields us the desired result is given as Corollary 5.5 in [7].

Theorem 4.74 (Auscher-McIntosh-Nahmod). *An ω-sectorial operator T has an* H$^\infty$*-functional calculus if there are $\alpha, \beta > 0$ s.t.*

$$\operatorname{dom}(T^\alpha) \subset \operatorname{dom}((T^*)^\alpha) \quad \textit{and} \quad \|T^\alpha u\| \lesssim \|(T^*)^\alpha u\| \quad \textit{and}$$
$$\operatorname{dom}\Big((T^*)^\beta\Big) \subset \operatorname{dom}\big(T^\beta\big) \quad \textit{and} \quad \Big\|(T^*)^\beta u\Big\| \lesssim \big\|T^\beta u\big\|\,.$$

In applications, it is sometimes possible to first establish that the spectral projectors $\chi^{\pm}(T)$ are bounded without first establishing a functional calculus. When that happens, we are able to relate the functional calculus of the ω-sectorial operator $|T|$ to the ω-bisectorial operator T as given in the following proposition.

Proposition 4.75. *If $\chi^{\pm}(T) \in \mathscr{B}(\mathcal{H})$ and $|T|$ has an H^{∞}-functional calculus, then T has an H^{∞}-functional calculus.*

Proof. Let $\psi \in \Psi\big(\mathrm{S}^{\circ}_{\mu}\big)$, then it is easy to see that

$$\psi(T) = \psi(|T| \operatorname{sgn}(T)) = \psi(|T|) \operatorname{sgn}(T)\,.$$

Since $\chi^{\pm}(T) \in \mathscr{B}(\mathcal{H})$ means that $\operatorname{sgn}(T) \in \mathscr{B}(\mathcal{H})$, we obtain

$$\|\psi(T)u\| \simeq \|\psi(|T|)u\| \leq C\|\psi\|_{\infty}\,. \qquad \square$$

Example 4.76. 1. Let T be a self-adjoint operator. Through earlier considerations, namely the construction of the spectral measure via the contour integrals, we are able to obtain that $\chi^{\pm}(T) \in \mathscr{B}(\mathcal{H})$. Note that to define $\chi^{\pm}(T)$, we need to consider the injective operator $T|_{\overline{\operatorname{ran}(T)}} : \overline{\operatorname{ran}(T)} \to \overline{\operatorname{ran}(T)}$. Consequently, we have that $|T|$ is a non-negative self-adjoint operator. By self-adjointness, we have that $|T| = |T^*|$ and therefore, $|T|$ has an H^{∞}-functional calculus. Using Proposition 4.75, we conclude that T also has an H^{∞}-functional calculus.

2. Let M' be a compact manifold with $\partial M' = \varnothing$. Fix an elliptic operator $D \in \operatorname{Diff}_m(E)$. Further, assume $\ker\big(\bar{D}\big) = 0$, $D\omega$-bisectorial, and $\chi^{\pm}\big(\bar{D}\big) \in \mathscr{B}(\mathrm{L}^2(M', E))$.

   As a consequence of elliptic regularity,

   $$\operatorname{dom}\big(\bar{D}\big) = \operatorname{dom}\big(\bar{D}^*\big) = \mathrm{H}^m(M', E)$$

   and therefore, both $\bar{D}$ and $\bar{D}^*$ have discrete spectrum and hence are invertible ω-bisectorial. Moreover, $\big|\bar{D}\big| = \bar{D}\operatorname{sgn}(D)$ and $\operatorname{sgn}\big(\bar{D}\big)$ commutes with $\bar{D}$.

   $$\operatorname{dom}\big(\big|\bar{D}\big|\big) = \operatorname{dom}\big(\big|\bar{D}^*\big|\big) = \mathrm{H}^m(M', E)\,.$$

   Since D is invertible, so is $\bar{D}$, and hence

   $$\|u\|_{\mathrm{H}^m} \simeq \|u\| + \big\|\bar{D}u\big\| \simeq \big\|\bar{D}u\big\| \simeq \big\|\big|\bar{D}\big|u\big\|\,,$$

   similarly

   $$\|u\|_{\mathrm{H}^m} \simeq \big\|\big|\bar{D}^*\big|u\big\|\,.$$

   On application of Theorem 4.74, we conclude that $\big|\bar{D}\big|$ has an H^{∞}-functional calculus. Since we have already assumed that $\chi^{\pm}(\bar{D}) \in \mathscr{B}(\mathrm{L}^2(M', E))$, by Proposition 4.75, we obtain that $\bar{D}$ has an H^{∞}-functional calculus. It is easy to see that the same conclusions hold for D^* by considering $D^{\dagger}$ in place of D.

4.9 The model problem

Let us start by recalling the neighbourhood U from the geometric and operator reduction statements found in Lemmas 4.2 and 4.6. There, we have that U is diffeomorphic to $Z_R := [0,R) \times \partial M$, and that $D = \sigma_t(\partial_t + A + R_t)$. As we saw in Definition 4.7, we extract out the model operator $D_0 := \sigma_0(\partial_t + A)$ on the infinite cylinder $Z := [0,\infty) \times \partial M$. The purpose of this section will be to understand the model operator D_0 on Z.

Proposition 4.77. *Let $D \in \mathrm{Diff}_1(E,F)$ be elliptic and A any adapted boundary operator. Then the following hold.*

I) $\mathrm{spec}(A) = \mathrm{spec}_{\mathrm{p}}(A)$ *is discrete with smooth finite dimensional generalised eigenspaces.*

II) There are $R_A > 0$ and $\omega < \frac{\pi}{2}$ such that

$$\mathrm{spec}(A) \subset \mathrm{S}_\omega \cup \overline{B_{R_A}(0)}\,.$$

III) There is a $c_A < \infty$ s.t. for all $\zeta \notin \mathrm{S}_\omega \cup \overline{B_{R_A}(0)}$ we have the resolvent estimates

$$|\zeta| \left\| (\zeta - A)^{-1} \right\| \leq c_A\,.$$

IV) There is a sequence $(r_j) \subset \mathbb{R}$ with $\lim_{j\to-\infty} r_j = -\infty$ and $\lim_{j\to\infty} r_j = \infty$ such that for each r_j there is an $\omega_j < \frac{\pi}{2}$ such that $A_{r_j} := A - r_j$ is ω_{r_j}-bisectorial and invertible.

Proof. We prove II) and III) first. Then I) follows from Corollary 4.24.

In actual fact, to prove II) and III), we need to first examine the spectrum of $\sigma_A(x,\xi)$ at each $x \in \partial M$ and $\xi \in T_x^*\partial M \setminus \{0\}$. Recall that $\sigma_A(x,\xi) : E_x \to E_x$ and E_x is a finite dimensional vector space, and therefore, the spectrum of $\sigma_A(x,\xi)$ is finite and consists only of eigenvalues.

a) Claim: $\mathrm{spec}(\sigma_A(x,\xi)) \cap \mathbb{R} = \varnothing$ for all $x \in \partial M, \xi \in T_x^*\partial M \setminus \{0\}$.

We prove by contradiction. Suppose there is a $\xi \in T_x^*\partial M \setminus \{0\}$ and a $\lambda \in \mathbb{R}$ and $v \neq 0$ s.t.

$$\lambda v = \sigma_A(x,\xi)v = \sigma_D(x,\tau)^{-1}\sigma_D(x,\xi)v\,.$$

By virtue of the fact that A is first-order, $\xi \mapsto \sigma_A(x,\xi)$ is an $\mathbb{R}$-linear map, and therefore, on applying $\sigma_D(x,\tau)$ to both sides and simplifying, we obtain

$$\sigma_D(x,\lambda\tau - \xi)v = 0\,.$$

Since τ and ξ are linearly independent, we have $\lambda\tau - \xi \neq 0$. But this implies that $\ker \sigma_D(x,\lambda\tau - \xi) \neq 0$ which contradicts the ellipticity of D.

b) Claim: There exists an $\omega < \pi/2$ such that $\mathrm{spec}(\imath\sigma_A(x,\xi)) \subset \mathrm{S}_\omega$ for all $x \in \partial M$ and $\xi \in T_x^*\partial M \setminus \{0\}$.

Fix an auxiliary metric $g_{\partial M}$ on ∂M, and let

$$S^*\partial M := \left\{ \xi \in T^*\partial M \;\middle|\; |\xi(x)|_{g_{\partial M}(x)} = 1 \right\},$$

the co-sphere bundle with respect to $g_{\partial M}$. Clearly $S^*\partial M$ is a compact set and therefore, the product $\partial M \times S^*\partial M$ is also compact. By a), we have that $\mathrm{dist}(\mathrm{spec}(\sigma_A(x,\xi)), \mathbb{R}) > 0$. The map

$$\partial M \times S^*\partial M \ni (x,\xi) \to \sigma_A(x,\xi)$$

is continuous, and therefore, the eigenvalues of $\sigma_A(x,\xi)$ vary continuously. Then,

$$\partial M \times S^*\partial M \ni (x,\xi) \to \mathrm{dist}(\mathrm{spec}(\sigma_A(x,\xi)), \mathbb{R}) > 0$$

is continuous and therefore, by compactness of $\partial M \times S^*\partial M$ we obtain a minimum $\min_{(x,\xi)}(\mathrm{dist}(\mathrm{spec}(\sigma_A(x,\xi)), \mathbb{R})) > 0$. Therefore, there exists an $\omega < \pi/2$ such that

$$\mathrm{spec}(\sigma_A(x,\xi)) \subset \imath \mathrm{S}_\omega \,.$$

The map $t \mapsto \sigma_A(x, t\xi)$ is homogeneous of degree 1 and therefore, since every ξ can be recovered as a point on the line trough 0 and $\xi/|\xi|_{g_{\partial M}}$,

$$\bigcup_{x \in \partial M} \bigcup_{\xi \in T^*_x \partial M} \mathrm{spec}(\imath \sigma_A(x,\xi)) \subset S_\omega \,.$$

c) Using b), the claims II) and III) follow from pseudodifferential methods. A detailed argument is beyond the scope of material here, but it follows in the compact setting by understanding the spectrum of the principal symbol. The passage from principal symbol spectra to the L^2-spectrum of the closed realisation of an elliptic differential operator can be found in Theorem 9.3 in [44] by Shubin.

For IV), we note that as a consequence of II), there is a sequence $\{x_j\}_{j \in \mathbb{Z}} \subset \mathrm{Re}\,\mathrm{spec}(A)$ such that $\lim_{j \to \pm\infty} x_j = \pm\infty$. Then, due to discreteness of the spectrum from I), we are able to find $r_j \in \mathbb{R}$ such that the vertical line $l_{r_j} := \{\zeta \in \mathbb{C} \mid \mathrm{Re}\,\zeta = r_j\}$ through r_j satisfies $l_{r_j} \subset \mathrm{res}(A)$. In fact, for each such r_j, there exists $\varepsilon_j > 0$ such that the closed ε-neighbourhood $l_{r_j,\varepsilon_j} := \{\zeta \in \mathbb{C} : |\mathrm{Re}\,\zeta - r_j| \leq \varepsilon\}$ of l_{r_j} satisfies $l_{r_j,\varepsilon_j} \subset \mathrm{res}(A)$. We leave it as an exercise to show that for reach such r_j, there exists an $\omega_j < \pi/2$ such that $A_{r_j} := A - r_j$ is invertible and ω_j-bisectorial. □

Remark 4.78. Note that it is unreasonable to expect that $\mathrm{spec}(A) \subset \mathrm{S}_\omega$ purely from knowing $\mathrm{spec}(\sigma_A(x,\xi)) \subset \mathrm{S}_\omega$. This can be seen from a very simple example. Suppose that A does, indeed, satisfy $\mathrm{spec}(A) \subset \mathrm{S}_\omega$. Then, fix any $r \in \mathbb{R}$ and note that $A_r := A - r$ satisfies

$$\sigma_{A_r}(x,\xi) = \sigma_A(x,\xi)\,.$$

However, since r can be chosen arbitrarily, on choosing some $\lambda \in \mathrm{spec}(A)$ such that $\mathrm{Im}\,\lambda \neq 0$ and setting $r = \mathrm{Re}\,\lambda$, we can see that $\mathrm{Im}\,\lambda \in \mathrm{spec}(A_r)$. Therefore, $\mathrm{spec}(A_r) \not\subset \mathrm{S}_\omega$ for any $\omega < \pi/2$.

This prompts us to formulate the following.

Definition 4.79 (Admissible spectral cut/admissible cut/spectral cut). For an adapted boundary operator A, if for $r \in \mathbb{R}$ there exists $\omega_r \in [0, \pi/2)$ such that $A_r := A - r$ is ω_r-bisectorial and invertible, then r is called an *admissible spectral cut* or simply an *admissible cut* or *spectral cut*.

From now on, we assume that A is ω-bisectorial and invertible. Recall that having fixed an adapted boundary operator A we obtain $D = \sigma_t(\partial_t + A + R_t)$. We always have an admissible spectral cut $r \in \mathbb{R}$ and

$$D = \sigma_t\big(\partial_t + A_{r_j} + (R_t + r_j)\big)\,.$$

Let us recall the functions $\chi^{\pm} \in \mathrm{H}^{\infty}(\mathrm{S}^{\circ}_{\mu})$ which we considered in Example 4.73. These were precisely the functions

$$\chi^{\pm}(\zeta) = \begin{cases} 1 & \text{if } \pm\mathrm{Re}(\zeta) > 0\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

It is easy to see from functional calculus considerations, when $\chi^{\pm}(A)$ can be defined and is a bounded operator, it is a projector to the spectral subspaces corresponding to the spectrum located in the complex plane with positive and negative real parts. Understanding these projectors is of vital importance.

Proposition 4.80. *Let A be an ω-bisectorial and invertible adapted boundary operator. Then, for each $\alpha \in \mathbb{R}$,*

$$\chi^{\pm}(A), \chi^{\pm}(A^*) \in \mathscr{B}(\mathrm{H}^{\alpha}(\partial M, E))\,.$$

Proof. As we saw in the proof of Proposition 4.77, the crucial point was to obtain that $\mathrm{spec}(\sigma_A(x,\xi)) \subset \mathrm{S}_{\omega}$. Here, this is also of vital importance, coupled with the fact that A is invertible. Invertibility means that $0 \in \mathrm{res}(A)$ and since $\mathrm{res}(A)$ is open, we have an $\epsilon > 0$ such that $B_{\epsilon}(0) \subset \mathrm{res}(A)$. This coupled with the bisectoriality of A means that, possibly passing to some $\epsilon' \in (0, \epsilon)$, the line neighbourhood $l_{0,\epsilon}$ of $\imath\mathbb{R}$ satisfies $l_{0,\epsilon} \subset \mathrm{res}(T)$. Therefore, when $\chi^{\pm}(A)$ are bounded operators, $\mathrm{id} = \chi^{+}(A) + \chi^{-}(A)$.

That $\chi^{\pm}(A)$ are bounded on $\mathrm{H}^{\alpha}(\partial M, E)$ requires the use of pseudodifferential methods, beyond the scope of the material here. Its statement and proof can be found as the main theorem in the paper [24] by Grubb. The idea is to relate these projectors to logarithms of A, which shows that these projectors are so-called pseudodifferential operators of order 0. Such operators are bounded on all Sobolev scales by the compactness of ∂M. □

Remark 4.81. Note that $\chi^{\pm}(A)^* = \chi^{\pm}(A^*)$.

Theorem 4.82. *For A an adapted boundary operator that is invertible and bisectorial, the operators $|A|, |A^*| = |A|^*, A, A^*$ enjoy an H^∞-functional calculus. In particular, for every $\alpha > 0$,*

$$\int_0^\infty \left\| t^\alpha |A|^\alpha \mathrm{e}^{-t|A|} u \right\|^2 \frac{\mathrm{d}t}{t} \simeq \|u\|^2 \qquad \text{for all } u \in \mathrm{L}^2(\partial M, E).$$

Proof. Recall that $|A| = A \operatorname{sgn}(A)$. Since $\operatorname{sgn}(A) = \chi^+(A) - \chi^-(A) \in \mathscr{B}(\mathrm{L}^2(\partial M, E)$, we have that $\operatorname{dom}(|A|) = \operatorname{dom}(A)$ and $|A|$ is ω-sectorial and invertible. Also, $\operatorname{dom}(|A|) = \operatorname{dom}(|A|^*)$ by elliptic regularity for A and A^*. Invertibility then yields

$$\||A|u\| \simeq \||A|u\| + \|u\| \simeq \|u\|_{\mathrm{H}^1(\partial M, E)}$$

and therefore, $\||A|u\| \simeq \||A|^* u\|$. We then apply Theorem 4.74 to conclude that $|A|$ (and $|A|^*$) both enjoy an H^∞-functional calculus. By Proposition 4.75, we obtain that A and A^* enjoy an H^∞-functional calculus.

For the quadratic estimate in the conclusion, note that whenever $\alpha > 0$,

$$\left(\zeta \mapsto \zeta^\alpha \mathrm{e}^{-\zeta}\right) \in \Psi(\mathrm{S}^\circ_{\mu+}) \,.$$

By Theorem 4.72, the desired estimate holds for all $u \in \overline{\operatorname{ran}(|A|)}$. However, recall from Proposition 4.50 IV),

$$\mathrm{L}^2(\partial M, E) = \ker(|A|) \oplus \overline{\operatorname{ran}(|A|)} \,.$$

Since $\ker(|A|) = \ker(A)$, the invertibility of A yields $\ker(|A|) = \{0\}$. Therefore, $\mathrm{L}^2(\partial M, E) = \overline{\operatorname{ran}(|A|)}$ which completes the proof. $\square$

Definition 4.83. We define the Czech and hat spaces of the model operator D_0, formally depending on the boundary adapted operator A:

$$\begin{aligned} \check{\mathrm{H}}_A(D_0) &:= \chi^-(A)\mathrm{H}^{\frac{1}{2}}(\partial M, E) \oplus \chi^+(A)\mathrm{H}^{-\frac{1}{2}}(\partial M, E)\,, \qquad \text{and} \\ \hat{\mathrm{H}}_A(D_0) &:= \chi^-(A^*)\mathrm{H}^{-\frac{1}{2}}(\partial M, E) \oplus \chi^+(A^*)\mathrm{H}^{\frac{1}{2}}(\partial M, E)\,. \end{aligned}$$

In due course, we will show that $\check{\mathrm{H}}_A(D_0) = \check{\mathrm{H}}(D_0)$. This gives a description of $\check{\mathrm{H}}(D_0)$ via the operator A which is on the boundary.

Proposition 4.84. $\left\langle \hat{\mathrm{H}}_A(D_0), \check{\mathrm{H}}_A(D_0) \right\rangle$ *is a reflexive perfect pair and* $\langle \cdot, \cdot \rangle|_{\mathrm{C}^\infty(\partial M, E)} = \langle \cdot, \cdot \rangle|_{\mathrm{L}^2(\partial M, E)}$.

Proof. From Proposition 3.67, we have that $\langle \mathrm{H}^\alpha(\partial M, E), \mathrm{H}^{-\alpha}(\partial M, E)\rangle$ is reflexive with $\langle \cdot, \cdot \rangle|_{\mathrm{C}^\infty(\partial M, E)} = \langle \cdot, \cdot \rangle|_{\mathrm{L}^2(\partial M, E)}$. By Proposition 2.63, since $\chi^\pm(A), \chi^\pm(A^*)$ are bounded projectors on $\mathrm{H}^\alpha(\partial M, E)$ for all $\alpha \in \mathbb{R}$, and $\chi^\pm(A^*) = \chi^\pm(A)^*$ w.r.t. $\langle \mathrm{H}^\alpha(\partial M, E), \mathrm{H}^{-\alpha}(\partial M, E)\rangle$, we obtain $\langle \chi^\pm(A)\mathrm{H}^\alpha(\partial M, E), \chi^\pm(A^*)\mathrm{H}^{-\alpha}(\partial M, E)\rangle$ reflexive and $\langle \cdot, \cdot\rangle|_{\mathrm{C}^\infty(\partial M,E)} = \langle \cdot, \cdot\rangle|_{\mathrm{L}^2(\partial M,E)}$.

We leave it as an exercise to then show that this yields $\left\langle \check{\mathrm{H}}_A(D_0), \hat{\mathrm{H}}_A(D_0)\right\rangle$ with $\langle \cdot, \cdot\rangle|_{\mathrm{C}^\infty(\partial M,E)} = \langle \cdot, \cdot\rangle|_{\mathrm{L}^2(\partial M,E)}$. □

Remark 4.85. We could have proved this more directly as follows. From Proposition 4.55 along with elliptic regularity, we can assert $\mathrm{dom}(|A|^\alpha) = \mathrm{H}^\alpha(\partial M, E)$ with $\||A|^\alpha u\| \simeq \|u\|_{\mathrm{H}^\alpha}$ for $\alpha \in [0,1]$. In particular, this means that, for $u, v \in \mathrm{C}^\infty(\partial M, E)$, $\langle u, v\rangle = \left\langle |A|^\alpha u, |A|^{-\alpha} v\right\rangle$ yields the desired paring. Then, using the density of $\mathrm{C}^\infty(\partial M, E)$ in $\mathrm{H}^\alpha(\partial M, E)$, along with the fact that $\chi^\pm(A) \in \mathscr{B}(\mathrm{H}^\alpha(\partial M, E)$, we can calculate to obtain the desired conclusion.

Now, note that since $D_0 = \sigma_0(\partial_t + A)$ we obtain that $\left(\sigma_0^{-1} D_0\right)^* = -\partial_t + A^*$.

Definition 4.86 (Extension operators on the cylinder). For $u \in \mathrm{L}^2(\partial M, E)$, define the extension operators $\mathcal{E}$ and $\mathcal{E}^*$ by

$$\mathcal{E}u := \mathrm{e}^{-t|A|}u \qquad \text{and} \qquad \mathcal{E}^* u := \mathrm{e}^{-t|A^*|}u\,.$$

Lemma 4.87. *For $u \in \mathrm{C}^\infty(\partial M, E)$ we have $\mathcal{E}u \in \mathrm{dom}(D_{0,\max})$ and $\mathcal{E}^*u \in \mathrm{dom}((\sigma_0^{-1} D_0)^*_{\max})$ along with the estimates*

$$\|\mathcal{E}u\|_{D_0} \lesssim \|u\|_{\check{\mathrm{H}}_A(D_0)} \qquad \textit{and} \qquad \|\mathcal{E}^*u\|_{\left(\sigma_0^{-1}D_0\right)^*} \lesssim \|u\|_{\hat{\mathrm{H}}_A(D_0)}\,.$$

Proof. The fact that $\mathcal{E}u \in \mathrm{dom}(D_{0,\max})$ and $\mathcal{E}^*u \in \mathrm{dom}((\sigma_0^{-1}D_0)^*)$ can be seen immediately from the definition.

Write $u = u_+ + u_-$ where $u_\pm := \chi^\pm(A)u$. Then

$$\|\mathcal{E}u\|_{D_0} \leq \|\mathcal{E}u_+\|_{D_0} + \|\mathcal{E}u_-\|_{D_0}$$

which justifies us bounding each of the terms on the right hand side by $\|u\|_{\check{\mathrm{H}}_A(D_0)}$. Moreover,

$$\partial_t \mathcal{E}u_\pm = \partial_t \mathcal{E}u_\pm = \partial_t \mathrm{e}^{-t|A|}u_\pm = -|A|\mathrm{e}^{-t|A|}u_\pm = |A|\mathcal{E}u_\pm\,.$$

Recall $A = |A|\,\mathrm{sgn}(A)$ where $\mathrm{sgn}(A) = \chi^+(A) - \chi^-(A)$.

a) We first analyse the term $\mathcal{E}u_+$. Recall that

$$\|\mathcal{E}u_+\|^2_{D_0} = \|\mathcal{E}u_+\|_{\mathrm{L}^2 fZ,E} + \|D_0\mathcal{E}u_+\|^2_{\mathrm{L}^2(Z,E)}\,.$$

Therefore, we compute

$$\begin{aligned} D_0\mathcal{E}u_+ &= \sigma_0(\partial_t + A)\mathcal{E}u_+ \\ &= \sigma_0(\partial_t + |A|\operatorname{sgn}(A))\mathrm{e}^{-t|A|}u_+ \\ &= \sigma_0\big(-|A|\mathrm{e}^{-t|A|}u_+ + |A|\operatorname{sgn}(A)\mathrm{e}^{-t|A|}u_+\big) \\ &= \sigma_0\big(-|A|\mathrm{e}^{-t|A|}u_+ + |A|\mathrm{e}^{-t|A|}u_+\big) \\ &= 0\,. \end{aligned}$$

Here, we have used the fact that

$$\operatorname{sgn}(A)u_+ = \big(\chi^+(A) - \chi^-(A)\big)\chi^+(A)u = \chi^+(a)^2u = \chi^+(a)u = u_+$$

and that

$$\begin{aligned} \mathrm{e}^{-t|A|}u_+ = \mathrm{e}^{-t|A|}\chi^+(A)u = \mathrm{e}^{-t|A|}\chi^+(A)^2u \\ = \chi^+(A)\mathrm{e}^{-t|A|}\chi^+(A)u = \chi^+(A)\mathrm{e}^{-t|A|}u_+ \end{aligned}$$

which shows that $\mathrm{e}^{-t|A|}u_+ \in \chi^+(A)\mathrm{L}^2(\partial M, E)$.

By what we have just proved, we have that

$$\begin{aligned} \|\mathcal{E}u_+\|_{D_0}^2 &= \|\mathcal{E}u_+\|_{\mathrm{L}^2(Z,E)}^2 + \underbrace{\|D_0\mathcal{E}u_+\|_{\mathrm{L}^2(Z,E)}^2}_{=0} \\ &= \int_0^\infty \big\|\mathrm{e}^{-t|A|}u_+\big\|_{\mathrm{L}^2(\partial M,E)}^2\,\mathrm{d}t \\ &= \int_0^\infty \Big\|t^{-\frac{1}{2}}t^{\frac{1}{2}}|A|^{\frac{1}{2}}|A|^{-\frac{1}{2}}\mathrm{e}^{-t|A|}u_+\Big\|_{\mathrm{L}^2(\partial M,E)}^2\,\mathrm{d}t \\ &= \int_0^\infty \Big\|t^{\frac{1}{2}}|A|^{\frac{1}{2}}\mathrm{e}^{-t|A|}|A|^{-\frac{1}{2}}u_+\Big\|_{\mathrm{L}^2(\partial M,E)}^2\,\frac{\mathrm{d}t}{t} \\ &\simeq \Big\||A|^{-\frac{1}{2}}u_+\Big\|^2 \\ &\simeq \|u_+\|_{\mathrm{H}^{-\frac{1}{2}}}^2\,, \end{aligned}$$

where the penultimate estimate used Theorem 4.82 and the ultimate estimate follows from Elliptic regularity along with Proposition 4.55.

b) Let us now estimate $\|\mathcal{E}u_-\|_{D_0}$. For that, note that

$$\operatorname{sgn}(A)u_- = \big(\chi^+(A) - \chi^-(A)\big)\chi^-(A)u = -\chi^-(A)^2u = -u_-\,,$$

and therefore,

$$\begin{aligned} D_0\mathcal{E}u_- &= \sigma_0\big(-|A|\mathrm{e}^{-t|A|}u_- + |A|\underbrace{\operatorname{sgn}(A)\mathrm{e}^{-t|A|}}_{=-\chi^-(A)}u_-\big) \\ &= -2\sigma_0\big(|A|\mathrm{e}^{-t|A|}u_-\big)\,. \end{aligned}$$

Using this, we obtain

$$
\begin{aligned}
\|\mathcal{E}u_-\|_{D_0}^2 &= \int_0^\infty \|\mathrm{e}^{-t|A|}u_-\|_{\mathrm{L}^2(\partial M,E)}^2 \, \mathrm{d}t + \|D_0\mathcal{E}u_-\|_{\mathrm{L}^2(Z,E)}^2 \\
&\simeq \|u_-\|_{\mathrm{H}^{-\frac{1}{2}}(\partial M,E)}^2 + \|D_0\mathcal{E}u_-\|_{\mathrm{L}^2(Z,E)}^2 \\
&\lesssim \|u_-\|_{\mathrm{H}^{\frac{1}{2}}(\partial M,E)} + \|D_0\mathcal{E}u_-\|_{\mathrm{L}^2(Z,E)}^2 ,
\end{aligned}
$$

where the last estimate is by the continuity of $\mathrm{H}^{\frac{1}{2}} \hookrightarrow \mathrm{H}^{-\frac{1}{2}}$.

It remains to estimate $\|D_0\mathcal{E}u_-\|_{\mathrm{L}^2(Z,E)}$:

$$
\begin{aligned}
\|D_0\mathcal{E}u_-\|_{\mathrm{L}^2(Z,E)}^2 &\simeq \int_0^\infty \||A|\mathrm{e}^{-t|A|}u_-\|_{\mathrm{L}^2(\partial M,E)}^2 \, \mathrm{d}t \\
&= \int_0^\infty \left\| t^{-\frac{1}{2}} t^{\frac{1}{2}} |A|^{\frac{1}{2}} \mathrm{e}^{-t|A|} |A|^{\frac{1}{2}} u_- \right\|_{\mathrm{L}^2(\partial M,E)}^2 \, \mathrm{d}t \\
&= \int_0^\infty \left\| t^{\frac{1}{2}} |A|^{\frac{1}{2}} \mathrm{e}^{-t|A|} |A|^{\frac{1}{2}} u_- \right\|_{\mathrm{L}^2(\partial M,E)}^2 \, \frac{\mathrm{d}t}{t} \\
&\simeq \left\| |A|^{\frac{1}{2}} u_- \right\|_{\mathrm{L}^2(\partial M,E)}^2 \\
&\simeq \|u_-\|_{\mathrm{H}^{\frac{1}{2}}(\partial M,E)}^2 .
\end{aligned}
$$

Therefore, we obtain that

$$
\|\mathcal{E}u\|_{D_0}^2 \lesssim \|u_-\|_{\mathrm{H}^{\frac{1}{2}}(\partial M,E)} .
$$

Combining a) and b),

$$
\begin{aligned}
\|\mathcal{E}u\|_{D_0} &\leq \|\mathcal{E}u_+\|_{\mathrm{L}^2(Z,E)} + \|\mathcal{E}u_-\|_{\mathrm{L}^2(Z,E)} \\
&\lesssim \|u_+\|_{\mathrm{H}^{-\frac{1}{2}}(\partial M,E)} + \|u_+\|_{\mathrm{H}^{\frac{1}{2}}(\partial M,E)} \\
&= \left\|\chi^+(A)u\right\|_{\mathrm{H}^{-\frac{1}{2}}(\partial M,E)} + \left\|\chi^-(A)u\right\|_{\mathrm{H}^{\frac{1}{2}}(\partial M,E)} \\
&\simeq \|u\|_{\check{\mathrm{H}}_A(D_0)} .
\end{aligned}
$$

The estimate $\|\mathcal{E}^*u\|_{\left(\upsigma_0^{-1}D_0\right)^*} \lesssim \|u\|_{\hat{\mathrm{H}}_A(D_0)}$ is obtained similarly. □

Remark 4.88. This lemma shows precisely the way in which 'parabolic' problems, i.e., heat equations, relate to elliptic equations in the presence of boundary. An elliptic problem in a cylinder can be seen as a parabolic problem on the boundary. As we will see in due course, the extension operators $\mathcal{E}$ and $\mathcal{E}^*$ are of fundamental importance in both understanding questions regarding regularity as well as to establish the surjectivity of the boundary restriction map.

Lemma 4.89. *For all $u \in \mathrm{C}^\infty_\mathrm{c}(Z,E)$ we have*

$$\left\| u|_{\partial M} \right\|_{\check{\mathrm{H}}_A(D_0)} \lesssim \|u\|_{D_0} \qquad \textit{and} \qquad \left\| u|_{\partial M} \right\|_{\hat{\mathrm{H}}_A(D_0)} \lesssim \|u\|_{\left(\sigma_0^{-1} D_0\right)^*}.$$

Proof. Recall the Green's formula from Proposition 4.9: for $u \in \mathrm{C}^\infty_\mathrm{c}(Z,E)$ and $v \in \mathrm{C}^\infty_\mathrm{c}(Z,F)$,

$$\langle D_0 u, v\rangle - \left\langle u, D_0^\dagger v\right\rangle = -\left\langle \sigma_0 u|_{\partial M}, v|_{\partial M}\right\rangle_{\mathrm{L}^2(\partial M,F)}.$$

Given $w \in \mathrm{C}^\infty(\partial M, E)$, set $v := \left(\sigma_0^{-1}\right)^* \mathcal{E}^* w$. Then,

$$\begin{aligned}
\left\langle D_0 u, \left(\sigma_0^{-1}\right)^* \mathcal{E}^* w\right\rangle - \langle u, \underbrace{\left(\sigma_0^{-1} D_0\right)^*}_{=D^\dagger\left(\sigma_0^{-1}\right)^*} \mathcal{E}^* w\rangle
&= -\left\langle \sigma_0 u|_{\partial M}, \left(\sigma_0^{-1}\right)^* w\right\rangle_{\mathrm{L}^2(\partial M,F)} \\
&= -\left\langle u|_{\partial M}, w\right\rangle_{\mathrm{L}^2(\partial M,E)}.
\end{aligned}$$

Now estimating the right hand by the left using the Cauchy-Schwartz inequality,

$$\left|\left\langle u|_{\partial M}, w\right\rangle\right| \lesssim \left\|\sigma_0^{-1} D_0 u\right\|_{\mathrm{L}^2(Z,E)} \|\mathcal{E}^* w\| + \|u\|_{\mathrm{L}^2(\partial M,E)} \left\|\left(\sigma_0^{-1}\right)^* \mathcal{E}^* w\right\|_{\mathrm{L}^2(\partial M,E)}.$$

Since $\sigma_0 \in \mathrm{C}^\infty(\partial M, \mathrm{End}(E))$,

$$\left\|\sigma_0^{-1} D_0 u\right\| \lesssim \|D_0 u\| \lesssim \|u\|_{D_0}$$

by the definition of the graph norm. Moreover, for similar reasons,

$$\|u\| \lesssim \|u\|_{D_0} \qquad \text{and} \qquad \|\mathcal{E}^* w\|_{\mathrm{L}^2(Z,E)} \lesssim \|\mathcal{E}^* w\|_{\left(\sigma_0^{-1} D_0\right)^*}.$$

Therefore,

$$\left|\left\langle u|_{\partial M}, w\right\rangle\right| \lesssim \|u\|_{D_0} \|\mathcal{E}^* w\|_{\left(\sigma_0^{-1} D_0\right)^*} \lesssim \|u\|_{D_0} \|w\|_{\hat{\mathrm{H}}_A(D_0)},$$

where the ultimate inequality follows from Lemma 4.87. This yields

$$\frac{\left|\left\langle u|_{\partial M}, w\right\rangle\right|}{\|w\|_{\hat{\mathrm{H}}_A(D_0)}} \lesssim \|u\|_{D_0},$$

for all $w \in \mathrm{C}^\infty(\partial M, E)$.

By Proposition 4.84, we have that $\left\langle \check{\mathrm{H}}_A(D_0), \hat{\mathrm{H}}_A(D_0)\right\rangle$ and that $\mathrm{C}^\infty(\partial M, E)$ is dense in $\check{\mathrm{H}}_A(D_0)$ and $\hat{\mathrm{H}}_A(D_0)$ from the density of $\mathrm{C}^\infty(\partial M, E)$ in $\mathrm{H}^{\pm\frac{1}{2}}(\partial M, E)$. Consequently, we obtain

$$\left\| u|_{\partial M} \right\|_{\check{\mathrm{H}}_A(D_0)} = \sup_{w \in \mathrm{C}^\infty(\partial M, E)} \frac{\left|\left\langle u|_{\partial M}, w\right\rangle\right|}{\|w\|_{\hat{\mathrm{H}}_A(D_0)}} \lesssim \|u\|_{D_0}.$$

The estimate $\left\| v|_{\partial M} \right\| \lesssim \|v\|_{((\sigma_0)^{-1} D_0)^*}$ is proved similarly. □

Lemma 4.90. *The space* $\mathrm{C}_{\mathrm{c}}^{\infty}(Z, E)$ *is dense in* $\mathrm{dom}(D_{0,\max})$.

Proof. On $\mathbb{R}$, we can readily obtain a sequence $\eta_j \in \mathrm{C}_{\mathrm{c}}^{\infty}([0,\infty))$ such that $\eta_j \to 1$ as $j \to \infty$ in $\mathrm{L}^{\infty}([0,\infty))$ and $(\partial_t \eta_j) \leq 1$. Now, given $u \in \mathrm{dom}(D_{0,\max})$, let $u_j = \eta_j u$. It is readily verified that $u_j \in \mathrm{dom}(D_{0,\max})$ and that

$$D_{0,\max} u_j = \sigma_0(\partial_t + A) u_j = \eta_j \sigma_0 D_{0,\max} u - \sigma_0(\partial_t \eta_j) u \,.$$

Therefore, we have that $u_j \to u$ in $\mathrm{dom}(D_{0,\max})$ with $\mathrm{spt}\, u_j$ is compact. Mollifying u_j to obtain $v_j^{\epsilon} \in \mathrm{C}_{\mathrm{c}}^{\infty}(Z, E)$, we have $v_j^{\epsilon} \to u_j$ in $\mathrm{dom}(D_{0,\max})$ as $\epsilon \to 0$. Combining these facts, we obtain the required conclusion. □

Theorem 4.91. *For an invertible bisectorial adapted boundary operator A for D and $D_0 = \sigma_0(\partial_t + A)$, the model operator built from A, we have that*

$$\check{\mathrm{H}}(D_0) = \check{\mathrm{H}}_A(D_0) = \chi^-(A)\mathrm{H}^{\frac{1}{2}}(\partial M, E) \oplus \chi^+(A)\mathrm{H}^{-\frac{1}{2}}(\partial M, E)$$

in the sense of Banach spaces (i.e. set equality with equivalent norms).

Proof. a) Claim: $\check{\mathrm{H}}(D_0) \subset \check{\mathrm{H}}_A(D_0)$.

Let $u \in \mathrm{dom}(D_{0,\max})$. From Lemma 4.90, there is a sequence $(u_n) \subset \mathrm{C}_{\mathrm{c}}^{\infty}(M, E)$ s.t. $u_n \to u$ in the D_0-norm. By Lemma 4.89 we have

$$\left\| u_n|_{\partial M} - u_m|_{\partial M} \right\|_{\check{\mathrm{H}}_A(D_0)} \lesssim \|u_n - u_m\|_{D_{0,\max}}$$

and since $\{u_n\}$ is D_0-norm Cauchy, $u|_{\partial M} = \lim_{n\to\infty} u_n|_{\partial M}$ is well-defined. Moreover,

$$\left\| u|_{\partial M} \right\|_{\check{\mathrm{H}}_A(D_0)} = \lim_{n\to\infty} \left\| u_n|_{\partial M} \right\|_{\check{\mathrm{H}}_A(D_0)} \lesssim \lim_{n\to\infty} \|u_n\|_{D_0} = \|u\|_{D_0} \,.$$

b) Claim: $\check{\mathrm{H}}_A(D_0) \subset \check{\mathrm{H}}(D_0)$.

Let $w \in \check{\mathrm{H}}_A(D_0)$. Then by Lemma 4.87 we have $\mathcal{E}w \in \mathrm{dom}(D_{0,\max})$, so $\mathcal{E}w|_{\partial M} = w$.

c) A priori $\ker\Big(u \mapsto u|_{\partial M}\Big) = \mathrm{dom}(D_{0,\min})$, and $u \mapsto u|_{\partial M} : \mathrm{dom}(D_{0,\max}) \to \check{\mathrm{H}}_A(D_0)$ is a bounded surjection with $\ker\Big(u \mapsto u|_{\partial M}\Big) = \mathrm{dom}(D_{0,\min})$. Since we have already shown that $\check{\mathrm{H}}_A(D_0) = \check{\mathrm{H}}(D_0)$, we conclude that the norms are comparable. □

To emphasise that we are not restricting ourselves, let us first consider the reduction of a model operator for a general adapted boundary operator to an invertible bisectorial one.

Corollary 4.92. *If A is any boundary adapted operator and let r be an admissible spectral cut, i.e. $A_r := A - r$ is bisectorial and invertible. Let $D = \sigma_0(\partial_t + A)$ and $D_0 := \sigma_0(\partial_t + A_r)$ the model operator for D with the adapted boundary operator A_r. Then,*

$$\check{\mathrm{H}}(D) = \check{\mathrm{H}}(D_0) = \check{\mathrm{H}}_{A_r}(D_0)\,.$$

Proof. We show that $\mathrm{dom}(D_{0,\max}) = \mathrm{dom}(D_{0,\max})$ with $\|u\|_D \simeq \|u\|_{D_0}$. From Lemma 4.90, we have that $\mathrm{C}_{\mathrm{c}}^\infty(Z, E)$ is dense in $\mathrm{dom}(D_{0,\max})$. The same argument yields that $\mathrm{C}_{\mathrm{c}}^\infty(Z, E)$ is dense in $\mathrm{dom}(D_{r,\max})$. Therefore, it suffices to prove $\|u\|_D \simeq \|u\|_{D_0}$ for $u \in \mathrm{C}_{\mathrm{c}}^\infty(Z, E)$. So, fix $u \in \mathrm{C}_{\mathrm{c}}^\infty(Z, E)$ and note that

$$\|u\|_{D_0}^2 = \|u\|^2 + \|D_0 u\|^2 \simeq \|u\|^2 + \|Du + \sigma_0 r u\|^2 \simeq \|u\|^2 + \|Du\|^2 = \|u\|_D^2\,. \qquad \square$$

Remark 4.93. Although not straightforward, it is possible to show that

$$\chi^+(A)\mathrm{H}^{-\frac{1}{2}}(\partial M, E) = \gamma \ker(D_{0,\max})\,.$$

The key point is that we have described the Hardy space of solutions of D_0 in terms of an operator on the boundary.

Fix $u \in \mathrm{C}_{\mathrm{cc}}^\infty(Z, E)$ and $v \in \mathrm{C}_{\mathrm{cc}}^\infty(Z, F)$. Then, we compute

$$\begin{aligned}
\langle D_0 u, v\rangle &= \langle \sigma_0(\partial_t + A)u, v\rangle \\
&= \langle (\partial_t + A)u, \sigma_0^* v\rangle \\
&= \langle u, (-\partial_t + A^*)\sigma_0^* v\rangle \\
&= \left\langle u, -\sigma_0^*\left(\partial_t - (\sigma_0^*)^{-1} A^* \sigma_0^*\right) v\right\rangle\,.
\end{aligned}$$

This shows us that $D_0^\dagger = -\sigma_0^*\left(\partial_t - (\sigma_0^*)^{-1} A^* \sigma_0^*\right)$.

Definition 4.94 (Induced formal adjoint adapted boundary operator from A). Given an adapted boundary operator A for D, we define the induced adapted boundary operator from A for $D^\dagger$ to be

$$\tilde{A} := -(\sigma_0^*)^{-1} A^* \sigma_0^*\,.$$

Remark 4.95.

1. Note here that A^* is really $A^\dagger$, but as we have remarked before, on the manifold ∂M, as it has no boundary, a differential operator has a unique extension. Therefore, this is only a slight abuse of notation.

2. From inspection, it is easy to see that the induced formal adjoint adapted boundary operator $\tilde{A} = (\sigma_0^{-1})^* A^* \sigma_0^*$ is an adapted boundary operator for $D^\dagger$.

Lemma 4.96. *The map* $\left(\sigma_0^{-1}\right)^* : E|_{\partial M} \to F|_{\partial M}$ *induces an isomorphism*

$$\hat{\mathrm{H}}_A(D_0) \xrightarrow{\cong} \check{\mathrm{H}}_{\tilde{A}}\left(D_0^\dagger\right),$$

where $\tilde{A}$ *is the induced formal adjoint adapted boundary operator from* A.

The pairing $\beta(u,v) := -\langle \sigma_0 u, v\rangle_{\mathrm{L}^2(\partial M,F)}$ *for* $u \in \mathrm{C}^\infty(\partial M,E), v \in \mathrm{C}^\infty(\partial M,F)$ *extends to a reflexive* $\left\langle \check{\mathrm{H}}_A(D_0), \check{\mathrm{H}}_{\tilde{A}}\left(D_0^\dagger\right)\right\rangle$.

Proof. Proposition 4.84 yields that $\left\langle \hat{\mathrm{H}}_A(D_0), \check{\mathrm{H}}_A(D_0)\right\rangle$ is reflexive and its restriction to $\mathrm{C}^\infty(\partial M,E)$ agrees with the L^2-inner product. If $\left(\sigma_0^{-1}\right)^* : \hat{\mathrm{H}}_A(D_0) \to \check{\mathrm{H}}_{\tilde{A}}\left(D_0^\dagger\right)$ is an isomorphism, then clearly β is a perfect pairing.

We prove that $\left(\sigma_0^{-1}\right)^*$ is an isomorphism. Clearly $(\sigma_0^{-1})^* : \mathrm{C}^\infty(\partial M,E) \to \mathrm{C}^\infty(\partial M,F)$ is a bijection. By the density of $\mathrm{C}^\infty(\partial M,E)$ in $\hat{\mathrm{H}}_A(D_0)$ and the density of $\mathrm{C}^\infty(\partial M,F)$ in $\check{\mathrm{H}}_{\tilde{A}}(D_0^\dagger)$, it suffices to prove $\|(\sigma_0^{-1})^* u\|_{\check{\mathrm{H}}_{\tilde{A}}(D_0^\dagger)} \simeq \|u\|_{\hat{\mathrm{H}}_{\tilde{A}}(D_0)}$ when $u \in \mathrm{C}^\infty(\partial M,E)$.

For $u \in \mathrm{C}^\infty(\partial M,E)$, we estimate

$$\begin{aligned}
\left\|\left(\sigma_0^{-1}\right)^* u\right\|^2_{\check{\mathrm{H}}_{\tilde{A}}(D_0)} &\lesssim \left\|\left(\sigma_0^{-1}\right)^* \mathcal{E}^* u\right\|^2_{D_0^\dagger} \\
&= \left\|D^\dagger\left(\sigma_0^{-1}\right)^* \mathcal{E}^* u\right\|^2_{\mathrm{L}^2(Z)} + \left\|\left(\sigma_0^{-1}\right)^* \mathcal{E}^* u\right\|^2_{\mathrm{L}^2(Z)} \\
&= \left\|-\sigma_0^*\left(\partial_t - \left(\sigma_0^{-1}\right)^* A^* \sigma_0\right)\left(\sigma_0^{-1}\right)^* \mathcal{E}^* u\right\|^2_{\mathrm{L}^2(Z)} + \left\|\left(\sigma_0^{-1}\right)^* \mathcal{E}^* u\right\|^2_{\mathrm{L}^2(Z)} \\
&= \|(-\partial_t - A^*)\mathcal{E}^* u\|^2_{\mathrm{L}^2(Z)} + \left\|\left(\sigma_0^{-1}\right)^* \mathcal{E}^* u\right\|^2_{\mathrm{L}^2(Z)} \\
&= \|\mathcal{E}^* u\|^2_{\left(\sigma_0^{-1} D\right)^*} \\
&\lesssim \|u\|^2_{\hat{\mathrm{H}}_A(D_0)},
\end{aligned}$$

where the ultimate inequality follows from Lemma 4.87.

To obtain the reverse inequality, let $\widetilde{\mathcal{E}}u := e^{-t|\tilde{A}|}u$ be the extension operator for $D_0^\dagger$. We have

$$\|(\sigma_0^*)v\|_{\hat{\mathrm{H}}_A(D_0)} \lesssim \left\|\sigma_0^* \widetilde{\mathcal{E}} v\right\|_{\left(\sigma_0^{-1} D_0\right)^*} \lesssim \left\|\widetilde{\mathcal{E}} v\right\|_{D_0^\dagger} \lesssim \|v\|_{\check{\mathrm{H}}_{\tilde{A}}(D_0^\dagger)},$$

where the first inequality follows from Lemma 4.89 since $(\sigma_0^* \mathcal{E} u)|_{\partial M} = \sigma_0^* u$, and the last inequality from applying Lemma 4.87 with $D_0^\dagger$ and $\check{\mathrm{H}}_{\tilde{A}}(D_0^\dagger)$ in place of D_0 and $\check{\mathrm{H}}_A(D_0)$. □

It is important to emphasise the significance of this lemma. For this, let us consider what we have achieved so far. Given an invertible bisectorial adapted boundary operator A for D, we are able to consider the model problem D_0 and its formal adjoint $D_0^\dagger$. The boundary value problems that interest us are now extensions of $D_{0,\min}$ contained in $D_{0,\max}$. As we have seen already, we are able to study the space $\check{\mathrm{H}}_A(D_0)$

instead. A boundary condition is a closed subspace $B \subset \check{\mathrm{H}}_A(D_0)$, and the operator we obtain is $D_{0,B}$ with domain $\mathrm{dom}(D_B) = \left\{u \in \mathrm{dom}(D_{0,\max}) : u|_{\partial M} \in B\right\}$.

Now, consider now the adjoint operator $D^*_{0,B}$. Again, as we have seen $D^*_{0,B} \subset D^\dagger_{0,\max}$. In order to study $D_{0,B}$ and its adjoint $D^*_{0,B}$, in the absence of Lemma 4.96, we would be required to study $\check{\mathrm{H}}_A(D_0)$ as well as $\check{\mathrm{H}}_{\tilde{A}}(D^\dagger_0)$. In particular, we have to study function spaces over two potentially distinct bundles.

However, Lemma 4.96 simplifies this picture significantly. Since $\check{\mathrm{H}}_{\tilde{A}}(D^\dagger_0)$ is isomorphic to $\hat{\mathrm{H}}_A(D_0)$, and the isomorphism is the concrete object $(\sigma_0^{-1})^*$, we can instead study spaces induced by A and over only one bundle, namely $E|_{\partial M}$. More concretely, recall that

$$B^\dagger = \left\{ v|_{\partial M} \;\middle|\; v \in \mathrm{dom}(D^*_{0,B}) \subset \mathrm{dom}\left(D^\dagger_{0,\max}\right)\right\} \subset \check{\mathrm{H}}_{\tilde{A}}\left(D^\dagger_0\right)$$

is the boundary condition for $D^*_{0,B}$, which is a subspace of a function space over F. However, by Lemma 4.96, we can instead study the object

$$\sigma_0^* B^\dagger \subset \hat{\mathrm{H}}_A(D_0)\,,$$

which is a subspace of a function space over E. Therefore, in what follows, we attempt to obtain a more refined understanding of the action of the map $(\sigma_0^{-1})^*$.

Lemma 4.97. *We have* $\left(\sigma_0^{-1}\right)^* : \chi^{\pm}(A^*)\mathrm{H}^{\pm\frac{1}{2}}(\partial M, E) \to \chi^{\mp}\left(\tilde{A}\right)\mathrm{H}^{\pm\frac{1}{2}}(\partial M, F)$.

Proof. By virtue of the fact that $(\zeta - U^{-1}AU)^{-1} = U(\zeta - A)^{-1}U$, through functional calculus, we obtain

$$\begin{aligned}\chi^{\mp}\left(\tilde{A}\right) &= \chi^{\mp}(-\left(\sigma_0^{-1}\right)^* A^* \sigma_0^*) \\ &= \left(\sigma_0^{-1}\right)^* \chi^{\mp}(-A^*)\sigma_0^* \\ &= \left(\sigma_0^{-1}\right)^* \chi^{\pm}(A^*)\sigma_0^*\,.\end{aligned}$$

The map $\sigma_0^* : \mathrm{H}^\alpha(\partial M, E) \to \mathrm{H}^\alpha(\partial M, F)$ is a Banach space isomorphism, and therefore, the conclusion follows. ∎

This result is certainly to be expected. Recall that $\hat{\mathrm{H}}_A(D_0) = \chi^-(A)\mathrm{H}^{-\frac{1}{2}}(\partial M, E) \oplus \chi^+(A)\mathrm{H}^{\frac{1}{2}}(\partial M, E)$. By Lemma 4.96, we know that $\hat{\mathrm{H}}_A(D_0) \cong \check{\mathrm{H}}_{\tilde{A}}(D^\dagger_0)$ via $(\sigma_0^{-1})^*$. Given that $\chi^\pm(A)\mathrm{H}^\alpha(\partial M, E)$ is infinite dimensional, we would certainly expect that the $\mathrm{H}^{\pm\frac{1}{2}}(\partial M, E)$ part of $\hat{\mathrm{H}}_A(D_0)$ be mapped in a nontrivial way to the $\mathrm{H}^\pm(\partial M, F)$.

Proposition 4.98. *For* $u \in \mathrm{dom}(D_{0,\max})$ *and* $v \in \mathrm{dom}\left(D^\dagger_{0,\max}\right)$,

$$\langle D_{0,\max} u, v\rangle_{\mathrm{L}^2(Z,F)} - \left\langle u, D^\dagger_{0,\max} v\right\rangle_{\mathrm{L}^2(Z,E)} = -\left\langle u|_{\partial M}, (\sigma_0^*) v|_{\partial M}\right\rangle_{\check{\mathrm{H}}_A(D_0)\times\hat{\mathrm{H}}_A(D_0)}\,.$$

Proof. For $u \in \mathrm{C}^\infty_c(\partial M, E)$ and $v \in \mathrm{C}^\infty_c(\partial M, F)$, we have by the Greens formula in Proposition 4.9 that the required formula holds. Since $\beta(x,y) = \langle x, \sigma_0^* y \rangle_{\mathrm{L}^2(\partial M, E)}$ from Lemma 4.96 and the spaces $\mathrm{C}^\infty_c(\partial M)$ and $\mathrm{C}^\infty_c(\partial M)$ are dense in $\mathrm{dom}(D_{0,\max})$ and $\mathrm{dom}\left(D^\dagger_{0,\max}\right)$ respectively by Lemma 4.90, the conclusion follows. □

Proposition 4.99. *Let $B \subset \check{\mathrm{H}}_A(D_0)$ be a boundary condition, i.e. a closed subspace. Then the adjoint boundary condition $B^\dagger$ is precisely*

$$B^\dagger = (\sigma_0^*)^{-1} B^{\perp, \langle \hat{\mathrm{H}}_A(D_0), \check{\mathrm{H}}_A(D_0) \rangle}.$$

Proof. This is a straightforward using Proposition 4.98 and Lemma 4.96 □

Remark 4.100. By Corollary 4.92, the discussion we had above holds for a general boundary adapted operator A and its associated model operator by first passing to A_r where r is an admissible spectral cut.

4.10 Maximal regularity

Consider the situation that M' is a compact manifold without boundary, and that D is an m-th order differential operator. We have already mentioned that, due to elliptic regularity,$\mathrm{dom}(D) = \mathrm{H}^m(M', E)$. Moreover, it is classical fact that $D : \mathrm{H}^{m+k}(M', E) \to \mathrm{H}^k(M', E)$ for $k \geq 0$. In making these assertions, we are implicitly identifying the operator D with $D_{\max} = D_{\min} = \overline{D}$.

When M is a compact manifold with boundary, we have already seen that $D_{\max} \neq D_{\min}$ and in fact, generally, ${}^{\mathrm{dom}(D_{\max})}\!/\!{}_{\mathrm{dom}(D_{\min})}$ is infinite dimensional. Therefore, we expect mapping properties for D for Sobolev spaces up to the boundary on M, which mirrors that of the the boundaryless scenario, would be captured by a good class of boundary conditions. To understand such boundary conditions, we need to study $\mathrm{dom}(D_{\max}) \cap \mathrm{H}^{m+k}(M, E)$ and characterise this space via information on the boundary.

We restrict our considerations to $m = 1$ and first consider the model operator D_0 on Z given an invertible adapted boundary operator A for D. In order to do that, we need to understand a more abstract result called the *maximal regularity* for injective sectorial operators.

Let $I \subset \mathbb{R}$ be an interval and $\mathcal{B}$ a Banach space. Define

$$\mathrm{L}^p(I, \mathcal{B}) := \left\{ f : I \to \mathcal{B} \;\middle|\; \int_I \|f(t)\|^p_{\mathcal{B}} \, \mathrm{d}t < \infty \right\}.$$

Theorem 4.101 (L^2**-maximal regularity, Theorem 2.1 in [21]).** *Let T be an injective densely-defined ω-sectorial operator for an $\omega \in \left[0, \frac{\pi}{2}\right)$ on a Hilbert space $\mathcal{H}$. Fix $\tau \in (0, \infty]$. Then for every $f \in \mathrm{L}^2([0,\tau), \mathcal{H})$, there exists a unique solution to*

$$\begin{cases} \partial_t u + Tu = f & \text{a.e. on } (0,\tau)\,, \\ u(0) = 0 \end{cases} \tag{4.7}$$

satisfying

I) $u \in \mathrm{L}^2([0,\tau_0], \mathrm{dom}(T))$ *whenever* $\tau_0 < \infty$ *and* $\tau_0 \leq \tau$.

II) $\partial_t u \in \mathrm{L}^2([0,\tau), \mathcal{H})$.

III) There is a constant $C_{\mathrm{MR}} < \infty$ *such that*

$$\int_0^\tau \|\partial_t u(t)\|_{\mathcal{H}}^2 \, \mathrm{d}t + \int_0^\tau \|Tu(t)\|_{\mathcal{H}}^2 \, \mathrm{d}t \leq C_{\mathrm{MR}} \int_0^\tau \|f(t)\|_{\mathcal{H}}^2 \, \mathrm{d}t\,. \tag{4.8}$$

The constant depends on ω, T and $C_{\mathrm{H}^\infty}(T)$, the constant appearing in the H^∞-functional calculus of T. However, it is independent of τ and f.

The solution generator is

$$W(\cdot;\cdot) : [0,\tau) \times \mathrm{L}^2([0,\tau), \mathcal{H}) \to \mathrm{L}^2([0,\tau_0], \mathrm{dom}(T))\,,$$

given by

$$u(t) := W(t; f) := \int_0^t \mathrm{e}^{-(t-s)T} f(s) \, \mathrm{d}s\,.$$

Section 9.3 in [26] has a more detailed discussion surrounding this this topic.

Remark 4.102. 1. The condition I) is only non-trivial when $\tau = \infty$. There, it is saying that, for each finite τ_0 we have that

$$\|u\|_{\mathrm{L}^2([0,\tau_0),\mathrm{dom}(T))}^2 = \int_0^{\tau_0} \left(\|u(t)\|_{\mathcal{H}}^2 + \|Tu(t)\|_{\mathcal{H}}^2 \, \mathrm{d}t\right) < \infty\,.$$

It may very well be that,

$$\lim_{\tau_0 \to \infty} \|u\|_{\mathrm{L}^2([0,\tau_0),\mathrm{dom}(T))} = \infty\,,$$

so we cannot expect that $\|u\|_{\mathrm{L}^2([0,\infty),\mathrm{dom}(T))}$ to be finite, even though for each $\tau_0 < \infty$, we have $u \in \mathrm{L}^2([0,\tau_0), \mathrm{dom}(T))$.

We see from (4.8) that

$$\int_0^\infty \|Tu(t)\|_{\mathcal{H}}^2 \, \mathrm{d}t \leq \int_0^\infty \|f(t)\|_{\mathcal{H}}^2 \, \mathrm{d}t < \infty\,,$$

and so the failure for $u \in \mathrm{L}^2([0,\infty), \mathrm{dom}(T))$ is due to the fact that, in general,

$$\int_0^\infty \|u(t)\|_{\mathcal{H}}^2 \, \mathrm{d}t = \infty\,.$$

That is, in general, $u \notin \mathrm{L}^2([0,\infty),\mathcal{H})$.

If, however, T is invertible, then

$$\|u\| \lesssim \|Tu\|$$

and therefore, from the maximal regularity estimate (4.8), we find $u \in \mathrm{L}^2((0,\infty),\mathcal{H})$.

2. The role of τ is to be able to be flexible with the kind of functions that can be considered as solutions. More precisely, for a small τ, there would be more functions $f : [0,\tau) \to \mathcal{H}$ than those arising as $f = \tilde{f}|_{[0,\tau]}$ where $\tilde{f} : [0,\infty) \to \mathcal{H}$.

3. We have only considered maximal regularity for Hilbert spaces, but it is also of importance to consider this on Banach spaces.

   On a Hilbert space, maximal regularity is a consequence of the fact that that T generates an analytic semigroup. Since $\omega < \pi/2$, the semigroup is analytic on $\mathrm{S}^\circ_{(\omega-\frac{\pi}{2})+}$. However, in a general Banach space, the actual condition underlying the theory of maximal regularity is actually a 'wave equation'. This notion of a wave equation can be conceptually understood in the Hilbert space setting. A sectorial operator T on a Hilbert space has an H^∞-functional calculus iff there is $\vartheta \in [0,\infty)$ and $k \in (0,\infty)$ such that

$$\|T^{\imath s}\| \leq k e^{\vartheta|s|}$$

   for all $s \in \mathbb{R}$. The latter property is termed *Bounded Imaginary Powers (BIP)*. Under sufficiently general conditions, $T^{\imath s} = \mathrm{e}^{\log(T)\imath s}$. This is a generator for solutions to the wave equation

$$\partial_t u + \imath \log(T) u = 0\,.$$

   In general (i.e. in a Banach space), having an H^∞-functional calculus is only sufficient for BIP.

In light of Theorem 4.101, we truncate and consider a finite cylinder. That is, we fix a $\varrho \in (0,\infty)$, which will be chosen later in application, and restrict our attention to

$$Z_{\bar{\varrho}} := [0,\varrho] \times \partial M\,.$$

We now consider D and E restricted to $Z_{\bar{\varrho}}$. However, now

$$\partial Z_{\bar{\varrho}} = (\{0\} \times \partial M) \sqcup (\{\varrho\} \times \partial M)\,,$$

and therefore, it now has an extra 'fake' boundary in addition to the original 'real' boundary $\{0\} \times \partial M$, which we previously denoted as ∂M and continue to do so.

In the fullness of time, we will see that this 'fake' boundary is purely a technical necessity. When applying the results of this section, we always institute a cutoff

$\eta \in \mathrm{C}_c^\infty([0,\infty))$ satisfying

$$\eta(t) = \begin{cases} 1 & \text{if } t \in \left[0, \frac{1}{2}\right), \\ 0 & \text{if } t \in \left[\frac{3}{4}, \infty\right). \end{cases}$$

Then, for a given $u \in \mathrm{dom}(D_{0,\max})$,

$$u|_{\partial M} = u(0) = \eta(0)u(0) = \eta u|_{\partial M},$$

and therefore, we have that $\eta u \in \mathrm{dom}(D_{0,\max} \cap) \, \mathrm{dom}(D_{0,\max}, Z_{\bar{\varrho}})$. Moreover,

$$(1 - \eta u)(0) = (\eta u)(\varrho) = 0$$

and therefore, $(1 - \eta u) \in \mathrm{dom}(D_{0,\min})$. This shows that, in order to understand properties of u up to the boundary, it suffices to study ηu.

From here on, for an invertible ω-bisectorial adapted boundary operator A, in order to apply Theorem 4.101, we set $T := |A|$, $\tau_0 := \varrho$ to be chosen later, $\tau := \infty$, $\mathcal{H} := \mathrm{L}^2(Z_{\bar{\varrho}}, E)$.

Definition 4.103. Define:

$$(S_0 u)(t) := \int_0^t e^{-(t-s)|A|} \chi^+(A) u(s) \, \mathrm{d}s - \int_t^\varrho e^{-(s-t)|A|} \chi^-(A) u(s) \, \mathrm{d}s$$

and

$$(C_\varrho u)(s) := u(\varrho - s).$$

Lemma 4.104. *The following hold.*

I) $S_0 u(t) = W(t; \chi^+(A)u) - W(\varrho - t; \chi^-(A) C_\varrho u)$.

II) $\chi^+(A)(S_0 u)(0) = 0$ *and* $\chi^-(A)(S_0 u)(\varrho) = 0$.

III) $\sigma_0^{-1} D_0 S_0 = (\partial_t + A) S_0 = \mathrm{id}$.

Proof. a) Ad I). The first term is clear, and for the second term, apply coordinate transform $s \mapsto \varrho - s$ and compute.

b) Ad II). From Proposition 4.70 IV), we have that the functional calculus commutes with itself. Moreover, we can interchange the application of the projector and integral.

c) Ad III). Write $u := u_+ - u_-$ where $u_\pm = \chi^\pm(A)u$. It suffices to show $(\partial_t + A)S_0 u_\pm = u_\pm$. Using $|A| = A \,\mathrm{sgn}(A)$, we get

$$\partial_t S_0 u_+ = -A \,\mathrm{sgn}(A) S_0 u_+ + u_+,$$

since $u_+ = \chi^+(A)u$ we have $\mathrm{sgn}(A)u_+ = u_+$ and therefore $(\partial_t + A)S_0 u_+ = u_+$.

The proof $(\partial_t + A)S_0 u_- = u_-$ is similar, but a little more involved. We leave it as an exercise. $\square$

Notation 4.105. For a differential operator X, we denote the domain of it for sections on $U \subset M$ as $\operatorname{dom}(\overline{X}; U)$.

Since the operator $|A|^l : \mathrm{L}^2(\partial M, E) \to \mathrm{L}^2(\partial M, E)$ with $\operatorname{dom}(|A|) = \mathrm{H}^l(\partial M, E)$, we can also consider $A : \mathrm{L}^2(Z_{\overline{\varrho}}, E) \to \mathrm{L}^2(Z_{\overline{\varrho}}, E)$ with domain

$$\operatorname{dom}(|A|^l; Z_{\overline{\varrho}}) = \left\{ u \in \mathrm{L}^2(Z_{\overline{\varrho}}, E) \;\middle|\; u(t) \in \operatorname{dom}(|A|^l) \right\}$$

with the induced norm

$$\left\| |A|^l u \right\|^2_{\mathrm{L}^2(Z_{\overline{\varrho}}, E)} = \int_0^{\rho} \left(\|u(t)\|^2_{\mathrm{L}^2(\partial M, E)} + \| |A| u(t) \|^2_{\mathrm{L}^2(\partial M, E)} \right).$$

Lemma 4.106. *We have*

$$\mathrm{H}^k(Z_{\bar{\varrho}}, E) = \bigcap_{l=0}^{k} \operatorname{dom}\left(\partial_t^{k-l} |A|^l; Z_{\overline{\varrho}} \right) \qquad \text{and} \qquad \|u\|_{\mathrm{H}^k(Z_{\bar{\varrho}}, E)} \simeq \sum_{l=0}^{k} \left\| \partial_t^{k-l} |A|^l u \right\|^2_{\mathrm{L}^2(Z_{\bar{\varrho}}, E)}.$$

Proof. Using the fact that A is elliptic, as we have already stated, we have that $\operatorname{dom}(|A|^k) = \mathrm{H}^k(\partial M, E)$. Through localisation, it is a routine calculation to show that, for $u \in \mathrm{C}^\infty(Z_{\overline{\varrho}}, E)$,

$$\|u\|_{\mathrm{H}^k(Z_{\bar{\varrho}}, E)} \simeq \sum_{l=0}^{k} \left\| \partial_t^{k-l} |A|^l u \right\|^2_{\mathrm{L}^2(Z_{\bar{\varrho}}, E)}.$$

The space $\mathrm{C}^\infty(Z_{\overline{\varrho}}, E)$ is dense in both $\mathrm{H}^k(Z_{\bar{\varrho}}, E)$ and $\operatorname{dom}\left(\partial_t^{k-l} |A|^l; Z_{\overline{\varrho}} \right)$ and so the desired conclusion follows. □

Lemma 4.107. *$S_0 : \mathrm{H}^k(Z_{\bar{\varrho}}, E) \to \mathrm{H}^{k+1}(Z_{\bar{\varrho}}, E)$ is a bounded operator and the bound is independent of ϱ.*

Proof. a) Case $k = 0$.

From Lemma 4.106, we have that when $v \in \mathrm{H}^1(Z_{\bar{\varrho}}, E)$,

$$\|v\|^2_{\mathrm{H}^1(Z_{\bar{\varrho}})} \simeq \int_0^{\varrho} \|\partial_t v\|^2_{\mathrm{L}^2(\partial M, E)} \, \mathrm{d}t + \int_0^{\varrho} \| |A| v \|^2_{\mathrm{L}^2(\partial M, E)} \, \mathrm{d}t.$$

Recalling that $\mathrm{C}^\infty(Z_{\bar\varrho},E)$ is dense in $\mathrm{H}^k(Z_{\bar\varrho},E)$ for all k let $u\in\mathrm{C}^\infty(Z_{\bar\varrho},E)$ and $v := S_0u$ and compute

$$\begin{aligned}
\|S_0u\|^2_{\mathrm{H}^1} &\simeq \int_0^\varrho \Big(\|\partial_t S_0u(t)\|^2_{\mathrm{L}^2(\partial M,E)} + \||A|u(t)\|^2_{\mathrm{L}^2(\partial M,E)}\Big)\,\mathrm{d}t\\
&\lesssim \int_0^\varrho \big\|\partial_t W\big(t;\chi^+(A)u\big)\big\|^2_{\mathrm{L}^2(\partial M,E)}\,\mathrm{d}t\\
&\quad + \int_0^\varrho \big\||A|W\big(t;\chi^+(A)u\big)\big\|^2_{\mathrm{L}^2(\partial M,E)}\,\mathrm{d}t\\
&\quad + \int_0^\varrho \big\|\partial_t W\big(\varrho-t;\chi^-(A)C_\varrho u\big)\big\|^2_{\mathrm{L}^2(\partial M,E)}\,\mathrm{d}t\\
&\quad + \int_0^\varrho \big\||A|W\big(\varrho-t;\chi^-(A)C_\varrho u\big)\big\|^2_{\mathrm{L}^2(\partial M,E)}\,\mathrm{d}t\\
&\lesssim \int_0^\varrho \big\|\chi^+(A)u(t)\big\|^2_{\mathrm{L}^2(\partial M,E)}\,\mathrm{d}t + \int_0^\varrho \big\|\chi^-(A)(C_\varrho u)(t)\big\|^2_{\mathrm{L}^2(\partial M,E)}\,\mathrm{d}t\\
&\lesssim \int_0^\varrho \Big(\big\|\chi^+(A)u(t)\big\|^2_{\mathrm{L}^2(\partial M,E)} + \big\|\chi^-(A)u(t)\big\|^2_{\mathrm{L}^2(\partial M,E)}\Big)\,\mathrm{d}t\\
&\simeq \int_0^\varrho \|u(t)\|^2_{\mathrm{L}^2(\partial M,E)}\,\mathrm{d}t\\
&= \|u\|_{\mathrm{L}^2(Z_{\bar\varrho},E)}\,.
\end{aligned}$$

where the third inequality follows from Theorem 4.101 and the fourth from the fact that $C_\varrho:\mathrm{L}^2(Z_{\bar\varrho})\to\mathrm{L}^2(Z_{\bar\varrho})$ is an isometry and commutes with $\chi^\pm(A)$. By density, $S_0:\mathrm{L}^2(Z_{\bar\varrho},E)\to\mathrm{H}^1(Z_{\bar\varrho},E)$.

b) Case $k>0$.

Let $f\in\mathrm{dom}\Big(|A|^l\Big)=\mathrm{H}^l(\partial M,E)$. Since by Theorem 4.101 we have that

$$\partial_t W(t;f) + |A|W(t;f) = f(t)\,,$$

and using the fact that ∂_t and $|A|^l$ commutes, we obtain

$$\begin{cases}\partial_t^l W(t;f) = (-1)^l W\Big(t;|A|^l f\Big)(t) + \sum_{m=0}^{l-1}\partial_t^{l-1-m}|A|^m f(t)\,,\\ \partial_t^l W(\varrho-t;f) = (-1)^{l+1} W\Big(\varrho-t;|A|^l f\Big) + \sum_{m=0}^{l-1}\partial_t^{l-1-m}|A|^m f(t)\,.\end{cases}\tag{4.9}$$

Again by density, assume $f\in\mathrm{C}^\infty(Z_{\bar\varrho},E)$. By the Lemma 4.106, it suffices to show that for $l\in[0,k+1]$ we have

$$\Big\|\partial_t^l|A|^{k+1-l}W(\cdot;f)\Big\|_{\mathrm{L}^2(Z_{\bar\varrho},E)} \lesssim \|f\|_{\mathrm{H}^k(Z_{\bar\varrho},E)}\,.$$

It is readily verified that $|A|^k$ commutes with Banach-valued integration in t, and by the regularity of f, it particularly satisfies $f\in\mathrm{dom}(|A|^{l-1})$. Therefore,

$$\begin{aligned}
|A|^l W(t;f) = |A|^l \int_0^\varrho e^{-(t-s)|A|} f(s)\,\mathrm{d}s&\\
&= |A|\int_0^\varrho e^{-(t-s)|A|}|A|^{l-1}f(s)\,\mathrm{d}s = |A|W\Big(t;|A|^{l-1}f\Big)
\end{aligned}$$

For $l = 0$:

$$\begin{aligned}\left\| |A|^{k+1} W(\cdot; f) \right\|^2_{\mathrm{L}(Z_{\bar{\varrho}}, E)} &= \left\| |A| W\left(\cdot; |A|^k f\right) \right\|_{\mathrm{L}^2(Z_{\bar{\varrho}}, E)} \\ &\lesssim \left\| |A|^k f \right\|_{\mathrm{L}^2(Z_{\bar{\varrho}}, E)} \\ &\lesssim \|f\|_{\mathrm{H}^k(Z_{\bar{\varrho}}, E)},\end{aligned}$$

where the second inequality follows from Theorem 4.101 and the last inequality from Lemma 4.106.

For $l \geq 1$: apply $|A|^{k+1-l}$ to (4.9) to obtain

$$\partial_t^l |A|^{k+1-l} W(t; f) = -1|A|W\left(t; |A|^k f\right) + \sum_{m=0}^{l-1} \partial_t^{l-1-m} |A|^{k+1+m-l} f(t),$$

and then estimate

$$\begin{aligned}\left\| \partial_t^l |A|^{k+1-l} W(\cdot; f) \right\|_{\mathrm{L}^2(Z_{\bar{\varrho}}, E)} &\leq \left\| |A| W\left(\cdot; |A|^k f\right) \right\|_{\mathrm{L}^2(Z_{\bar{\varrho}}, E)} \\ &\quad + \sum_{m=0}^{l-1} \left\| \partial_t^{l-1-m} |A|^{k+1+m+l} f \right\|_{\mathrm{L}^2(Z_{\bar{\varrho}}, E)} \\ &\lesssim \left\| |A|^k f \right\|_{\mathrm{L}^2(Z_{\bar{\varrho}}, E)} + \|f\|_{\mathrm{H}^k(Z_{\bar{\varrho}}, E)} \\ &\lesssim \|f\|_{\mathrm{H}^k(Z_{\bar{\varrho}}, E)},\end{aligned}$$

where in the second inequality we have again used Theorem 4.101.

Similarly, we can estimate the term $W(C_\varrho(\cdot); f)$, noting $C_\varrho : \mathrm{H}^k \to \mathrm{H}^k$ isometrically. To finish the proof, we note that for $u \in \mathrm{C}^\infty(Z_{\bar{\varrho}}, E)$

$$\begin{aligned}\|S_0 u\|_{\mathrm{H}^{k+1}(Z_{\bar{\varrho}}, E)} &\leq \left\| W\left(\cdot; \chi^+(A)u\right) \right\|_{\mathrm{H}^{k+1}(Z_{\bar{\varrho}}, E)} + \left\| W\left(C_\varrho(\cdot); \chi^-(A) C_\varrho u\right) \right\|_{\mathrm{H}^{k+1}(Z_{\bar{\varrho}}, E)} \\ &\lesssim \left\| \chi^+(A)u \right\|_{\mathrm{H}^k(Z_{\bar{\varrho}})} + \left\| \chi^-(A)u \right\|_{\mathrm{H}^k(Z_{\bar{\varrho}}, E)} \\ &\simeq \|u\|_{\mathrm{H}^k(Z_{\bar{\varrho}}, E)}.\end{aligned}$$

Since $\mathrm{C}^\infty(Z_{\bar{\varrho}}, E)$ is dense in $\mathrm{H}^k(Z_{\bar{\varrho}})$, we obtain $S_0 : \mathrm{H}^k(Z_{\bar{\varrho}}, E) \to \mathrm{H}^{k+1}(Z_{\bar{\varrho}}, E)$.

c) Lastly, to prove the assertion that the constant in the bound for S_0 is independent of ϱ, note that this constant is built up of universal constants and the constant arising from the invocation of Theorem 4.101. An important consequence of this theorem is that the constant is independent of ϱ. □

We now want to consider the way in which D_0 maps between Sobolev scales as a Banach space isomorphism. This can only happen by imposing an appropriate boundary condition, both at the real and artificial boundaries $\{0\} \times \partial M$ and $\{\varrho\} \times \partial M$ respectively.

Definition 4.108. Define

$$B_0 := \chi^-(A)\mathrm{H}^{\frac{1}{2}}(\{0\} \times \partial M, E) \times \chi^+(A)H^{\frac{1}{2}}(\{\varrho\} \times \partial M, E)\,,$$
$$\mathrm{H}^k(Z_{\bar{\varrho}}, E; B_0) := \left\{u \in \mathrm{H}^k(Z_{\bar{\varrho}}, E) \;\middle|\; (u(0), u(\varrho)) \in B_0\right\}.$$

Remark 4.109. Let $A_{0,\rho} = A_0 \times A_\varrho$ be the structure of an adapted boundary operator for D_0 on $Z_{\bar{\varrho}}$, where A_0 acts on $0 \times \partial M$ and A_ϱ acts on $\{\varrho\} \times \partial M$. The Czech space also splits as

$$\check{\mathrm{H}}_{A_{0,\rho}}(D_0) = \check{\mathrm{H}}_{A_0}(D_0) \times \check{\mathrm{H}}_{A_\varrho}(D_0)\,.$$

Denote the inward pointing vectorfield along the 'real' boundary $\{0\} \times \partial M$ by τ_0. Since we need an inward pointing vectorfield τ_ρ on the 'fake' boundary also, a reasonable choice would be $-\tau_0$ by first identifying points at the boundary $\{0\} \times \partial M$ and $\{\varrho\} \times \partial M$ in an appropriate manner. More precisely, we can take $\tau_\rho(\rho, x) = -\tau_0(0, x)$.

By choosing the adapted operator on the 'real' boundary to be A, we would then write $A_\varrho = -A$. With this choice, the Czech space would then be

$$\begin{aligned}\check{\mathrm{H}}_{A_{0,\rho}}(D_0) = \chi^-(A)\mathrm{H}^{\frac{1}{2}}(\{0\} \times \partial M, E) \oplus \chi^+(A)\mathrm{H}^{-\frac{1}{2}}(\{0\} \times \partial M, E)&\\ \times \chi^+(A)\mathrm{H}^{\frac{1}{2}}(\{\varrho\} \times \partial M, E) \oplus \chi^-(A)\mathrm{H}^{-\frac{1}{2}}(\{\varrho\} \times \partial M, E)&\,.\end{aligned}$$

From this, we can read of the subspaces of $\check{\mathrm{H}}_{A_{0,\rho}}(D_0)$ which has $\mathrm{H}^{\frac{1}{2}}(\{0,\rho\}\times\partial M, E)$ regularity. Therefore, it is clear the reasons for which B_0 is constructed from $\chi^-(A)\mathrm{H}^{\frac{1}{2}}(\{0\}\times\partial M, E)$ and $\chi^+(A)\mathrm{H}^{\frac{1}{2}}(\{\varrho\}\times\partial M, E)$, rather than $\chi^-(A)\mathrm{H}^{\frac{1}{2}}(\{\varrho\}\times\partial M, E)$ as would be the naïve expectation.

Recalling Notation 4.105, we let $\mathrm{dom}(D_{0,\max}; Z_{\bar{\varrho}}) \subset \mathrm{L}^2(Z_{\bar{\varrho}}, E)$ denote the maximal domain of D_0 in $\mathrm{L}^2(Z_{\bar{\varrho}}, E)$.

Proposition 4.110. *The following hold:*

I) For all $u \in \mathrm{dom}(D_{0,\max}, Z_{\bar{\varrho}})$ with $\chi^-(A)(u(\varrho, \cdot)) = 0$ we have

$$\begin{aligned}\big(I - S_0\sigma_0^{-1}D_0\big)u(t,\cdot) &= (I - S_0(\partial_t + A))u(t,\cdot)\\ &= \mathrm{e}^{-t|A|}\chi^+(A)(u(0))\,.\end{aligned}\tag{4.10}$$

II) For all $k \in \mathbb{N}$,

$$D_0 : \mathrm{H}^{k+1}(Z_{\bar{\varrho}}, E; B_0) \to \mathrm{H}^k(Z_{\bar{\varrho}}, F)$$

is a Banach space isomorphism with inverse $S_0\sigma_0^{-1}$.

Remark 4.111. The assumption $\chi^-(A)(u(\varrho)) = 0$ is always satisfied in application. As we said before, $\{\varrho\} \times \partial M$ is the 'fake' boundary, which arises from restricting our considerations to a finite cylinder. In applications, we localise our problem to near the boundary. More precisely, we institute a cutoff to reduce to the case of u with $\operatorname{spt}(u) \subset [0, \varrho)$. This implies $u(\varrho) = 0$ and in particular that $\chi^-(A)(u(\varrho)) = 0$. Without this assumption, the formula would contain the term

$$e^{-(\varrho - t)|A|}\chi^-(A)(u(\varrho))$$

in 4.10.

Proof. a) Ad I). By linearity of D_0, we can write $D_0 u = D_0 u_+ + D_0 u_-$ with $u_\pm(t) = \chi^\pm(A)u(t)$. Moreover,

$$\sigma_0^{-1} D_0 u_\pm(s) = \partial_s u_\pm(s) \pm |A| \operatorname{sgn}(A) u_\pm(s)$$

since $A = |A| \operatorname{sgn}(A)$. Therefore,

$$\begin{aligned} S_0 \sigma_0^{-1} u_+(t) &= \int_0^t e^{-(t-s)|A|}(\partial_s u_+(s) + |A| u_+(s))\, \mathrm{d}s \\ &= \int_0^t \partial_s\big(e^{-(t-s)|A|}u_+\big)(s)\, \mathrm{d}s \\ &= \big[e^{-(t-s)|A|}u_+(s)\big]_{s=0}^t \\ &= u_+(t) - e^{-t|A|}u_+(0)\,, \end{aligned}$$

where in the third equality, we use a Banach-valued fundamental theorem of calculus that is readily verified. A broad outline of these ideas can be found in the MSc thesis of Kreuter in [34]. The book [15] by Cazenave and Haraux is a standard reference that gives a detailed account. From this calculation, we deduce that

$$\big(I - S_0 \sigma_0^{-1} D_0\big) u_+(t) = e^{-t|A|}\chi^+(A)(u(0))\,.$$

A similar calculation together with $\chi^-(A)u(\varrho) = 0$ yields that

$$\big(I - S_0 \sigma_0^{-1} D_0\big) u_- = 0\,.$$

Together, obtain I).

b) Ad II). If $u \in \mathrm{H}^{k+1}(Z_{\bar\varrho}, E; B_0)$, then $\chi^+(A)u(0) = 0$ and $\chi^-(A)u(\varrho) = 0$, so by I), we have that $u = S_0 \sigma_0^{-1} D_0 u$. Since σ_0 smooth and invertible,

$$D_0 : \mathrm{H}^{k+1}(Z_{\bar\varrho}, E; B_0) \to \mathrm{H}^k(Z_{\bar\varrho}, E)$$

is a bounded isomorphism. □

We are now concerned with proving regularity up to the boundary in the cylinder. For that, we first note the following important theorem.

Theorem 4.112 (Interior elliptic regularity). *Let D be a first-order differential operator on a manifold M with boundary. Let $u \in \operatorname{dom}(D_{\max})$ with $\operatorname{spt} u \subset M \setminus U$, where U is a neighbourhood of the boundary. Then if $Du \in \mathrm{H}^k_{\mathrm{loc}}(M,F)$, $u \in \mathrm{H}^{k+1}_{\mathrm{loc}}(M,E)$*

Theorem 4.113 (Regularity up to boundary in Z).

$$\operatorname{dom}(D_{0,\max}) \cap \mathrm{H}^{k+1}_{\mathrm{loc}}(Z,E) = \Big\{ u \in \operatorname{dom}(D_{0,\max}) \ \Big|\ D_0 u \in \mathrm{H}^k_{\mathrm{loc}}(Z,E) \text{ and } \chi^+(A)\Big(u|_{\partial M}\Big) \in \mathrm{H}^{k+\frac{1}{2}}(\partial M, E) \Big\}.$$

Remark 4.114. Note that the set to the right hand side is precisely

$$\Big\{ u \in \operatorname{dom}(D_{0,\max}) \ \Big|\ D_0 u \in \mathrm{H}^k_{\mathrm{loc}}(Z,E) \text{ and } u|_{\partial M} \in \mathrm{H}^{k+\frac{1}{2}}(\partial M,E) \Big\}.$$

That is, from the theorem, we deduce that $\chi^+(A)\Big(u|_{\partial M}\Big) \in H^{k+\frac{1}{2}}(\partial M,E)$ is automatic.

Proof. a) '$\subset$':

Let $u \in \operatorname{dom}(D_{0,\max}) \cap \mathrm{H}^{k+1}_{\mathrm{loc}}(Z,E)$. Fix some $\varrho < \infty$ and let $\eta \in \mathrm{C}^\infty_c([0,\infty))$ s.t.

$$\eta(t) = \begin{cases} 1 & \text{if } t \in \big[0, \frac{1}{2}\varrho\big), \\ 0 & \text{if } t \in \big[\frac{3}{4}\varrho, \infty\big). \end{cases}$$

Then $u = \eta u + (1-\eta)u$ and $\eta u|_{\partial M} = u|_{\partial M}$. Since $(1-\mu)u|_{\partial M} = 0$, using interior elliptic regularity from Theorem 4.112, we obtain $D_0(1-\eta)u \in \mathrm{H}^k_{\mathrm{loc}}(E)$.

Let $v := \eta u \in \mathrm{H}^k(Z_{\bar\varrho}, E)$ and $u|_{\partial M} = v|_{\partial M} \in \mathrm{H}^{k+\frac{1}{2}}(\partial M,E)$ by hypothesis. In particular, $\chi^+(A)\Big(u|_{\partial M}\Big) \in \mathrm{H}^{k+\frac{1}{2}}(\partial M,E)$. The assertion $D_0 v \in \mathrm{H}^k(Z_{\bar\varrho},E)$ is immediate.

b) '$\supset$':

Let $u \in \operatorname{dom}(D_{0,\max})$ with $D_0 u \in \mathrm{H}^k_{\mathrm{loc}}(Z,E)$ and $\chi^+(A)\Big(u|_{\partial M}\Big) \in \mathrm{H}^{k+\frac{1}{2}}(\partial M,E)$. Using cutoff η from above, we see that $\operatorname{spt}((1-\eta)u) \subset \mathring{Z}$ so in particular, $(1-\eta)u \in \operatorname{dom}(D_{0,\min})$. Since D_0 is a first-order differential operator, we have that

$$D_0(1-\eta)u = D_0 u - \sigma_{D_0}(\cdot, d\eta)u - \eta D_0$$

and since $D_0 u \in \mathrm{H}^k(Z,E)$, we have that $D_0(1-\eta u) \in \mathrm{H}^k_{\mathrm{loc}}(Z,E)$. By ellipticity of D_0, using interior elliptic regularity as before from Theorem 4.112, we obtain $(1-\eta)u \in \mathrm{H}^{k+1}_{\mathrm{loc}}(Z,E)$.

Now, let us consider the term which contains the boundary information. For that, set $v := \eta u$. It is easy to see that

$$v(0) = u|_{\partial M} \qquad \text{and} \qquad v(\varrho) = 0\,.$$

Using Proposition 4.110 (4.10),

$$v(t) = S_0 \sigma_0^{-1} D_0 v(t) + e^{-t|A|}\big(\chi^+(A)v(0)\big) =: v_0(t) + v_1(t)\,.$$

From Lemma 4.107, we have that $S_0 : \mathrm{H}^k(Z_{\bar\varrho}, E) \to \mathrm{H}^{k+1}(Z_{\bar\varrho}, E)$. Since $D_0 u \in \mathrm{H}^k_{\mathrm{loc}}(Z, E)$, by the compactness of $Z_{\bar\varrho}$ we obtain that $D_0 v \in \mathrm{H}^k(Z_{\bar\varrho}, E)$. Moreover, $\sigma_0^{-1} : \mathrm{H}^k(Z_{\bar\varrho}, E) \to \mathrm{H}^k(Z_{\bar\varrho}, E)$ so we get $v_0 \in \mathrm{H}^{k+1}(Z_{\bar\varrho}, E)$.

Fix $l \in [0, k+1]$. Then,

$$\begin{aligned}
\left\|\partial_t^l |A|^{k+1-l} v_1\right\|^2_{\mathrm{L}^2(Z_{\bar\varrho},E)} &= \left\||A|^{k+1-l}\partial_t v_1\right\|^2_{\mathrm{L}^2(Z_{\bar\varrho},E)} \\
&= \left\||A|^{k+1-l}\partial_r^l \mathrm{e}^{-t|A|}\big(\chi^+(A)v(0)\big)\right\|^2_{\mathrm{L}^2(Z_{\bar\varrho},E)} \\
&= \left\||A|^{k+1-l}|A|^l \mathrm{e}^{-t|A|}\big(\chi^+(A)v(0)\big)\right\|^2_{\mathrm{L}^2(Z_{\bar\varrho},E)} \\
&= \left\||A|^{k+1} \mathrm{e}^{-t|A|}\big(\chi^+(A)v(0)\big)\right\|^2_{\mathrm{L}^2(Z_{\bar\varrho},E)} \\
&= \int_0^\varrho \left\||A|^{k+1} \mathrm{e}^{-t|A|}\big(\chi^+(A)v(0)\big)\right\|^2_{\mathrm{L}^2(\partial M,E)} \,\mathrm{d}t \\
&= \int_0^\varrho \left\|t^{\frac12}|A|^{\frac12} \mathrm{e}^{-t|A|}\Big(|A|^{k+\frac12}\chi^+(A)v(0)\Big)\right\|^2_{\mathrm{L}^2(\partial M,E)} \frac{\mathrm{d}t}{t} \\
&\lesssim \left\||A|^{k+\frac12}\chi^+(A)v(0)\right\|^2_{\mathrm{L}^2(\partial M,E)} \\
&\simeq \left\|\chi^+(A)v(0)\right\|^2_{\mathrm{H}^{k+\frac12}(\partial M)}\,,
\end{aligned}$$

where the third equality follows from the repeated differentiation of the semigroup using the commutativity of ∂_t and $|A|$, and the penultimate inequality from the H^∞ functional calculus via Theorem 4.82.

This shows that $v_1 \in \bigcap_{l=0}^{k+1} \mathrm{dom}\Big(\partial_t^l |A|^{k+1-l}; [0,\varrho]\Big) = \mathrm{H}^{k+1}(Z_{\bar\varrho}, E)$. Since $v = v_0 + v_1$ and we have already established that $v_0 \in\in \mathrm{H}^{k+1}(Z_{\bar\varrho}, E)$, we obtain that $v \in \mathrm{H}^{k+1}(Z, E)$. Also, we have that $(1-\eta) \in \mathrm{H}^{k+1}_{\mathrm{loc}}(Z, E)$ and therefore,

$$u = (1-\eta)u + \eta u = (1-\eta)u + v \in \mathrm{H}^{k+1}_{\mathrm{loc}}(Z, E)\,. \qquad \square$$

4.11 The general problem

Let us now return back to the original problem. That is, we consider $D : \mathrm{C}^\infty(M, E) \to \mathrm{C}^\infty(M, F)$ an elliptic first-order differential operator, acting between two hermitian bundles (E, h^E) and (F, h^F) with (M, μ) a measure manifold with compact boundary equipped with a interior pointing vectorfield T along the boundary ∂M. As before, we emphasise that only ∂M is assumed to be compact.

Let $R > 0$ be the constant in the operator reduction Lemma 4.6. Then, we know that for a fixed boundary adapted operator A, which we assume to be invertible

ω-bisectorial, there is an open set U containing ∂M with U diffeomorphic to Z_R for which $D = \sigma_t(\partial_t + A + R_t)$ with R_t a smoothly varying family of first-order differential operators on ∂M of at most order one one for which, given $R' \in (0, R)$, there exists a constant $C' < \infty$ such that

$$\|R_t u\|_{\mathrm{L}^2(\partial M, E)} \leq C'\Big(t\|Au\|_{\mathrm{L}^2(\partial M,E)} + \|u\|_{\mathrm{L}^2(\partial M,E)}\Big)$$

whenever $u \in \mathrm{C}^\infty(\partial M, E)$.

This will be a key ingredient allowing us to reduce boundary value problems for D to D_0. Therefore, from here on, let us fix

$$R' = \frac{R}{2}\,.$$

Let us first highlight the following corollary of the previous theorem.

Corollary 4.115. *There is a constant $c_1 < \infty$ such that whenever $\varrho < R'$ and $u \in \mathrm{C}^\infty_{\mathrm{c}}(Z_{\overline{\varrho}}, E)$ with $\chi^+(A)u(0) = 0$ and* $\mathrm{spt}(u) \subset Z_\varrho = [0, \varrho) \times \partial M$*,*

$$\|u\|^2_{\mathrm{H}^1\left(Z_{\overline{R'}}, E\right)} \leq c_1^2\Big(\|(\partial_t + A)u\|^2_{\mathrm{L}^2\left(Z_{\overline{\varrho}}, E\right)} + \|u\|^2_{\mathrm{L}^2\left(Z_{\overline{\varrho}}, E\right)}\Big)\,.$$

Proof. Under the hypothesis of the corollary, we have that $\chi^+(A)(u(0)) = 0$ and $\chi^-(A)(u(\varrho)) = 0$. Therefore, on applying Proposition 4.110 II), with $\rho = \varrho$, we obtain $D_0 : \mathrm{H}^1(Z_{\overline{\varrho}}, E; B_0) \to \mathrm{L}^2(Z_{\overline{\varrho}}, F)$ is an Banach space isomorphism. The estimate in the conclusion is precisely a quantification of this fact. □

Exercise 4.116. Assuming A is self-adjoint, calculate this directly without utilising Proposition 4.110.

This immediately gives us the following regularity for the minimal operator.

Corollary 4.117. *We have* $\mathrm{dom}(D_{0,\min}) \subset \Big\{u \in \mathrm{H}^1_{\mathrm{loc}}(M, E) \;\Big|\; u|_{\partial M} = 0\Big\}$.

In order to reduce questions regarding D to D_0, we need to know that the closures of D and D_0 in a sufficiently small neighbourhood of the boundary yield the same domains. This ensures that on multiplying a section in $\mathrm{dom}(D_{\max})$ by an appropriate cutoff, we are guaranteed to be in a sufficiently 'large' closure of D_0 which is respectively contained in $\mathrm{dom}(D_{0,\max})$. For this, we need to use the following abstract result.

Lemma 4.118. *Let T, S be closable operators on a Banach space $\mathcal{B}$ with* $\mathrm{dom}(S) \subset \mathrm{dom}(T)$*. Suppose there exist constants $c \in [0,1)$ and $C < \infty$ satisfying*

$$\|Tu\| \leq c\|Su\| + C\|u\| .$$

Then

$$\mathrm{dom}\big(\bar{S}\big) = \mathrm{dom}\big(\overline{T+S}\big) \qquad \textit{and} \qquad \|u\|_S \simeq \|u\|_{T+S} .$$

This lemma suggests that we should look for such a bound for our choice of operators D_0 and D.

Lemma 4.119. *There is a constant $c_2 > 0$ s.t. for all $\varrho \in (0, R')$ and $u \in$* $\mathrm{C}^\infty(Z_\varrho, E)$ *with* $\mathrm{spt}(u) \subset [0, \varrho) \times \partial M$ *we have*

$$\|D - D_0\|_{\mathrm{L}^2(Z_{\bar{\varrho}})} \leq c_2\Big(\varrho\|D_0 u\|_{\mathrm{L}(Z_{\bar{\varrho}})} + \|u\|_{\mathrm{L}^2(Z_{\bar{\varrho}})}\Big) .$$

Proof. Recall that, using Lemma 4.6, (4.2) yields

$$D - D_0 = (\sigma_t - \sigma_u)\sigma_0^{-1} D_0 + \sigma_t R_t . \tag{4.11}$$

Since $t \mapsto \sigma_t$ is smooth, there is a $\tilde{c}_2$ such that whenever $t \leq R'$ we have the Lipschitz estimate

$$|(\sigma_t - \sigma_0)v| \leq \tilde{c}_2 t|v| .$$

In particular, if we assume $t \leq \varrho$, then

$$|(\sigma_t - \sigma_0)v| \leq \tilde{c}_2\varrho|v| .$$

The first term in (4.11) is then estimated as:

$$\big\|(\sigma_t - \sigma_0)\sigma_0^{-1} D_0 u\big\|_{\mathrm{L}^2(Z_{\bar{\varrho}},E)} \leq \tilde{c}_2\varrho\big\|\sigma_0^{-1} D_0 u\big\|_{\mathrm{L}^2(Z_{\bar{\varrho}},E)} \leq \tilde{\tilde{c}}_2\varrho\|D_0 u\|_{\mathrm{L}^2(Z_{\bar{\varrho}},E)} ,$$

where the constant $\tilde{\tilde{c}}_2$ contains the norm of σ_0^{-1}. Now consider consider the second term in (4.11). We estimate this as

$$\begin{aligned}
\|\sigma_t R_t u\|^2_{\mathrm{L}^2(Z_{\bar{\varrho}},E)} &\leq c_3^2 \int_0^\varrho \|R_t u(t)\|^2_{\mathrm{L}(\partial M,E)}\, \mathrm{d}t \\
&\leq c_4^2 \int_0^\varrho t^2\|Au(t)\|^2_{\mathrm{L}^2(\partial M,E)}\, \mathrm{d}t + c_4\|u\|^2_{\mathrm{L}^2(Z_{\bar{\varrho}},E)} \\
&= c_4^2 \int_0^\varrho \|A(tu(t))\|^2_{\mathrm{L}^2(\partial M,E)}\, \mathrm{d}t + c_4\|u\|^2_{\mathrm{L}^2(Z_{\bar{\varrho}})} ,
\end{aligned}$$

where the second inequality follows from Lemma 4.6 and the constant c_4 contains the constant arising from the invocation of Lemma 4.6 using R' as we defined previously. In particular, this constant is independent of ϱ. Let

$$v(t) := tu(t) .$$

Then $\operatorname{spt}(v) \subset [0,\varrho) \times \partial M$ and so in particular, $v(0) = 0$ which yields $\chi^+(A)(v(0)) = 0$. Therefore,

$$\int_0^{\varrho} \Big(\|\partial_t v(t)\|^2_{\mathrm{L}(\partial M,E)} + \|Av(t)\|^2_{\mathrm{L}^2(\partial M,E)} \Big) \, \mathrm{d}t \le c_0^2 \|v\|^2_{\mathrm{H}^1(Z_{\bar\varrho},E)}$$
$$\le c_1^2 c_0^2 \Big(\|(\partial_t + A)v(t)\|^2_{\mathrm{L}^2(Z_{\bar\varrho},E)} + \|u\|^2_{\mathrm{L}^2(Z_{\bar\varrho},E)} \Big) \,,$$

where the first inequality follows from Lemma 4.106 and the second inequality from Corollary 4.115. Moreover,

$$(\partial_t + A)v = (\partial_t + A)(tu) = u + t(\partial_t + A)u$$

and by using this, we get

$$\int_0^{\infty} \|(\partial_t + A)v\|^2_{\mathrm{L}^2(\partial M,E)} \, \mathrm{d}t = \int_0^{\varrho} \|u + t(\partial_t + A)u\|^2_{\mathrm{L}^2(\partial M,E)} \, \mathrm{d}t$$
$$\le 2\Big(\|u\|^2_{\mathrm{L}^2(Z_{\bar\varrho},E)} + \varrho^2 \|(\partial_t + A)u\|^2_{\mathrm{L}^2(Z_{\bar\varrho},E)} \Big)$$
$$\le c_5^2 \varrho^2 \|D_0 u\|^2 + 2\|u\|^2_{\mathrm{L}^2(Z_{\bar\varrho},E)} \,.$$

On combining these estimates, we obtain

$$\|\sigma_t R_t u\|^2_{\mathrm{L}^2(Z_{\bar\varrho},E)} \le (c_0 c_1 c_4 c_5)^2 \varrho^2 \|D_0 u\|^2_{\mathrm{L}^2(Z_{\bar\varrho},E)} + (c_0^2 c_1^2 + c_4^2 + 2)\|u\|^2_{\mathrm{L}^2(Z_{\bar\varrho},E)} \,.$$

Therefore, we can choose c_2 and C_2 appropriately to obtain

$$\|(D - D_0)u\| \le c_2 \varrho \|D_0 u\|_{\mathrm{L}^2(Z_{\bar\varrho})} + C_2 \|u\|_{\mathrm{L}^2(Z_{\bar\varrho})} \,. \qquad \square$$

Let D_{cl} and $D_{0,\mathrm{cl}}$ be the closures of D and D_0 respectively with domain

$$\mathcal{D} := \{ u \in \mathrm{C}^\infty(Z_{\bar\varrho}, E) \mid \operatorname{spt}(u) \subset [0,\varrho) \times \partial M \} \,.$$

Corollary 4.120. *Whenever $\varrho < R'$ further satisfies $\varrho < \frac{1}{c_2}$, we have that*

$$\operatorname{dom}(D_{\mathrm{cl}}) = \operatorname{dom}(D_{0,\mathrm{cl}}) \qquad \textit{and} \qquad \|u\|_{D_{\mathrm{cl}}} \simeq \|u\|_{D_{0,\mathrm{cl}}} \,.$$

Proof. For this choice of ϱ, we have from the Lemma 4.119 that

$$\|(D - D_0)u\|_{\mathrm{L}^2(Z_{\bar\varrho})} \le b\|D_0 u\| + C_2\|u\| \,, i$$

where $b := c_2 \varrho < c_2 \frac{1}{c_2} < 1$.

Let $T := D - D_0$ and $S := D_0$ with domain

$$\mathcal{D} = \{ u \in \mathrm{C}^\infty(Z_{\bar\varrho}, E) \mid \operatorname{spt}(u) \subset [0,\varrho) \times \partial M \} \,.$$

It is clear that

$$\bar{S} = D_{0,\mathrm{cl}} \qquad \text{and} \qquad \overline{T + S} = \overline{(D - D_0 + D_0)|_{\mathcal{D}}} = D_{\mathrm{cl}} \,. \qquad \square$$

The conclusion then follows from invoking Lemma 4.118 with this choice of T and S.

In order to continue, to consider the general case, we need to make the following additional assumption.

[FO6] D and $D^\dagger$ are *complete*.

That is, $\mathrm{dom}_c(D) := \{u \in \mathrm{dom}(D_{\max}) \mid \mathrm{spt}(u) \text{ compact}\}$ and $\mathrm{dom}_c\big(D^\dagger\big) := \big\{v \in \mathrm{dom}\big(D^\dagger_{\max}\big) \bigm| \mathrm{spt}(v) \text{ compact}\big\}$ are dense in $\mathrm{dom}(D_{\max})$ and $\mathrm{dom}\big(D^\dagger_{\max}\big)$ respectively.

Exercise 4.121. If D is complete, then $\mathrm{C}^\infty_c(M, E)$ is dense in $\mathrm{dom}(D_{\max})$.

Theorem 4.122. *Let D be complete and A an ω-bisectorial invertible adapted boundary operator for D. Then*

$$\check{\mathrm{H}}(D) = \check{\mathrm{H}}_A(D_0)\,.$$

Proof. Choose $\varrho > 0$ small such that it satisfies the hypothesis of Corollary 4.120 and let $U_\varrho = \Psi^{-1}([0,\rho) \times \partial M)$ where Ψ is the diffeomorphism from Lemma 4.2. This choice of ϱ is reasonable since in the hypothesis of Corollary 4.120, we automatically assume that $\varrho < R' < R$, where the R is the constant in Lemma 4.2. Let $\eta \in \mathrm{C}^0_c(M, [0,1])$ with $\mathrm{spt}(\eta) \subset U_\varrho$.

Now, fix $u \in \mathrm{C}^\infty(M, E)$. Then, we write

$$u = (1-\eta)u + \eta u =: u_0 + u_1\,.$$

Clearly, $u_i \in \mathrm{dom}(D_{\max})$ and furthermore, $u_0|_{\partial M} = 0$. Therefore, $u_0 \in \mathrm{dom}(D_{\min})$.

To study u_1, we identify U_ϱ and Z_ϱ. Since we assume that D is complete, there exists a sequence of $u_1^j \in \mathrm{C}^\infty_c(M, E)$ with $u_1^j \to u_1$ in $\mathrm{dom}(D_{\max})$. It is easy to verify that $\eta u_j^1 \to u_1$ in $\mathrm{dom}(D_{\max})$ by the choice of η. Since $\eta u_j^1 \in \mathcal{D}$ we have that $u_1 \in \mathrm{dom}(D_{\mathrm{cl}})$. Therefore,

$$\eta u = u_1 \in \mathrm{dom}(D_{\mathrm{cl}}) = \mathrm{dom}(D_{0,\mathrm{cl}}) \subset \mathrm{dom}(D_{0,\max})\,. \tag{4.12}$$

By invoking Theorem 4.91, we find that $u_1|_{\partial M} \in \check{\mathrm{H}}_A(D_0)$. Moreover,

$$u|_{\partial M} = (\eta u)|_{\partial M} = u_1|_{\partial M}\,,$$

and therefore,

$$\begin{aligned}\Big\|u|_{\partial M}\Big\| &\lesssim \|\eta u\|_{D_{0,\max}} = \|\eta u\|_{D_{0,\mathrm{cl}}} \simeq \|\eta u\|_{D_{\mathrm{cl}}} \\ &\le \|u\|_{\mathrm{L}^2(Z_\varrho,E)} + \|\sigma_D(\cdot, d\eta)u\|_{\mathrm{L}^2(Z_\varrho,E)} + \|D_{\max}u\| \lesssim \|u\|_{D_{\max}}\,,\end{aligned}$$

where the first estimate follows from Theorem 4.91, the third from (4.12) together with Corollary 4.120, and the penultimate inequality from the product rule. This shows that $u \mapsto u|_{\partial M} : \mathrm{dom}(D_{\max}) \to \check{\mathrm{H}}_A(D_0)$ is bounded.

It remains to show that this $u \mapsto u|_{\partial M} : \mathrm{dom}(D_{\max}) \to \check{\mathrm{H}}_A(D_0)$ is a surjection. For that, define the extension operator $\mathcal{E}_\varrho : \check{\mathrm{H}}_A(D_0) \to \mathrm{dom}(D_{0,\max})$ defined by

$$\mathcal{E}_\varrho v = \eta \mathcal{E} v\,.$$

Note, however, that $\operatorname{spt} \mathcal{E}_\varrho v \subset Z_\varrho = [0,\varrho) \times \partial M$. Therefore,

$$\mathcal{E}_\varrho v \in \operatorname{dom}(D_{0,\mathrm{cl}}) = \operatorname{dom}(D_{\mathrm{cl}}) \subset \operatorname{dom}(D_{\max})$$

with the equality following from Corollary 4.120. Furthermore, we also obtain

$$\|\mathcal{E}_\varrho v\|_{D_{\max}} = \|\mathcal{E}_\varrho v\|_{D_{\mathrm{cl}}} \simeq \|\mathcal{E}_\varrho v\|_{D_{0,\mathrm{cl}}} \lesssim \|v\|_{\check{\mathrm{H}}_A(D_0)},$$

Therefore $\mathcal{E}_\varrho : \check{\mathrm{H}}_A(D_0) \to \operatorname{dom}(D_{\max})$ is a bounded map. Together with previous estimate we have that $u \mapsto u|_{\partial M} : \operatorname{dom}(D_{\max}) \to \check{\mathrm{H}}_A(D_0)$ is a bounded surjection.

This shows that $\check{\mathrm{H}}(D) = \check{\mathrm{H}}_A(D_0)$ with comparable norms. □

Remark 4.123. Let us recall our desire [Req 3]:

Describe $\check{\mathrm{H}}(D)$, and in particular its topology, from data only living on ∂M.

It is this theorem that allows us to actualise this desire, to the extent that it is realistically possible. This theorem tells us that, by fixing an invertible bisectorial adapted boundary operator A, we are able to describe and understand $\check{\mathrm{H}}(D)$ purely in terms of spectral information of A. Indeed, this operator is not only determined by boundary information, but it has only a nodding acquaintance with the operator D. It only relies upon the principal symbol of D and therefore, there are many possible choices for A. This freedom allows us to choose A as suited to the particular question which we are attempting to answer.

Proposition 4.124. *Let $u \in \operatorname{dom}(D_{\max})$ and $v \in \operatorname{dom}\left(D^\dagger_{\max}\right)$. Then, given any adapted boundary operator A that is ω-bisectorial and invertible,*

$$\langle D_{\max} u, v\rangle_{\mathrm{L}^2(M,F)} - \left\langle u, D^\dagger_{\max} v\right\rangle_{\mathrm{L}^2(M,E)} = -\left\langle u|_{\partial M}, \sigma_0^* v|_{\partial M}\right\rangle_{\check{\mathrm{H}}_A(D_0)\times\hat{\mathrm{H}}_A(D_0)}.$$

Proof. Since we assume that D and $D^\dagger$ are complete, it suffices to consider $u \in \mathrm{C}^\infty_{\mathrm{c}}(M,E)$ and $v \in \mathrm{C}^\infty_{\mathrm{c}}(M,F)$. Then, by Green's formula from Proposition 4.9, we have that

$$\langle Du, v\rangle - \left\langle u, D^\dagger v\right\rangle = -\left\langle u|_{\partial M}, \sigma_0^* v|_{\partial M}\right\rangle_{\mathrm{L}^2(\partial M,E)}.$$

Choose η as in the proof of Theorem 4.122. Then,

$$\begin{aligned} \eta u &\in \operatorname{dom}(D_{\mathrm{cl}}) = \operatorname{dom}(D_{0,\mathrm{cl}}) \subset \operatorname{dom}(D_{0,\max}), \\ \eta v &\in \operatorname{dom}\left(D^\dagger_{\mathrm{cl}}\right) = \operatorname{dom}\left(D^\dagger_{0,\mathrm{cl}}\right) \subset \operatorname{dom}\left(D^\dagger_{0,\max}\right). \end{aligned}$$

Moreover $\eta u|_{\partial M} = u|_{\partial M}$ and $\eta v|_{\partial M} = v|_{\partial M}$, so by Proposition 4.98,

$$\langle u|_{\partial M}, \sigma_0^* v|_{\partial M}\rangle_{\mathrm{L}^2(\partial M,E)} = \langle u|_{\partial M}, \sigma_0^* v|_{\partial M}\rangle_{\check{\mathrm{H}}_A(D_0)\times\hat{\mathrm{H}}_A(D_0)}.$$ □

Corollary 4.125. *If B is a boundary condition for D_B, then the adjoint operator satisfies $D_B^* = D^\dagger_{B^\dagger}$, where*

$$B^\dagger := \left\{ v|_{\partial M} \;\middle|\; v \in \operatorname{dom}(D_B^*) \right\} = (\sigma_0^*)^{-1} B^{\perp, \langle \hat{\mathrm{H}}_A(D_0), \check{\mathrm{H}}_A(D_0) \rangle} \,.$$

Proof. This follows immediately on combining Proposition 4.124 and Proposition 4.99. □

4.12 Regularity up to the boundary

Recall again the R arising from the geometric reduction lemma, Lemma 4.2. That is, U_R is diffeomorphic to $[0,R) \times \partial M$ satisfying the geometric properties in the conclusion of this lemma. Further fixing $R' < R$ from the operator reduction lemma, Lemma 4.6, say $R' := \frac{R}{2}$, we obtain the following analogous to Lemma 4.119.

Lemma 4.126. *For $k \in \mathbb{N}$, there exists $\varrho < R'$ such that*

$$(D - \sigma_0 R_0) : \mathrm{H}^{k+1}(Z_{\bar\varrho}, E; B_0) \to \mathrm{H}^k(Z_{\bar\varrho}, F)$$

is an isomorphism, where

$$B_0 := \chi^-(A)\mathrm{H}^{\frac12}(\{0\} \times \partial M, E) \times \chi^+(A)\mathrm{H}^{\frac12}(\{\varrho\} \times \partial M, E) \,.$$

With this, we can now prove the following, generalising Theorem 4.113.

Theorem 4.127. *We have that*

$$\operatorname{dom}(D_{\max}) \cap \mathrm{H}^{k+1}_{\mathrm{loc}}(M,E) = \Big\{ u \in \operatorname{dom}(D_{\max}) \;\Big|\; Du \in \mathrm{H}^k_{\mathrm{loc}}(M,E) \text{ and } \chi^+(A)\Big(u|_{\partial M}\Big) \in \mathrm{H}^{k+\frac12}(\partial M, E) \Big\} \,.$$

Proof. The direction '$\subset$' is clear.

We prove '$\supset$'. Fix $k \in \mathbb{N}$ and let ϱ be given from Lemma 4.126. Let $\eta \in \mathrm{C}^\infty_{\mathrm{c}}(M,[0,1])$ such that $\operatorname{spt}(\eta) \subset U_\varrho \cong [0,\varrho) \times \partial M$ and $\eta = 1$ in $\Psi^{-1}\big([0,\frac{\varrho}{2}) \times \partial M\big)$ and $\eta = 0$ outside $\Psi^{-1}\big([0,\frac{3\varrho}{4}) \times \partial M\big)$. Let $u \in \operatorname{dom}(D_{\max})$ with $Du \in \mathrm{H}^k_{\mathrm{loc}}(M,E)$ and $\chi^+(A)\Big(u|_{\partial M}\Big) \in \mathrm{H}^{k+\frac12}(\partial M, E)$. Without loss of generality, by induction, we can further suppose that $u \in \mathrm{H}^k_{\mathrm{loc}}(M,E)$.

Since $\operatorname{spt}(1-\eta)u \subset M \setminus U_\varrho$, by interior elliptic regularity, i.e. Theorem 4.112, we obtain that $(1-\eta)u \in \mathrm{H}^{k+1}_{\mathrm{loc}}(M,E)$.

It remains to consider $v := \eta u$. Note that $v(0) = (\eta u)(0) = u|_{\partial M}$. Therefore,

$$Dv = \sigma_D(\pi_{T^*M}(d\eta), d\eta)u + \eta Du \in \mathrm{H}^k(Z_\varrho, E)$$

and by hypothesis,

$$\chi^+(A)(v(0)) = \chi^+(A)\Big(u|_{\partial M}\Big) \in \mathrm{H}^{k+\frac{1}{2}}(\partial M, E)\,.$$

From construction, we have that $v \in \mathrm{dom}(D_{\max})$ and $\mathrm{spt}(v) \subset Z_\varrho$. Therefore, $v \in \mathrm{dom}(D_{\mathrm{cl}}) = \mathrm{dom}(D_{0,\mathrm{cl}}) \subset \mathrm{dom}(D_{0,\max})$ and using Proposition 4.110 I), we write

$$v(t) = S_0\sigma_0^{-1}D_0v(t) + \mathrm{e}^{-t|A|}\big(\chi^+(A)v(0)\big) =: v_0(t) + v_1(t)\,. \tag{4.13}$$

Since $\chi^+(A)(v(0)) \in \mathrm{H}^{k+\frac{1}{2}}(\partial M, E)$, by the exact same calculation in the proof of Theorem 4.113, we obtain that $v_1 \in \mathrm{H}^{k+1}(Z_\varrho, E)$. In fact, from the hypothesis, we also know regularity information about v. It is the regularity of v_0 that is unknown to us.

We show that $v_0 \in \mathrm{H}^{k+1}(Z_\varrho, E)$. First

$$(D - \sigma_0 R_0)v_0 = (D - \sigma_0 R_0)v - (D - \sigma_0 R_0)v_1\,.$$

Using that $\sigma_0 R_0 : \mathrm{H}^l(Z_\varrho, E) \to \mathrm{H}^l(Z_\varrho, F)$ boundedly for each l, $Du \in \mathrm{H}^k_{\mathrm{loc}}(M, E)$ implies $Dv \in \mathrm{H}^k(Z_\varrho, E)$ and $u \in \mathrm{H}^k_{\mathrm{loc}}(M, E)$ implies $v \in \mathrm{H}^k(Z_\varrho, F)$, we obtain $(D - \sigma_0 R_0)v \in \mathrm{H}^k(Z_\varrho, F)$. Therefore, $(D - \sigma_0 R_0)v_0 \in \mathrm{H}^k(Z_\varrho, F)$. Now, note that $\chi^-(A)(v_0(\varrho)) = 0$ and by Lemma 4.126, we have that $(D - \sigma_0 R_0) : \mathrm{H}^{k+1}(Z_\varrho, E; B_0) \to \mathrm{H}^k(Z_\varrho, F)$ isomorphically. Therefore, $v_0 \in \mathrm{H}^{k+1}(Z_\varrho, E)$.

By (4.13), since $v_1 \in \mathrm{H}^{k+1}(Z_\varrho, E)$, we obtain that $v \in \mathrm{H}^{k+1}(Z_\varrho, E)$. Since $\mathrm{spt}\, v \subset Z_\varrho$, we have that $v \in \mathrm{H}^{k+1}_{\mathrm{loc}}(M, E)$. The conclusion then follows. $\square$

5 The Atiyah-Patodi-Singer Index Theorem

The purpose of this chapter is to give a description of the celebrated index theorem of Atiyah, Patodi and Singer. It first appeared in the the series of seminal papers [3, 4, 5, 6] by these three authors.

Throughout this chapter, as in the last, $D \in \mathrm{Diff}_1(E, F)$ elliptic, where $(E, h^E) \to M$ and $(F, h^F) \to M$ are hermitian bundles.

5.1 Index and elliptically regular boundary conditions

Let us recall the notion of an elliptically regular boundary condition from Definition 3.75. Applying this to the case when $m = 1$, we obtain that a boundary condition $B \subset \check{\mathrm{H}}(D)$ is elliptically regular if

$$B \subset \mathrm{H}^{\frac{1}{2}}(\partial M, E) \qquad \text{and} \qquad B^\dagger \subset \mathrm{H}^{\frac{1}{2}}(\partial M, F)\,.$$

Proposition 5.1. *Let A be an adapted boundary operator that is invertible bisectorial. Then the following are equivalent.*

(I) B is an elliptically regular boundary condition.

(II) B is a boundary condition, $B \subset \mathrm{H}^{\frac{1}{2}}(\partial M, E)$ and $B^{\perp,\langle \hat{\mathrm{H}}_A(D_0), \check{\mathrm{H}}_A(D_0)\rangle} \subset \mathrm{H}^{\frac{1}{2}}(\partial M, F)$.

(III) $\mathrm{dom}(D_B) \subset \mathrm{H}^1_{\mathrm{loc}}(M, E)$ and $\mathrm{dom}\left(D^\dagger_{B^\dagger}\right) = \mathrm{dom}(D^*_B) \subset \mathrm{H}^1_{\mathrm{loc}}(M, F)$.

Proof. These equivalences are easily verified. Note that for (II), we use

$$B^\dagger = (\sigma_0^*)^{-1} B^{\perp,\langle \hat{\mathrm{H}}_A(D_0), \check{\mathrm{H}}_A(D_0)\rangle}$$

from Proposition 4.99 and Theorem 4.122, along with the fact that $(\sigma_0^*)^{-1} : \mathrm{H}^\ell(\partial M, E) \to \mathrm{H}^\ell(\partial M, F)$ is bounded. For (III) simply use previous Theorem 4.127. □

Before we consider the Fredholmness properties of such boundary conditions, let us first present some abstract facts. Recall that a closed operator $T : \mathcal{B}_1 \to \mathcal{B}_2$ is said to be Fredholm if $\mathrm{ran}(T)$ is closed and $\ker T$ and $\operatorname{coker} T := \mathcal{B}_2 / \mathrm{ran}\, T$ are both finite dimensional.

Definition 5.2 (Analytical index). The *analytical index* of a Fredholm operator $T : \mathcal{B}_1 \to \mathcal{B}_2$ is defined as

$$\mathrm{ind}(D_B) := \dim(\ker(D_B)) - \dim(\operatorname{coker}(D_B)) \in \mathbb{Z}\,.$$

Remark 5.3. By the closed range theorem (c.f. Chapter 4, Section 5 in [52]), we have that $\mathrm{ran}(T^*)$ is automatically closed when $\mathrm{ran}(T)$ is closed.

Lemma 5.4. *Let $T : \mathrm{H} \to \mathrm{H}$ be a densely-defined closed operator on a Hilbert space with closed range. Then $\operatorname{coker}(T) \cong \ker(T^*)$. In particular, if T is Fredholm,*

$$\mathrm{ind}(T) = \dim(\ker(T)) - \dim(\ker(T^*))\,.$$

Proof. From Proposition 4.16 III), we have that $\mathcal{H} = \ker(T^*) \oplus \mathrm{ran}(T)$ since $\mathrm{ran}(T)$ is assumed to be closed. Then, clearly

$$\operatorname{coker}(T) = \mathcal{H} / \mathrm{ran}(T) \cong \ker(T^*)\,.$$

□

The following is an important result due to Hörmander, which can be found as Proposition 19.1.3 in [31].

Lemma 5.5. *Suppose we have Banach spaces $\mathcal{B}_1, \mathcal{B}_2$ and $L : \mathcal{B}_1 \to \mathcal{B}_2$ a bounded operator. Then, the following are equivalent:*

I) There exists a Banach space $\mathcal{B}_3$, a compact map $K : \mathcal{B}_1 \to \mathcal{B}_3$, and a constant $c < \infty$ such that

$$\|u\|_{\mathcal{B}_1} \le c\big(\|Lu\|_{\mathcal{B}_2} + \|Ku\|_{\mathcal{B}_3}\big)\,.$$

II) $\ker(L)$ is finite dimensional and $\mathrm{ran}(L)$ is closed.

With the aid of this lemma, we can assert the following cornerstone result for our considerations in this chapter.

Proposition 5.6. *Suppose M is compact and B is an elliptically regular boundary condition for D. Then D_B is Fredholm.*

Proof. We apply Lemma 5.5 on setting $\mathcal{B}_1 := \mathrm{dom}(D_B)$, $\mathcal{B}_2 := \mathcal{B}_3 := \mathrm{L}^2(M,E)$, $L := D_B$ and $K := \mathrm{id}$. By Proposition 5.1 (III), we have $\mathrm{dom}(D_B) \subset \mathrm{H}^1_{\mathrm{loc}}(M,E) = \mathrm{H}^1(M,E)$, where the second equality follows since M is compact. Now

$$\|u\|_{\mathcal{B}_1} = \|u\|_{D_B} \simeq \|D_B u\|_{\mathrm{L}^2(M,E)} + \|u\|_{\mathrm{L}^2(M,E)}\,,$$

so in particular,

$$\|u\|_{\mathcal{B}_1} \lesssim \|D_B u\|_{\mathcal{B}_2} + \|u\|_{\mathcal{B}_3}.$$

Again, due to the compactness of M, the inclusion map $\mathrm{H}^1(M, E) \hookrightarrow \mathrm{L}^2(M, E)$ is compact. Therefore, $K = \mathrm{id} : \mathrm{dom}(D_B) \subset \mathrm{H}^1(M, E) \to \mathrm{L}^2(M, E)$ is compact. Invoking Lemma 5.5, we conclude that D_B has finite dimensional kernel and closed range.

Since B is elliptically regular, Proposition 5.1 (III) also guarantees us that $\mathrm{dom}(D_B^*) \subset \mathrm{H}^1(M, F)$. Therefore, we can repeat the same argument for $D^\dagger_{B^\dagger}$ in place of D_B to obtain that $\ker\left(D^\dagger_{B^\dagger}\right)$ is finite dimensional and $\mathrm{ran}\left(D^\dagger_{B^\dagger}\right)$ is closed.

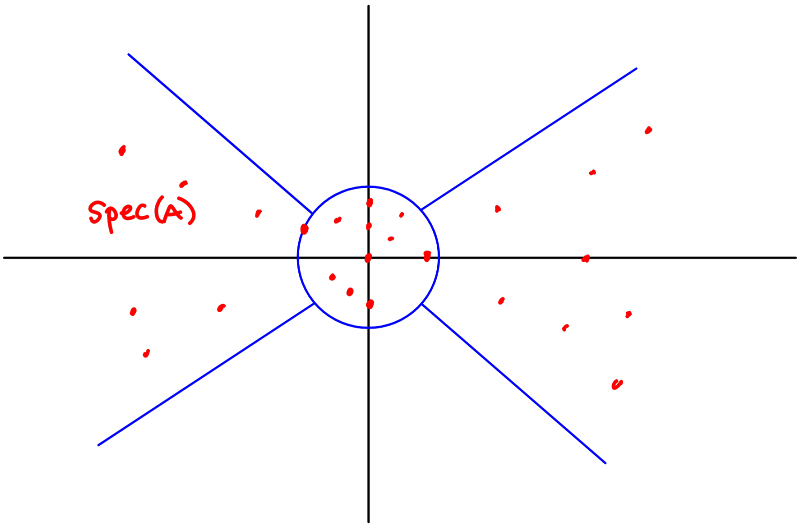
We have that $\mathrm{ran}(D_B)$ is closed and that $\ker(D_B)$ is finite dimensional. By Lemma 5.4 we conclude that $\mathrm{coker}(D_B) \cong \ker(D_B^*)$ and we have proved that the latter space is also finite dimensional. Therefore, D_B is a Fredholm operator. □

Remark 5.7. If we assumed that B was only semi-elliptically regular, i.e. that $B \subset \mathrm{H}^{\frac{1}{2}}(\partial M, E)$ but not necessarily that $B^\dagger \subset \mathrm{H}^{\frac{1}{2}}(\partial M, F)$, then the argument in the proof of Proposition 5.6 holds for D_B. That is, we obtain that D_B has finite dimensional kernel and closed range. Now, from Remark 5.3, we automatically have that $\mathrm{ran}(D_B^*)$ is also closed. Moreover, from Lemma 5.4, we still have that $\mathrm{coker}(D_B) \cong \ker(D_B^*)$. Therefore, in this setting, the failure for D_B to be a Fredholm operator can only arise from $\ker(D_B^*) = \ker(D^\dagger_{B^\dagger})$ having an infinite dimensional kernel. Elliptic regularity for a boundary condition is a symmetric relation on B and $B^\dagger$ and it ensures that both kernels $\ker(D_B)$ and $\ker(D_B^*)$ are finite dimensional.

5.2 Atiyah-Patodi-Singer boundary conditions

We now consider a very special class of boundary conditions which can be defined given an adapted boundary operator. To begin with, let us consider a general adapted boundary operator A. That is, we do not assume it is bisectorial invertible.

However, from Proposition 4.77 I) and II), we know that the spectrum of A is discrete and that it is contained in a region $\mathrm{S}_\omega \cup \overline{B_{R_A}(0)}$ for some $\omega \in [0, \frac{\pi}{2})$ and $R_A \in [0, \infty)$. In particular, $\imath\mathbb{R} \cap \mathrm{spec}(A)$ is just a finite set of points.



The boundary condition we want to define is the sum of all generalised eigenspaces with negative real part sitting inside $\mathrm{H}^{\frac{1}{2}}(\partial M, E)$. To do this rigorously, let us first consider the real part of the spectrum of A.

Lemma 5.8. *The real part of the spectrum of A can be written as*

$$\mathrm{Re}(\mathrm{spec}(A)) = \{-\infty < \cdots < \lambda_{-n} < \cdots < \lambda_{-1} < 0 = \lambda_0 < \lambda_1 < \dots\} .$$

Proof. By the discreteness of the spectrum, we are guaranteed there are no finite accumulation points. Therefore, the real part of the spectrum must take such a form. □

Note that in the way we have enumerated the set $\mathrm{Re}\,\mathrm{spec}(A)$ is without counting for multiplicity. In fact, we may very well have two points $z_1, z_2 \in \mathrm{spec}(A)$ such that $\mathrm{Re}\, z_1 = \mathrm{Re}\, z_2$. However, the important point here is that for all $r \in (\lambda_{-1}, 0)$, we obtain $A_r = A - r$ is ω_r-bisectorial and invertible. The same conclusion holds for $r(0, \lambda_1)$.

Definition 5.9 (Atiyah-Patodi-Singer (APS) boundary condition). Let $a := \frac{1}{2}\min\{-\lambda_{-1}, \lambda_1\}$ and define the APS boundary condition for the operator A to be

$$B_{\mathrm{APS}}(A) := \chi^{-}(A_a)\mathrm{H}^{\frac{1}{2}}(\partial M, E) .$$

Proposition 5.10. *Given an adapted boundary operator A for D, the APS-boundary condition $B_{\mathrm{APS}}(A)$ is elliptically regular.*

Proof. Since A_a is an invertible bisectorial adapted boundary operator, on invoking Theorem 4.122, we have that $\check{\mathrm{H}}(D) = \check{\mathrm{H}}_{A_a}(D_0)$ with D_0 the induced model operator from A_a. Moreover,

$$\check{\mathrm{H}}_{A_a}(D_0) = \chi^-(A_a)\mathrm{H}^{\frac{1}{2}}(\partial M, E) \oplus \chi^+(A_a)\mathrm{H}^{-\frac{1}{2}}(\partial M, E) \quad = B_{\mathrm{APS}}(A) \oplus \oplus \chi^+(A_a)\mathrm{H}^{-\frac{1}{2}}(\partial M, E)\,,$$

which shows that $B_{\mathrm{APS}}(A)$ is closed in $\check{\mathrm{H}}_{A_a}(D_0)$ and hence, a boundary condition. Moreover, by construction $B_{\mathrm{APS}}(A) \subset \mathrm{H}^{\frac{1}{2}}(\partial M, E)$ and therefore, it is semi-elliptically regular.

By Proposition 5.1, it suffices to prove that

$$B_{\mathrm{APS}}(A)^{\perp, \langle \hat{\mathrm{H}}_{A_0}(D_0), \check{\mathrm{H}}_{A_a}(D_0)\rangle} \subset \mathrm{H}^{\frac{1}{2}}(\partial M, E)\,.$$

We compute

$$\begin{aligned}
B_{\mathrm{APS}}(A)^{\perp, \langle \hat{\mathrm{H}}_{A_a}(D_0), \check{\mathrm{H}}_{A_a}(D_0)\rangle} &= \Big(\chi^-(A_a)\mathrm{H}^{\frac{1}{2}}(\partial M, E)\Big)^{\perp, \mathrm{H}^{-\frac{1}{2}}(\partial M, E)} \cap \hat{\mathrm{H}}_{A_a}(D_0) \\
&= \chi^+(A_a^*)\mathrm{H}^{-\frac{1}{2}}(\partial M, E) \cap \Big(\chi^+(A_a^*)\mathrm{H}^{\frac{1}{2}}(\partial M, E) \oplus \chi^-(A_a^*)\mathrm{H}^{-\frac{1}{2}}(\partial M, E)\Big) \\
&= \chi^+(A_a^*)\mathrm{H}^{-\frac{1}{2}}(\partial M, E) \cap \chi^+(A_a^*)\mathrm{H}^{\frac{1}{2}}(\partial M, E) \\
&= \chi^+(A_a^*)\mathrm{H}^{\frac{1}{2}}(\partial M, E) \subset \mathrm{H}^{\frac{1}{2}}(\partial M, E)\,. \qquad \square
\end{aligned}$$

Remark 5.11. Given we have the freedom to compute the annihilator with respect to any bisectorial invertible adapted boundary operator, we could have instead $B_{\mathrm{APS}}(A)^{\perp, \langle \hat{\mathrm{H}}_{A_r}(D_0), \check{\mathrm{H}}_{A_r}(D_0)\rangle}$ for some other admissible spectral cut $r \in \mathbb{R}$. For the sake of argument, let us assume that $r > a$. Then, when we compute the annihilator, we would have to additionally take care of the subspace

$$\chi^-(A_r^*)\mathrm{H}^{-\frac{1}{2}} \cap \chi^+(A_a^*)\mathrm{H}^{-\frac{1}{2}}\,.$$

A quick sketch of this situation reveals that this subspace is the sum of generalised eigenspaces corresponding to finite spectral points. Therefore, this is a finite dimensional subspace. Moreover, each generalised eigenspace consists of smooth sections and therefore,

$$\chi^+(A_r^*)\mathrm{H}^{-\frac{1}{2}} \cap \chi^-\big(A_{-a}^*\big)\mathrm{H}^{-\frac{1}{2}} \subset \mathrm{C}^\infty(\partial M, E)\,.$$

Since any two norms are comparable on a finite dimensional vector space, we are able to write

$$\|u\|_{\mathrm{H}^{-\frac{1}{2}}(\partial M, E)} \simeq \|u\|_{\mathrm{H}^{\frac{1}{2}}(\partial M, E)}$$

for all $u \in \chi^+(A_r^*)\mathrm{H}^{-\frac{1}{2}} \cap \chi^-\big(A_{-a}^*\big)\mathrm{H}^{-\frac{1}{2}}$. This remark gives a conceptual basis as to the reason all subspaces $\chi^-(A_r)\mathrm{H}^{\frac{1}{2}}(\partial M, E)$ for an arbitrary spectral cut $r \in R$ should be elliptically regular. Clearly, these are the APS conditions $B_{\mathrm{APS}}(A_r)$.

Finally, we end this subsection with the following important consequence of our considerations here. This corollary allows us to use methods arising from index theory to understand both boundary value problems and other important structural aspects of adapted boundary operators.

Corollary 5.12. *Let M be compact and A any adapted boundary operator for D. Then $D_{B_{\mathrm{APS}}(A)}$ is a Fredholm operator.*

5.3 Spectral asymmetry of self-adjoint operators

Let Σ be a compact manifold without boundary, and $E_\Sigma \to \Sigma$ a hermitian bundle over Σ. We focus our attention now on $A \in \mathrm{Diff}_1(E_\Sigma)$ which is assumed to be elliptic and formally self-adjoint. Note that since Σ is boundaryless and compact, we have that A has a unique closure $\overline{A} = A_{\min} = A_{\max}$. Therefore, with slight abuse of notation, throughout, we will identify A and $\overline{A}$.

From Theorem 3.38, $\mathrm{dom}(A) = \mathrm{H}^1(\Sigma, E_\Sigma)$ and since it has a nonempty resolvent, we have that $\mathrm{spec}(A)$ is discrete. Counting the nonzero eigenvalues with multiplicity, we obtain that

$$\mathrm{spec}(A) = \{-\infty < \cdots \leq \lambda_{-n} \leq \cdots \leq \lambda_{-1} < \lambda_0 = 0 \\ < \lambda_1 \leq \cdots \leq \lambda_n \leq \cdots < \infty\} \subset \mathbb{R}\,.$$

The fact that A is self-adjoint means that the generalised eigenspaces are, in fact, eigenspaces. Moreover, $\mathrm{Eig}_A(\lambda) \subset \mathrm{C}^\infty(\Sigma, E_\Sigma)$ for all $\lambda \in \mathrm{spec}(A)$, and they are finite dimensional.

With this in mind, we define the following important gadget that was first introduced in the seminar paper [4] by Atiyah-Patodi-Singer.

Definition 5.13 (Eta-function). Given $s \in \mathbb{C}$, define

$$\eta_A(s) := \sum_{\lambda \in \mathrm{spec}(A) \setminus \{0\}} \frac{\mathrm{sgn}(\lambda)}{|\lambda|^s}$$

whenever it exists, where implicitly this sum is understood to be counting multiplicities.

While this definition seems like something ad-hoc, it is well motivated by pre-existing concepts in the literature. Consider

$$\zeta(s) := \sum_{i=1}^{\infty} i^{-s}\,,$$

the celebrated *Riemann zeta-function*. It is an important object in number theory as well as a number of other areas in mathematics as well as mathematical physics. It is meromorphic on $\mathbb{C}$ and in fact, it is holomorphic on $\mathbb{C} \setminus \{1\}$ with a simple pole at $s = 1$. Let us list some interesting properties of the zeta-function.

1. The heavily popularised expression '$1 + 2 + 3 + \cdots = -\frac{1}{12}$' due to Ramanujan is justified by the fact that $\zeta(-1) = -\frac{1}{12}$.

2. The Riemann hypothesis, which has consequences to the distribution of prime numbers, is given in terms of the zeta-function. More precisely, the hypothesis is that the nontrivial zeros of ζ are in $\frac{1}{2} + \imath\mathbb{R}$.

For B non-negative self-adjoint and with pure point spectrum, the *spectral zeta-function* of B is

$$\zeta_B(s) := \sum_{\lambda \in \mathrm{spec}(B) \setminus \{0\}} |\lambda|^{-s} .$$

As in the expression for the eta-function, this sum is understood to count multiplicities.

Like their distant cousin the Riemann zeta-function, which is of prominence in number theory, spectral zeta-functions play an important role in geometry. From the defining expression, it is easy to see that the eta-function generalises the spectral zeta-function to general self-adjoint operators, which admit two sided spectrum. When the spectrum of A is symmetric counting multiplicities, or equivalently when $\mathrm{Eig}_A(\lambda) \cong \mathrm{Eig}_A(-\lambda)$, it is clear that $\eta_A(s) = 0$. Therefore, in the general self-adjoint operator situation, the eta-function measures a certain 'spectral asymmetry'. In fact, in [4] where this was first introduced, the central theme was that of 'spectral asymmetry'.

5.3.1 An outline to study eta-functions

Let us the eta-function in the following convenient manner, that separates the sum into negative and positive spectral parts:

$$\eta_A(s) = \sum_{i=1}^{\infty} \lambda_i^{-s} + \sum_{i=1}^{\infty} \lambda_{-i}^{-s} .$$

Since we assume A to be self-adjoint, let us consider the APS boundary condition in this context. It is easy to see that, as a consequence of the self-adjointness of A, $\chi^{\pm}(A)^* = \chi^{\pm}(A)$. Moreover, each eigenspace is also orthogonal and therefore,

$$B_{\mathrm{APS}}(A) = \chi^-(A_a)\mathrm{H}^{\frac{1}{2}}(\Sigma, E) = \bigoplus_{\lambda<0}^{\perp} \mathrm{Eig}_A(\lambda) \cap \mathrm{H}^{\frac{1}{2}}(\Sigma, E) .$$

Letting $E = \pi^* E_\Sigma$, where $\pi : [0, \infty) \times \Sigma \to \Sigma$ is the projection map, we build a 'model' operator $D_0 \in \mathrm{Diff}_1(E)$ out of A, simply by writing $D_0 = (\partial_t + A)$. The $D_0^\dagger = -\partial_t + A^* = -(\partial_t - A)$ is then precisely $-A$. From this, it is clear that $\sigma_0 = -\mathrm{id}$ so that $\sigma_0^* = -\mathrm{id}$ also. Moreover,

$$\hat{\mathrm{H}}_A(D_0) = \chi^-(A)\mathrm{H}^{-\frac{1}{2}}(\Sigma, E) \oplus \chi^+(A)\mathrm{H}^{\frac{1}{2}}(\Sigma, E) .$$

Therefore, from Proposition 4.99,

$$\begin{aligned} -B_{\mathrm{APS}}^\dagger(A) = (\sigma_0^*) B_{\mathrm{APS}}^\dagger(A) &= B^{\perp, \langle \hat{\mathrm{H}}_A(D_0), \check{\mathrm{H}}_A(D_0) \rangle} \\ &= \chi^+(A_a)\mathrm{H}^{\frac{1}{2}}(\Sigma, E) = \bigoplus_{\lambda \geq 0}^{\perp} \mathrm{Eig}_A(\lambda) \cap \mathrm{H}^{\frac{1}{2}}(\partial M, E) \end{aligned}$$

Since B_{APS} are precisely the eigenspaces of A associated with negative spectrum, and $-B^{\dagger}_{\mathrm{APS}}$ are the eigenspaces of A with positive spectrum, it is suggestive that these boundary conditions 'encode' information regarding η_A. This then leads us to postulate the following approach to the study of η_A.

1. Use the index of an operator to study η_A - consider a compact manifold M, with $\partial M = \Sigma$, along with an operator D which in a neighbourhood of the boundary takes the form $D = \sigma_0(\partial_t + A)$. We have seen from Corollary 5.12 that $D_{B_{\mathrm{APS}}(A)}$ is now Fredholm operator so this operator has a well-defined index.

2. Prove an 'index formula' which connects the index of $D_{B_{\mathrm{APS}}(A)}$ to η_A.

To explore this approach further, it is useful to recall the following class of operators.

Definition 5.14. Let $S \in \mathscr{B}(\mathcal{H})$. Then S is said to be *trace class* if for an orthonormal basis (e_k) for the Hilbert space $\mathcal{H}$,

$$\mathrm{tr}(S) := \sum_k \langle S e_k, e_k \rangle < \infty \,.$$

Remark 5.15. The trace is independent of basis. However, in application, it is worthwhile to use a basis arising from the spectral theory associated to S in some meaningful way.

Remark 5.16. The trace of an operator already allows us to see the eta-invariant from an operator theoretic point of view. Since Σ is compact, by using Lemma 5.5, we obtain that A is Fredholm. By the self-adjointness of A, we have that $\mathrm{L}^2(\Sigma, E) = \ker(A) \oplus^{\perp} \mathrm{ran}(A)$. We know that $A|_{\mathrm{ran}(A)} : \mathrm{ran}(A) \to \mathrm{ran}(A)$ is unbounded self-adjoint. Moreover, since the eigenspaces of A are orthogonal, we have that $\mathrm{spec}\Big(A|_{\mathrm{ran}(A)}\Big) = \mathrm{spec}(A) \setminus \{0\}$. In particular, $\Big|A|_{\mathrm{ran}(A)}\Big|$ is invertible, and $\Big|A|_{\mathrm{ran}(A)}\Big|^{-s}$ exists at least for non-negative real values of s. Putting together these facts, we obtain that

$$\eta_A(s) = \mathrm{tr}\Bigg(\Big|A|_{\mathrm{ran}(A)}\Big|^{-s} \mathrm{sgn}\Big(A|_{\mathrm{ran}(A)}\Big)\Bigg) \,.$$

The reason we introduce the notion of trace is to afford us with a larger class of operators to 'access' the index. More precisely, want want to access η_A via a trace of a combination of 'heat kernels'.

Let $T \in \mathscr{C}(\mathcal{H})$ densely-defined and suppose that T^*T and TT^* have discrete spectrum with finite dimensional eigenspaces, e.g. for T^*T, TT^* compact.

Clearly T^*T and TT^* are non-negative self-adjoint.

Lemma 5.17. *Let $T \in \mathscr{C}(\mathcal{H})$ be densely-defined. Then, T^*T and TT^* are non-negative self-adjoint and*

$$\ker(T) = \ker(T^*T)$$

Proof. We know that T^* is also densely-defined from Proposition 4.16. Consider the operator $T : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$.

$$\tilde{T} = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix},$$

with domain $\mathrm{dom}(\tilde{T}) = \mathrm{dom}(T) \oplus \mathrm{dom}(T^*)$. It is easy to see that $\tilde{T}$ is self-adjoint operator, and so in particular, it is 0-bisectorial and has an H^∞-functional calculus. Therefore,

$$\left|\tilde{T}\right| = \tilde{T}\,\mathrm{sgn}(\tilde{T})$$

exists and it is densely-defined non-negative self-adjoint operator with the same domain as $\mathrm{dom}(\tilde{T})$. In particular, it is 0-sectorial. By application of Proposition 4.61, we obtain that $\cap_{j=1}^{\infty} \mathrm{dom}(|T|^j)$ is a dense subspace of $\mathcal{H} \oplus \mathcal{H}$. Therefore,

$$T^2 = \begin{pmatrix} T^*T & 0 \\ 0 & TT^* \end{pmatrix}.$$

is densely-defined and easily verified to be non-negative self-adjoint. By projecting off the components, we have that T^*T and TT^* are both non-negative self-adjoint.

Now, to prove that $\ker(T) = \ker(T^*T)$, we first note the trivial direction $\ker(T) \subset \ker(T^*T)$. We prove the reverse containment. Let $u \in \ker(T^*T)$. Then $v := Tu \in \ker(T^*)$. From Proposition 4.16 III), we have $\mathcal{H} = \ker(T^*) \oplus^\perp \overline{\ker(T)}$. Since $v \in \mathrm{ran}(T) \cap \ker(T^*) = 0$, so $u \in \ker(T)$. □

Lemma 5.18. *Let $T \in \mathscr{C}(\mathcal{H})$ be densely-defined and suppose that T^*T and TT^* have discrete spectrum with finite dimensional eigenspaces. Then,*

$$\lambda \in \mathrm{spec}(T^*T) \setminus \{0\} \Leftrightarrow \lambda \in \mathrm{spec}(TT^*) \setminus \{0\}$$

and $\mathrm{Eig}_{T^*T}(\lambda) \cong \mathrm{Eig}_{TT^*}(\lambda)$.

Proof. Let $\lambda \in \mathrm{spec}(T^*T) \setminus \{0\}$. Then there is a $\varphi \in \mathrm{dom}(T^*T)$ satisfying $T^*T\varphi = \lambda\varphi \in \mathrm{dom}(T^*T) \subset \mathrm{dom}(T)$. So

$$TT^*T\varphi = (TT^*)(T\varphi) = \lambda(T\varphi).$$

Therefore, $\lambda \in \mathrm{spec}(TT^*) \setminus \{0\}$. Since by Lemma 5.17 we have that $\ker(T^*T) = \ker(T)$, the map

$$\varphi \mapsto T\varphi : \mathrm{Eig}_{T^*T}(\lambda) \to \mathrm{Eig}_{TT^*}(\lambda)$$

is an injection.

Now suppose $\psi \in \mathrm{Eig}_{TT^*}(\lambda)$. Then $TT^*\psi = \lambda\psi \in \mathrm{dom}(T^*)$ and $T^*T(T^*\psi) = \lambda T^*\psi$. This means we can act on this by T^* and on doing so, we obtain $T^*\psi \in \mathrm{Eig}_{T^*T}(\lambda)$. Therefore $\varphi \mapsto T\varphi$ is a bijection and by linearity, it is an isomorphism. □

Proposition 5.19. *Let $T \in \mathscr{C}(\mathcal{H})$ with T^*T and TT^* discrete with finite dimensional eigenspaces and suppose that $f \in \mathrm{C}_\mathrm{b}([0,\infty])$. Then, $f(TT^*)$ is trace class iff $f(T^*T)$ is trace class and in that case*

$$f(0)\,\mathrm{ind}(T) = \mathrm{tr}(f(T^*T)) - \mathrm{tr}(f(TT^*))\,.$$

Proof. First, write

$$\begin{aligned}\mathrm{tr}(f(T^*T)) &= \sum_{\lambda\in\mathrm{spec}(T^*T)} f(\lambda) = f(0)\dim(\ker(T^*T)) + \sum_{\lambda>0} f(\lambda)\,,\\ \mathrm{tr}(f(TT^*)) &= \sum_{\mu\in\mathrm{spec}(TT^*)} f(\mu) = f(0)\dim(\ker(TT^*)) + \sum_{\mu>0} f(\mu)\,,\end{aligned}$$

allowing these sums to take the value ∞ if the trace is nonexistent. From Lemma 5.18, fixing $N \in \mathbb{N}$,

$$\sum_{0<\lambda<N} f(\lambda) = \sum_{0<\mu<N} f(\mu)\,.$$

Therefore, the limit $N \to \infty$ exists for one if and only if it exists for the other sum. Moreover, we assume that $\mathrm{Eig}_{T^*T}(\lambda)$ and $\mathrm{Eig}_{TT^*}(\lambda)$ are finite dimensional, $\dim(\ker(T^*T)) + \dim(\ker(TT^*)) < \infty$. This shows the assertion that $f(T^*T)$ is trace class if and only if $f(TT^*)$ is trace class.

Now, suppose that these are trace class. Then,

$$\begin{aligned}\mathrm{tr}(f(T^*T)) - \mathrm{tr}(f(TT^*)) &= f(0)(\dim(\ker(T^*T)) - \dim(\ker(TT^*)))\\ &= f(0)(\dim(\ker(T)) - \dim(\ker(T^*)))\\ &= f(0)\,\mathrm{ind}(T)\,,\end{aligned}$$ □

where the second equality follows from Lemma 5.17.

Remark 5.20. This proposition affords us with a huge amount of freedom in order to compute the index of an operator, up to a constant multiple. We begin with an operator T with the only assumption that is Fredholm. However, the spectral theory of this operator is completely unknown to us. Now, via this proposition, we can instead move to 'second order' non-negative self-adjoint operators, T^*T and TT^*, built out of this operator. There are many functions in $f \in \mathrm{C}_\mathrm{b}([0,\infty])$. Given T, we simply need to find a good class of such functions so that either $f(T^*T)$ or $f(TT^*)$ can be verified to be trace class. With this alone, we can now access the index of the operator up to an identifiable multiple of $\mathrm{ind}(T)$, namely $f(0)$.

Now we have some extra tools available to us. Letting $D_{\mathrm{APS}} = D_{B_{\mathrm{APS}}(A)}$, with the choice of $T = D_{\mathrm{APS}}$, we are lead to the operators $D_{\mathrm{APS}}^* D_{\mathrm{APS}}$ and $D_{\mathrm{APS}} D_{\mathrm{APS}}^*$ via Proposition 5.19. Given the success of 'heat equations' in the boundaryless situation for index theorems, we hope for the following.

(I) Clearly the spectrum of $D_{\mathrm{APS}}^* D_{\mathrm{APS}}$ and $D_{\mathrm{APS}} D_{\mathrm{APS}}^*$ are discrete with finite dimensional eigenspaces since, $\mathrm{dom}(D_{\mathrm{APS}}^* D_{\mathrm{APS}}) \subset \mathrm{dom}(D_{\mathrm{APS}}) \subset \mathrm{H}^1(\partial M, E)$ and $\mathrm{dom}(D_{\mathrm{APS}} D_{\mathrm{APS}}^*) \subset \mathrm{dom}(D_{\mathrm{APS}}^*) \subset \mathrm{H}^1(\partial M, F)$, and these latter spaces embed compactly into $\mathrm{L}^2(\partial M, E)$ and $\mathrm{L}^2(\partial M, F)$ respectively. Since $D_{\mathrm{APS}}^* D_{\mathrm{APS}}$ and $D_{\mathrm{APS}} D_{\mathrm{APS}}^*$ are both self-adjoint, there is a nonempty resolvent set, and therefore, we obtain a compact resolvent.

(II) Moreover, the heat semigroup $\mathrm{e}^{-\tau D_{\mathrm{APS}}^* D_{\mathrm{APS}}}$, which exists because $D_{\mathrm{APS}}^* D_{\mathrm{APS}}$ is non-negative self-adjoint. What we hope for is that this is is trace class for $\tau > 0$. Then by Proposition 5.19, $\mathrm{e}^{-\tau D_{\mathrm{APS}} D_{\mathrm{APS}}^*}$ trace class and

$$\mathrm{ind}(D_{\mathrm{APS}}) = \mathrm{tr}\left(\mathrm{e}^{-\tau D_{\mathrm{APS}}^* D_{\mathrm{APS}}}\right) - \mathrm{tr}\left(\mathrm{e}^{-\tau D_{\mathrm{APS}} D_{\mathrm{APS}}^*}\right).$$

(III) Guided by the boundaryless index theorem, we hope that in the analysis,

$$\mathrm{tr}\left(\mathrm{e}^{-\tau D_{\mathrm{APS}}^* D_{\mathrm{APS}}}\right) - \mathrm{tr}\left(\mathrm{e}^{-\tau D_{\mathrm{APS}} D_{\mathrm{APS}}^*}\right)$$

is able to ‘see’ η_A through an appropriate asymptotic expansion near some $\tau_0 \in [0, \infty]$. As in the boundaryless case, we need to understand the *Schwartz kernel*, sometimes simply (and confusingly) called the *kernel* associated this object.

5.4 Schwartz kernels of operators

For manifolds M, N let $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ be the canonical projections.

Definition 5.21 (Outer tensor product). The *outer tensor product* of the vector bundles $E \to M$ and $F \to N$ is

$$E \boxtimes F := \pi_1^* E \otimes \pi_2^* F.$$

It is clear that the fibres $(E \boxtimes F)_{(x,y)} = E_x \otimes E_y$.

An important theorem in the theory of Schwartz kernels is that there is a bijection $\mathcal{D}'(M \times N, E \boxtimes F) \to \mathscr{B}(\mathcal{D}(M, E), \mathcal{D}'(M, E))$. This is the famous *Schwartz kernel theorem*, but we shall not require its full force and therefore, we will only mention it in passing. The objects in $\mathcal{D}'(M \times N, E \boxtimes F)$ are precisely the Schwartz kernels.

We describe the kernels directly through an integral, which mirrors the situation of the scalar valued case. Let $K \in \mathrm{MeasSect}(M \times M, E^* \boxtimes E)$. The associated operator to K is

$$(T_K u)(x) := \int_M K(x', x) u(x') \, \mathrm{d}\mu(x').$$

Let (φ_λ) be an orthonormal basis for $\mathrm{L}^2(M,E)$. Then, for $u \in \mathrm{L}^2(M,E)$, we have the Fourier expansion $u(x) = \sum_\lambda u^\lambda \varphi_\lambda(x)$. Further suppose that

$$(T_K u) \in \mathscr{B}\big(\mathrm{L}^2(M,E)\big)\,.$$

Then, we expand the kernel also in the Fourier basis to obtain

$$K(x',x) = K^\lambda \varphi_\lambda(x')^* \otimes \varphi_\lambda(x)\,,$$

where $\varphi_\lambda(x')^*[u(x')] := h^E_{x'}(u(x'),\varphi_\lambda(x))$.

Therefore,

$$\begin{aligned}(T_K u)(x) &= \int K^\lambda h^E_{x'}(u(x'),\varphi_\lambda(x'))\varphi_\lambda(x)\;\mathrm{d}\mu(x')\\ &= K^\lambda \langle u,\varphi_\lambda\rangle_{\mathrm{L}^2(M,E)}\varphi_\lambda(x)\\ &= K^\lambda u^\lambda \varphi_\lambda(x)\,.\end{aligned}$$

Now, let us consider the situation when T_K is a trace class operator. Using this basis (φ_λ) to compute the trace, we obtain

$$\begin{aligned}\mathrm{tr}(T_K) &= \sum_\lambda \langle T_K\varphi_\lambda,\varphi_\lambda\rangle\\ &= \sum_\lambda \Big\langle \int K(x,y)\varphi_\lambda(y)\;\mathrm{d}\mu(y),\varphi_\lambda\Big\rangle\\ &= \sum_\lambda \Big\langle \int K^\omega \varphi_\omega(y)^*[\varphi_\lambda(y)]\;\mathrm{d}\mu(y)\varphi_\omega,\varphi_\lambda\Big\rangle\\ &= \sum_\lambda \Big\langle K^\omega\Big(\int h(\varphi_\omega(y)),\varphi_\lambda(y)\;\mathrm{d}\mu(y)\Big)\varphi_\lambda,\varphi_\lambda\Big\rangle\\ &= \sum_\lambda \int K^\lambda |\varphi_\lambda(y)|^2_{h(y)}\;\mathrm{d}\mu(y)\underbrace{\langle\varphi_\lambda,\varphi_\lambda\rangle}_{=1}\\ &= \sum_\lambda K^\lambda\,.\end{aligned}$$

Let us now compare this with the trace of the Schwartz kernel. Since $K(x,y) = K^\lambda \varphi_\lambda(y)^* \otimes \varphi_\lambda(x)$, we have

$$\mathrm{tr}(K(x,x)) = K^\lambda\,\mathrm{tr}(\varphi_\lambda(x)^*\otimes\varphi_\lambda(x)) = K^\lambda\varphi_\lambda(x)^*[\varphi_\lambda(x)] = K^\lambda|\varphi_\lambda(x)|^2_{h_x}.$$

Integrating this expression, we find

$$\int_M \mathrm{tr}(K(x,x))\;\mathrm{d}\mu(x) = \int K^\lambda|\varphi_\lambda(x)|^2\;\mathrm{d}\mu(x) = \sum_\lambda K^\lambda = \mathrm{tr}(T_K)\,.$$

Therefore,

$$\mathrm{tr}\,T_K = \int_M \mathrm{tr}\,K(x,x)\;\mathrm{d}\mu(x)\,.$$

To conclude this section, we remark that we typically consider a family of operators which are trace class. Then, the kernels themselves are parametrised via some auxiliary manifold P. That is $P \ni p \mapsto K(p,\cdot,\cdot) \in \mathrm{MeasSect}(M \times M, E^* \boxtimes E)$. A Fourier expansion then is of the form

$$K(p,x,x') = K^\lambda(p)\varphi_\lambda(x')^* \otimes \varphi_\lambda(x)\,.$$

We will see that for a heat kernel, $P = (0,\infty)$ and we consider

$$(0,\infty) \ni \tau \mapsto \mathrm{e}^{-\tau D} u = \int_M K^\lambda(\tau,x',x)u(x')\;\mathrm{d}\mu(x')\,.$$

5.5 Analysis in the cylinder

Let (Σ,ν) be a compact measured manifold without boundary. Fix a self-adjoint $A \in \mathrm{Diff}_1(\Sigma,E_\Sigma)$. Let $E = \pi^* E_\Sigma$ where $\pi : Z = [0,\infty) \times \Sigma \to \Sigma$ and the induced density on Z is $\mu = |dt| \otimes |\nu|$.

In order to analyse η_A, we analyse the model operator $D_0 = (\partial_t + A)$. Indeed, contrary to our earlier discussion, this is not a compact space. However, the structure of the infinite cylinder presents advantages to an initial analysis, much like the way in which the analysis of the model problem in Section 4.9 allowed us to understand the general problem in Section 4.11. Later, we will localise to a precompact region of the cylinder.

We have already seen that

$$B_{\mathrm{APS}}(A) = \bigoplus_{\lambda<0}^{\perp} \mathrm{Eig}_A(\lambda) \cap \mathrm{H}^{\frac{1}{2}}(\Sigma,E)$$
$$-B^\dagger_{\mathrm{APS}}(A) = \bigoplus_{\lambda\geq 0}^{\perp} \mathrm{Eig}_A(\lambda) \cap \mathrm{H}^{\frac{1}{2}}(\Sigma,E)$$

Let $D_{0,\mathrm{APS}} = D_{0,B_{\mathrm{APS}}(A)}$ and define the Laplacians

$$\Delta_1 := D^*_{0,\mathrm{APS}} D_{0,\mathrm{APS}} \qquad \text{and} \qquad \Delta_2 := D_{0,\mathrm{APS}} D^*_{0,\mathrm{APS}}\,.$$

In the analysis to follow, we study

$$\mathrm{tr}\Big(\mathrm{e}^{-\tau\Delta_1} - \mathrm{e}^{-\tau\Delta_2}\Big)$$

and connect this to η_A. For that, we analyse Δ_1 and Δ_2 and their related heat equations. The following result, readily verified, is key to the analysis.

Lemma 5.22. *The spaces* $\mathrm{L}^2([0,\infty),\mathrm{L}^2(\Sigma,E_\Sigma)) = \mathrm{L}^2(Z,E)$ *and whenever* $u \in \mathrm{L}^2(Z,E)$,

$$u(t,x) = \sum_{\lambda\in\mathrm{spec}(A)} u^\lambda(t)\varphi_\lambda(x)\,.$$

5.5.1 Analysis of the heat equations

First, let us consider heat equation with respect to the Laplacian Δ_1. Since $D^\dagger_{\mathrm{APS}} = -(\partial_t + A)$, the map Δ_1 acts as

$$\Delta_1 = (-\partial_t + A)(\partial_t + A) = -\partial_t^2 - \partial_t A + A\partial_t + A^2 = -\partial_t^2 + A^2.$$

The domain of this operator is precisely

$$\mathrm{dom}(\Delta_1) = \Big\{ u \in \mathrm{dom}((D_0^\dagger D_0)_{\max}) \;\Big|\; u \in \mathrm{dom}(D_{0,\mathrm{APS}}) \\ \text{and } D_{0,\mathrm{APS}} u \in \mathrm{dom}\big(D^*_{0,\mathrm{APS}}\big) \Big\}\,.$$

We reduce our analysis to eigensections. For that, we use Lemma 5.22 to understand $u \in \mathrm{dom}(\Delta_1)$ in terms of a condition on its Fourier coefficients. So for that, write $u(t) = \sum_\lambda u^\lambda(t)\varphi_\lambda \in \mathrm{dom}(\Delta_1)$. Then

$$u \in \mathrm{dom}(D_{0,\mathrm{APS}}) \quad \Leftrightarrow \quad \forall \lambda \geq 0\colon u^\lambda(0) = 0\,.$$

Also, note that

$$(D_{0,\mathrm{APS}} u)(t) = (\partial_t + A)\left[\sum_\lambda u_\lambda(t)\varphi_\lambda\right] = \sum_\lambda (\partial_t + \lambda)(u_\lambda(t)\varphi_\lambda)\,.$$

Therefore,

$$D_{0,\mathrm{APS}} u \in \mathrm{dom}((D_{0,\mathrm{APS}})^*) \quad \Leftrightarrow \quad \forall \lambda < 0\colon (\partial_t + \lambda)u_\lambda = 0\,.$$

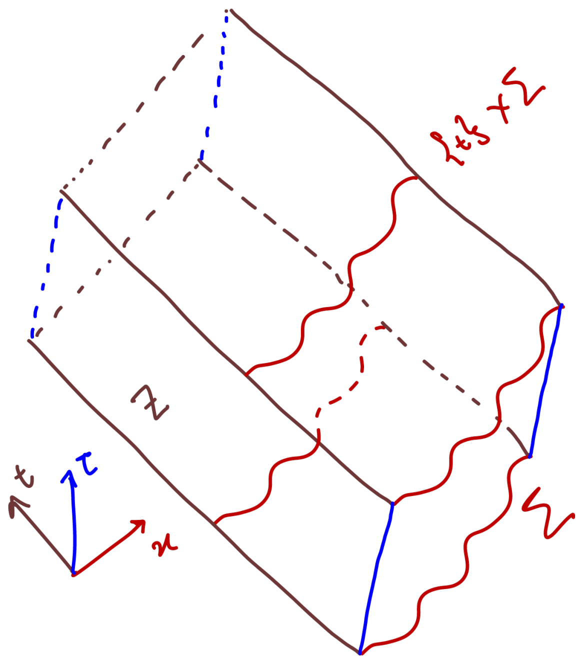
Via Fourier decomposition, the domain $\mathrm{dom}(\Delta_1)$ is described precisely to be

$$\mathrm{dom}(\Delta_1) = \big\{ u \in \mathrm{dom}\big(\big(D^\dagger D\big)_{\max}\big) \;\big|\; u_\lambda = 0 \text{ if } \lambda \geq 0 \text{ and } (\partial_t + \lambda)u_\lambda = 0 \text{ if } \lambda < 0 \big\}\,.$$

Consider the heat equation

$$(\partial_\tau + \Delta_1)u = 0 \qquad \text{with} \qquad u(0) = u_0\,.$$

Note here that τ is our variable for 'time' and t is the transversal variable in the cylinder, which leads us away from the boundary Σ to the interior.



The heat operator then takes the form

$$0 = (\partial_\tau + \Delta_1)u(\tau, t, x) = \sum_\lambda \big((\partial_\tau - \partial_t^2 + \lambda^2)u_\lambda(\tau, t)\big)\varphi_\lambda(x)$$

which is equivalent to requiring

$$0 = \big(\partial_\tau - \partial_t^2 + \lambda^2\big)u_\lambda(\tau, t) \qquad \text{and} \qquad u_\lambda(0, \cdot) = u_{\lambda,0}(\cdot)$$

for all $\lambda \in \operatorname{spec}(A)$. If $u(\tau, t, x) = \sum_\lambda u(\tau, t, x)$ solves heat equation $0 = (\partial_t + A)u$, then:

I) $u(\tau, \cdot) \in \operatorname{dom}(\Delta_1)$ for all $\tau > 0$,

II) $u_\lambda(\tau, \cdot) = 0$ for all $\tau > 0$ and $\lambda \geq 0$,

III) $(\partial_t + \lambda)u_\lambda(\tau, t)|_{t=0} = 0$ for all $\tau > 0$ and $\lambda < 0$.

That is, through the Fourier expansion, we have reduced our analysis to analysis of ODEs on $\mathbb{R}$.

We do not explicitly solve for the Schwartz kernels of these equations as they are classical and well-known. However, we present them here so that they can be verified to be the correct objects.

Fix $\lambda \in \mathbb{R}$ and let

$$\tilde{K}_1^\lambda(\tau, t, t') := \frac{\mathrm{e}^{-\lambda^2 \tau}}{\sqrt{4\pi\tau}}\left(\mathrm{e}^{-\frac{(t-t')^2}{4\tau}} - \mathrm{e}^{-\frac{(t-t')^2}{4\tau}}\right).$$

Recalling the complementary error function

$$\operatorname{erfc}(s) := \frac{2}{\sqrt{\pi}} \int_s^\infty \mathrm{e}^{-\xi^2} \, \mathrm{d}\xi\,,$$

define

$$K_1^\lambda(\tau,t,t') := \begin{cases} \tilde{K}_1^\lambda(\tau,t,t') & \text{if } \lambda \geq 0\,, \\ \tilde{K}_1^{-\lambda}(\tau,t,t') + \lambda \mathrm{e}^{-\lambda(t+t')}\operatorname{erfc}\left(\frac{t+t'}{2\sqrt{\tau}} - \lambda\sqrt{\tau}\right) & \text{if } \lambda < 0\,, \end{cases} i$$

The Schwartz kernel or heat kernel for $\mathrm{e}^{-\tau\Delta_1}$ is then readily verified to be

$$K_1(\tau,t,x,t',x') = \sum_\lambda K_1^\lambda(\tau,t,t')\varphi_\lambda(x')^* \otimes \varphi_\lambda(x)\,. \tag{5.1}$$

Next, we consider the heat equation of Δ_2. This is $u \in \mathrm{L}^2((0,\infty),\mathrm{dom}(\Delta_2))$ which satisfies $(\partial_\tau + \Delta_2)$ with $u(\tau,t) = 0$. This time, we find that

$$\mathrm{dom}(\Delta_2) = \Big\{ u \in \mathrm{dom}((D_0 D^\dagger)_{\max}) \;\Big|\; u_\lambda(\tau,0) = 0 \text{ if } \lambda < 0 \\ \text{and } (-\partial_t + \lambda)u_\lambda(\tau,t)|_{t=0} = 0 \text{ if } \lambda \geq 0 \Big\}$$

Define

$$K_2^\lambda := \begin{cases} \tilde{K}_1^\lambda(\tau,t,t') & \text{if } \lambda < 0\,, \\ \tilde{K}_1^\lambda(\tau,t,t') - \lambda \mathrm{e}^{\lambda(t+t')}\operatorname{erfc}\left(\frac{t+t'}{2\sqrt{\tau}} + \lambda\sqrt{\tau}\right) & \text{if } \lambda \geq 0\,. \end{cases}$$

Note that $K_2^\lambda = K_1^{-\lambda}(\tau,t,t')$ for $\lambda \geq 0$. Then, the heat kernel for the heat equation corresponding to Δ_2 is given by

$$K_2(\tau,t,x,t',x') := \sum_\lambda K_2^\lambda(\tau,t,t')\varphi_\lambda(x')^* \otimes \varphi_\lambda(x)\,. \tag{5.2}$$

Proposition 5.23. *For τ sufficiently small (as $\tau \to 0$ asymptotically), there is a $c < \infty$ s.t. for all $(t,x),(t',x') \in Z = [0,\infty) \times \Sigma$,*

$$|K_i(\tau,t,x,t',x')| \leq c\tau^{-\frac{1}{2}(n+1)}\mathrm{e}^{-\frac{(t-t')^2}{4\tau}}\,,$$

where $n = \dim(\Sigma)$. In particular, $K_i(\tau,t,x,t',x') \to 0$ exponentially as $\tau \to 0$ as long as $t \neq t'$.

Proof sketch. Since

$$\int_x^\infty \mathrm{e}^{-\xi^2}\,\mathrm{d}\xi < \mathrm{e}^{-x^2}\,,$$

we obtain the estimate

$$\left|K_i^\lambda(\tau,t,t')\right| < \left(\frac{\mathrm{e}^{-\frac{\lambda^2}{\tau}}}{\sqrt{\pi\tau}} + \frac{2|\lambda|\mathrm{e}^{-\frac{\lambda^2}{\tau}}}{\sqrt{\pi}}\right)\mathrm{e}^{-\frac{(t-\tau')^2}{4\tau}}\,.$$

Then

$$|K_i(\tau,t,,x,t',x')| \leq \frac{3}{2\sqrt{\pi\tau}}\left(\sum_\lambda \mathrm{e}^{-\frac{\lambda^2\tau}{2}}\left(|\varphi_\lambda(x)|^2 + |\varphi_\lambda(x)|^2\right)\right)\mathrm{e}^{-\frac{(t-t')^2}{4\tau}}\,.$$

Letting $\mathbb{K}(\tau, x, y)$ be the heat kernel for $\mathrm{e}^{-\tau A^2}$ on Σ, we note that

$$\mathrm{tr}\Big(\mathbb{K}\Big(\frac{\tau}{2}, x, x\Big)\Big) = \sum_{\lambda} \mathrm{e}^{-\frac{\lambda^2 \tau}{2}} |\varphi_\lambda(x)|^2 .$$

Since Σ is compact without boundary, and $A^2 \in \mathrm{Diff}_2(E_\Sigma)$ is self-adjoint (in particular domain H^2), we have asymptotically as $\tau \to 0$

$$\mathrm{tr}(\mathbb{K}(\tau, x, x)) \lesssim \tau^{-\frac{n}{2}} .$$

These estimates are contained in the paper [1] and its erratum [2] due to Atiyah-Bott-Patodi. The conclusion then follows. □

This proposition says that, off the diagonal $(t, t') \in \mathbb{R}_+ \times \mathbb{R}_+$, the contribution is asymptotically small. In any case, since we are interested in

$$\mathrm{tr}(\mathrm{e}^{-\tau \Delta_1} - \mathrm{e}^{-\tau \Delta_2}) ,$$

we are required to understand the differences of the kernels on the diagonal $t = t'$. Therefore,

$$K_{\Delta_1, \Delta_2} := K_1 - K_2$$

and define

$$K(\tau, t, x) := \mathrm{tr}(K_{\Delta_1, \Delta_1}(\tau, t, x, t, x)) .$$

To assist us in the calculations to follow, define

$$\mathbf{sgn}(s) = \begin{cases} \mathrm{sgn}(s) & s \neq 0, \\ 1 & s = 0 . \end{cases}$$

Through simplifying the expression for $K(\tau, t, x)$ using (5.1) and (5.2), we find

$$\begin{aligned} K(\tau, t, x) &= \sum_{\lambda} \mathbf{sgn}(\lambda) \Bigg(\frac{-\mathrm{e}^{-\lambda^2 \tau} \mathrm{e}^{-\frac{t^2}{\tau}}}{\sqrt{\pi t}} + |\lambda| \mathrm{e}^{2|\lambda| t} \,\mathrm{erfc}\Big(\frac{t}{\sqrt{\tau}} + |\lambda| \sqrt{t} \Big) \Bigg) |\varphi_\lambda(x)|^2 \\ &= \sum_{\lambda} \mathbf{sgn}(\lambda) \frac{d}{dt} \Big(\frac{1}{2} \mathrm{e}^{2|\lambda| t} \,\mathrm{erfc}\Big(\frac{t}{\sqrt{\tau}} + |\lambda| \sqrt{\tau} \Big) \Big) |\varphi_\lambda(x)|^2 . \end{aligned}$$

Integrating in (t, x) yields

$$K(\tau) := \int_0^\infty \int_\Sigma K(\tau, t, x) \;\mathrm{d}\nu(x) \;\mathrm{d}t = -\sum_{\lambda} \frac{\mathbf{sgn}(\lambda)}{2} \,\mathrm{erfc}\big(|\lambda| \sqrt{\tau}\big) .$$

Therefore

$$K'(\tau) = \frac{1}{\sqrt{4\pi\tau}} \sum_{\lambda} \lambda \mathrm{e}^{-\lambda^2 \tau} . \tag{5.3}$$

Now note that $\mathrm{erfc}(0) = 1$, and since we sum over λ counting multiplicities,

$$K(\tau) = -\frac{\dim(\ker(A))}{2} + \sum_{\lambda \neq 0} \mathrm{sgn}(\lambda) \,\mathrm{erfc}\big(|\lambda| \sqrt{\tau}\big) .$$

Let us now define

$$\mathrm{h} := \dim\ker(A)\,.$$

Since

$$\mathrm{erfc}(s) < \frac{2}{\sqrt{\pi}}\mathrm{e}^{-s^2}\,,$$

we obtain

I) $K(\tau) \to -\frac{1}{2}\mathrm{h}$ as $\tau \to \infty$,

II) $\left|K(\tau) + \frac{1}{2}\mathrm{h}\right| \leq \frac{1}{\sqrt{\pi}}\sum_{\lambda\neq 0}\mathrm{e}^{-\lambda^2\tau}$ as $\tau \to \infty$, and

III) $|K(\tau)| \leq c\tau^{-\frac{1}{2}n}$ as $\tau \to 0$ where $n = \dim(\Sigma)$.

These facts then imply that

$$\int_0^\infty \left(K(\tau) + \frac{1}{2}\mathrm{h}\right)\tau^{s-1}\,\mathrm{d}\tau < \infty$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ sufficiently large.

Using integration by parts and using equation 5.3, we obtain

$$\int_0^\infty \left(K(\tau) + \frac{1}{2}\mathrm{h}\right)\tau^{s-1}\,\mathrm{d}\tau = -\frac{\Gamma\left(s+\frac{1}{2}\right)}{2s\sqrt{\pi}}\sum_{\lambda\neq 0}\frac{\mathrm{sgn}(\lambda)}{|\lambda|^{2s}} = -\frac{\Gamma\left(s+\frac{1}{2}\right)}{2s\sqrt{\pi}}\eta_A(2s)\,.$$

Here, $\Gamma : \mathbb{C}\setminus(-\mathbb{N}) \to \mathbb{C}$ is the famous *Gamma function*, famously satisfying $k! = \Gamma(k+1)$, defined as

$$\Gamma(z) = \int_0^\infty s^{z-1}\mathrm{e}^{-s}\,\mathrm{d}s$$

when $\mathrm{Re}\,z > 0$ and analytically continued to $\mathbb{C}\setminus(-\mathbb{N})$.

5.6 The index formula

As we have already stated in Corollary 5.12, when the underlying manifold M is compact and $D \in \mathrm{Diff}_1(E,F)$ elliptic, for any given adapted boundary operator A, $D_{B_{\mathrm{APS}}(A)}$ is Fredholm. Therefore, to prove an index formula, it is reasonable to restrict our attention to compact M.

Indeed, throughout this section, we assume that M is compact with boundary $\partial M = \Sigma$, carrying a density μ. Let $\left(E, h^E\right), \left(F, h^F\right) \to M$ be Hermitian vector bundles and $D \in \mathrm{Diff}_1(E,F)$ elliptic. To utilise our analysis in Section 5.5, we further assume the following: U_ϱ is a diffeomorphic to $Z_\varrho = [0,\varrho)\times\Sigma$ such that

$$D = \sigma_0(\partial_t + A)$$

inside U_ϱ with A self-adjoint. Moreover, we assume that inside U_ϱ, the density $\mu = |dt|\otimes\nu$, where ν is the induced density from μ with respect to the vectorfield ∂_t on ∂M.

Let $D_A = \partial_t + A$. Note that this is not the model operator D_0 with respect to A, which is $D_0 = \sigma_0(\partial_t + A)$. From Theorem 4.122, we know $\check{\mathrm{H}}(D) = \check{\mathrm{H}}(D_A)$, where we write $D_A = \partial_t + A$ on the cylinder Z. Define

$$D_{\mathrm{APS}} = D_{\max}|_{\mathrm{dom}(D_{\mathrm{APS}})}$$

with domain

$$\begin{aligned}\mathrm{dom}(D_{\mathrm{APS}}) &= \left\{ u \in \mathrm{dom}(D_{\max}) \;\middle|\; u|_{\Sigma} \in B_{\mathrm{APS}}(A) \right\} \\ &= \left\{ u \in \mathrm{dom}(D_{\max}) \;\middle|\; u|_{\Sigma} \in \chi^{+}(A_a)\mathrm{H}^{-\frac{1}{2}}(\Sigma, E) \right\},\end{aligned}$$

where we chose $a = \frac{1}{2}\min\{-\lambda_{-1}, \lambda_1\}$ in Definition 5.9. As aforementioned, by Proposition 5.10. and Corollary 5.12, we know that $B_{\mathrm{APS}}(A)$ is elliptically regular and hence D_{APS} is Fredholm. Moreover, we have the following.

Lemma 5.24. *The operators $e^{-\tau L_1}$ and $e^{-\tau L_2}$ are trace class for $\tau > 0$.*

As in the analysis of the model operator $D_A = \partial_t + A$ in Section 5.5 (which was D_0 there), where we considered the Laplacians Δ_1 and Δ_2, let us now define the associated Laplacians for D_{APS}:

$$L_1 := D_{\mathrm{APS}}^* D_{\mathrm{APS}} \qquad \text{and} \qquad L_2 := D_{\mathrm{APS}} D_{\mathrm{APS}}^* .$$

Since $\mathrm{ind}(D_{\mathrm{APS}}) \in \mathbb{Z}$, by Lemma 5.24, we obtain

$$\begin{aligned}\mathrm{ind}(D_{\mathrm{APS}}) &= \dim(\ker(D_{\mathrm{APS}})) - \dim(\ker(D_{\mathrm{APS}}^*)) \\ &= \dim(\ker(D_{\mathrm{APS}}^* D_{\mathrm{APS}})) - \dim(\ker(D_{\mathrm{APS}} D_{\mathrm{APS}}^*)) \\ &= \mathrm{tr}\big(\mathrm{e}^{-\tau L_1}\big) - \mathrm{tr}\big(\mathrm{e}^{-\tau L_2}\big) .\end{aligned}$$

Given our particular geometric setup, let us briefly consider what can be expected.

1. Near the boundary, appropriately localising to inside U_ρ, given that it is diffeomorphic to Z_ρ, we anticipate that there are no 'curvature' contributions to the index, but only boundary contributions.

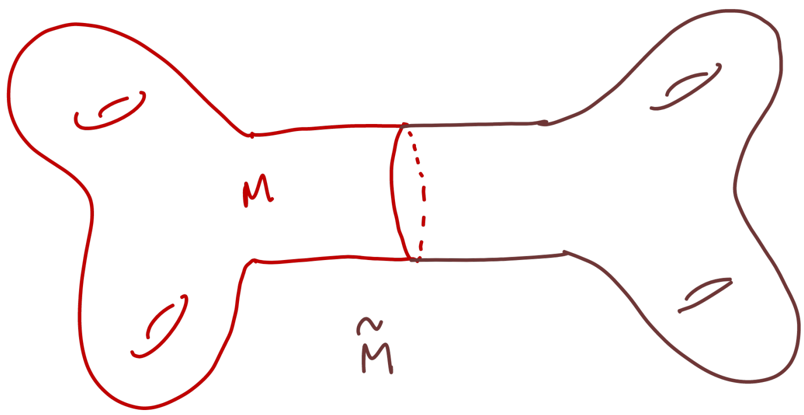
2. Away from the boundary, outside of U_ρ, the manifold might have geometry. We know from the boundaryless index theorem that it is precisely understood in terms of the geometry.

Moving forwards, we need to capture these two aspects. We anticipate that the first aspect to be captured in the analysis we did in Section 5.5.

To capture the second feature, lead by the fact that the cylindrical end near the boundary should not geometrically contribute to the index, we let $\tilde{M} := M \cup_{\Sigma} (-M)$, the double of M. Conceptually, this is the smooth manifold obtained by gluing two copies of M along the boundary. Formally,

$$\tilde{M} = {M \times \{0,1\}}\big/{\sim}$$

where $(x,0) \sim (x,1)$ for all $x \in \partial M$.



The bundles E and F double to $\tilde{E}$ and $\tilde{F}$ as expected, and since the manifold is product near the boundary, and the operator near the boundary is $D = \sigma_0(\partial_t + A)$, the operator doubles to an operator

$$\tilde{D} \in \mathrm{Diff}_1\left(\tilde{E}, \tilde{F}\right)$$

which is elliptic. Note that the fact that the operator is product near the boundary is important to assert that $\tilde{D}$ has smooth coefficients.

Again from before, we know that $\tilde{D}$ has a unique closure, i.e. $\tilde{D}_{\min} = \tilde{D}_{\max}$ since $\tilde{M}$ is compact with $\partial\tilde{M} = 0$. We will from now on write $\tilde{D}$ in place of $\overline{\tilde{D}}$.

Define

$$\tilde{L}_1 := \tilde{D}^*\tilde{D} \qquad \text{and} \qquad \tilde{L}_2 = \tilde{D}\tilde{D}^* .$$

Since the manifold $\tilde{M}$ is now a compact manifold without boundary, $\mathrm{ind}(\tilde{D}) \in \mathbb{Z}$ and $\mathrm{e}^{-\tau\tilde{L}_1}$ and $\mathrm{e}^{-\tau\tilde{L}_2}$ are trace class operators. Therefore,

$$\mathrm{ind}(\tilde{D}) = \mathrm{tr}\left(\mathrm{e}^{-\tau\tilde{L}_1} - \mathrm{e}^{-\tau\tilde{L}_2}\right).$$

It is here that we expect to find a geometric contribution to $\mathrm{ind}(D_{\mathrm{APS}})$.

By $e_1, e_2, \tilde{e}_1, \tilde{e}_2$, denote the the heat kernels of $\mathrm{e}^{-\tau L_1}, \mathrm{e}^{-\tau L_2}, \mathrm{e}^{-\tau\tilde{L}_1}, \mathrm{e}^{-\tau\tilde{L}_2}$ respectively. Let

$$K^{\mathrm{APS}}(\tau, y) := \mathrm{tr}(e_1 - e_2)(\tau, y, y) \qquad \text{and} \qquad \tilde{K}(\tau, y) := \mathrm{tr}(\tilde{e}_1 - \tilde{e}_2)(\tau, y, y) .$$

Now let $\xi_1, \xi_2 \in \mathrm{C}_c^\infty([0,\infty), [0,1])$ with

$$\xi_1(t) := \begin{cases} 1 & \text{if } t \in \left[0, \frac{\varrho}{2}\right), \\ 0 & \text{if } t \in \left[\frac{3}{4}, \infty\right), \end{cases} \qquad \text{and} \qquad \xi_2 := 1 - \xi_1 .$$

Since we are analysing the index via trace class operators, we are particularly interested in understanding the behaviour of these traces as $\tau \to 0$. To facilitate the analysis, it is useful to understand *asymptotic equivalence* as well as *asymptotic expansions* of the heat kernels associated to these objects. For that reason, let us recall the following standard notions.

Notation 5.25 (Little-o and Big-O). Let f be a complex valued function and g real-valued and positive.

1. We say that $f(s) = \mathrm{o}(g(s))$ as $x \to L$ if given $\epsilon > 0$, there exists $\delta > 0$ such that
$$|f(s)| \leq \epsilon g(s)$$
for all $|x - L| < \delta$. For the limit $L = \infty$, we say that there exists an $N > 0$ such that for all $x \geq N$, this inequality holds.

2. We say that $f(s) = \mathrm{O}(g(s))$ as $x \to L$ if there exists M and δ such that for all x satisfying $|x - L| < \delta$,
$$|f(s)| \leq M g(s)\,.$$

Notation 5.26 (Asymptotic equivalence and expansions). 1. For two functions f and g, we say
$$f(s) \sim g(s) \qquad \text{as } s \to L$$
if $f(x) = g(x)(1 + \mathrm{o}(1))$ as $x \to L$.

2. We say
$$f(s) \sim \sum_j a_j \varphi_j(s) \qquad \text{as } s \to L$$
if $\varphi_{j+1}(s) \in \mathrm{o}(\varphi_j(s))$ as $s \to L$ and
$$f(s) - \sum_{k=0}^{j-1} a_j \varphi_j(s) = \mathrm{O}(\varphi_j(s)) \qquad \text{as } s \to L,$$
or
$$f(s) - \sum_{k=0}^{j-1} a_j \varphi_j(s) = \mathrm{o}(\varphi_{j-1}(s)) \qquad \text{as } s \to L.$$
for all $j \in \mathbb{N}$. The convergence of $\sum_j a_j \varphi_j(s)$ is to be understood as converging for a fixed j in the limit $s \to L$.

Armed with this notation, we obtain as $\tau \to 0$

$$\begin{aligned}\operatorname{ind}(D_{\mathrm{APS}}) &= \operatorname{tr}\big(\mathrm{e}^{-\tau L_1} - \mathrm{e}^{-\tau L_2}\big)\\ &\sim \int_0^1 \int_{\partial M} K^{\mathrm{APS}}(\tau,t,x)\eta_1(t)\ \mathrm{d}\nu(x)\ \mathrm{d}t + \int_{\tilde{M}} \tilde{K}(\tau,y)\eta_2(t)\ \mathrm{d}\tilde{\mu}(y)\\ &\sim \int_0^1 \int_{\partial M} K(\tau,t,x)\eta_1(t)\ \mathrm{d}\nu(x)\ \mathrm{d}t + \int_M \tilde{K}(\tau,y)\ \mathrm{d}\mu(y)\\ &\sim \int_0^\infty \int_{\partial M} K(\tau,t,x)\ \mathrm{d}\nu(x)\ \mathrm{d}t + \int_M \tilde{K}(\tau,y)\ \mathrm{d}\mu(y)\,,\end{aligned}$$

where $K(\tau,t,x) = \operatorname{tr}(K_1(\tau,t,x,t,x) - K_2(\tau,t,x,t,x))$, where K_i is the heat kernel for $\mathrm{e}^{-\tau\Delta_i}$ from analysis on the cylinder $Z = [0,\infty) \times \partial M$.

From the heat kernel proof of the Atiyah-Singer index theorem on closed manifolds for elliptic first-order operators, we find

$$\tilde{K}(\tau,y) \sim \sum_{k \geq -n} \alpha_k(y)\tau^{\frac{1}{2}k}$$

for some $\alpha_n \in \mathrm{C}^\infty$. Recalling that

$$K(\tau) = \int_0^\infty \int_{\partial M} K(\tau,t,x)\ \mathrm{d}\nu(x)\ \mathrm{d}t\,,$$

we obtain

$$\operatorname{ind}(D_{\mathrm{APS}}) \sim K(\tau) + \int \tilde{K}(\tau,y)\ \mathrm{d}\mu(y)\,.$$

Rearranging this expression, we obtain

$$K(\tau) \sim \operatorname{ind}(D_{\mathrm{APS}}) - \sum_{k \geq -n} \tau^{\frac{1}{2}k} \int_M \alpha_k(x)\ \mathrm{d}\mu(x)\,.$$

Recall in that in subsection 5.5.1, we deduced the expression

$$\int_0^\infty \left(K(\tau) + \frac{1}{2}\mathrm{h}\right)\tau^{s-1}\ \mathrm{d}\tau = \frac{-\Gamma\big(s+\frac{1}{2}\big)}{2s\sqrt{\pi}}\eta_A(2s)\,.$$

Therefore,

$$\eta_A(2s) = \frac{-2s\sqrt{\pi}}{\Gamma\big(s+\frac{1}{2}\big)}\left(\frac{\frac{1}{2}\mathrm{h} + \operatorname{ind}(D_{\mathrm{APS}})}{s} - \sum_{k=-n}^{N} \int_M \frac{\alpha_k(x)}{\frac{1}{2}k + s}\ \mathrm{d}\mu(x) + \vartheta_N(s)\right),$$

where ϑ_N is a function that is holomorphic on $\big\{s \in \mathbb{C} \bigm| \operatorname{Re}(s) > -\frac{N+1}{2}\big\}$. Then, by complex analysis, the function η_A is meromorphic on $\mathbb{C}$, and takes a finite value at $s = 0$, given by the expression.

$$\eta_A(0) = 2\int_M \alpha_0(x)\ \mathrm{d}\mu(x) - \mathrm{h} + 2\operatorname{ind}(D_{\mathrm{APS}})\,.$$

This yields the following celebrated and remarkable theorem of Atiyah, Patodi, and Singer [**APS4**, 3, 4, 5, 6].

Theorem 5.27 (Atiyah-Patodi-Singer). *Let (M, μ) be a compact measured manifold with boundary and $D \in \mathrm{Diff}_1(E, F)$ such that in a neighbourhood $U_\varrho \cong [0, \varrho) \times \partial M = Z_\varrho$,*

$$D = \sigma_0(\partial_t + A)$$

where A is a self-adjoint adapted boundary operator and $d\mu = |dt| \otimes d\nu$. Let $D_{\mathrm{APS}} = D_{B_{\mathrm{APS}}(A)}$ where

$$B_{\mathrm{APS}}(A) = \sum_{\lambda < 0} \mathrm{Eig}_A(\lambda) \cap \mathrm{H}^{\frac{1}{2}}(\partial M, E)\,.$$

The operator D_{APS} is elliptically regular, hence Fredholm, and

$$\mathrm{ind}(D_{\mathrm{APS}}) = \int_M \alpha_0 \,\mathrm{d}\mu - \frac{\dim\ker(A) + \eta_A(0)}{2}\,.$$

In this expression, α_0 is the coefficient of the τ^0-term in the asymptotic expansion as $\tau \to 0$ of the kernel of

$$\mathrm{tr}\left(\mathrm{e}^{-\tau \tilde{D}^* \tilde{D}} - \mathrm{e}^{-\tau \tilde{D}\tilde{D}^*}\right), \tag{5.4}$$

where $\tilde{D}$ is the operator double on $\tilde{M}$, the double of the manifold. The function

$$\eta_A(s) = \sum_{\lambda \in \mathrm{spec}(A)\setminus\{0\}} |\lambda|^{-s}\,\mathrm{sgn}(\lambda)$$

converges absolutely for $\mathrm{Re}(s)$ large, is meromorphic on all of $\mathbb{C}$, and is finite at $s = 0$. Moreover, if the asymptotic expansion in (5.4) has no negative power, then η_A is holomorphic on $\left\{s \in \mathbb{C} \,\middle|\, \mathrm{Re}(s) > -\frac{1}{2}\right\}$.

Remark 5.28. The assumption that $D = \sigma_0(\partial_t + A)$ in a neighbourhood of ∂M can be dropped. However, this introduces an additional term on the right, called the *transgression term.* In particular, this measures the way in which the boundary ∂M is embedded into M through its dependency on the second fundamental form. However, the presentation of this theorem requires greater technicalities. It also hides the conceptual nature of this theorem, particular its proof where, as aforementioned, the doubled operator on the doubled manifold accounts for the geometric term, and the cylindricality of the operator near the boundary accounts for the eta function.

5.7 Dirac-Type operators

We have discussed the APS theorem in the abstract, under the hypothesis that the operator and measure are both product near the boundary. In this section, we show that these hypotheses are geometrically natural by demonstrating a large class of operators for which these assumptions are valid.

From here on, let us assume that (M, g) is a Riemannian manifold with compact boundary.

Definition 5.29. $D \in \mathrm{Diff}_1(E, F)$ is called Dirac-type if

$$\sigma_D(x,\xi)^*\sigma_D(x,\eta) + \sigma_D(x,\eta)^*\sigma_D(x,\xi) = 2g(\xi,\eta)\,\mathrm{id}_{E_x}\ ,$$
$$\sigma_D(x,\xi)\sigma_D(x,\eta)^* + \sigma_D(x,\eta)\sigma_D(x,\xi)^* = 2g(\xi,\eta)\,\mathrm{id}_{F_x}\ .$$

These conditions are called the *Clifford relations* of the symbol.

Lemma 5.30. *If $D \in \mathrm{Diff}_1(E, F)$ is Dirac-type, then it is elliptic and for $0 \neq \xi \in T^*M$,*

$$\sigma_D(x,\xi)^{-1} = |\xi|_{g(x)}^{-2}\sigma_D(x,\xi)^*\ .$$

Since we are in a Riemannian setting, we have the inward pointing unit normal vector as a unique and canonical choice of inward pointing transversal vectorfield along the boundary. Throughout our discussion of Dirac-type operators, we calculate with this vectorfield. Therefore, we fix the following notation.

Notation 5.31. Let $\vec{n}$ be the inward pointing unit normal vector field along ∂M with respect to g. Let $\vec{N} = \vec{n}^\flat$.

Proposition 5.32. *For $D \in \mathrm{Diff}_1(E, F)$ Dirac-type, whenever $x \in \partial M$ and $0 \neq \xi \in T^*\partial M$,*

$$\sigma_D\left(x,\vec{N}\right)^{-1} \circ \sigma_D(x,\xi) : E_x \to E_x \tag{5.5}$$

is skew-hermitian. Furthermore, there exists an adapted boundary operator A that is formally self-adjoint with principal symbol given by (5.5).

Proof. By Lemma 5.30, we obtain

$$\sigma_D\left(x,\vec{N}\right)^{-1} = \left|\vec{N}\right|^{-2}\sigma_D\left(x,\vec{N}\right)^* = \sigma_D\left(x,\vec{N}\right)^*\ .$$

Therefore,

$$\begin{aligned}\sigma_D\left(x,\vec{N}\right)^{-1}\sigma_D(x,\xi) &= \sigma_D\left(x,\vec{N}\right)^*\sigma_D(x,\xi)\\ &= -\sigma_D(x,\xi)^*\sigma_D\left(x,\vec{N}\right) + g\left(\xi,\vec{N}\right)\mathrm{id}_{E_x}\end{aligned}$$

Now, since we have chosen $\vec{n}$ to be the inward pointing unit normal, we have that $g\left(\xi,\vec{N}\right) = 0$ and therefore,

$$\begin{aligned}\sigma_D\left(x,\vec{N}\right)^{-1}\sigma_D(x,\xi) &= \sigma_D\left(x,\vec{N}\right)^*\sigma_D(x,\xi)\\ &= -\sigma_D(x,\xi)^*\sigma_D\left(x,\vec{N}\right) = -\left(\sigma_D\left(x,\vec{N}\right)^*\sigma_D(x,\xi)\right)^*\ .\end{aligned}$$

This shows that $\sigma_D\left(x,\vec{N}\right)^{-1}\circ\sigma_D(x,\xi)$ is skew-hermitian.

Now, we show there exists a formally self-adjoint adapted boundary operator. For that, fix $A_0\in\mathrm{Diff}_1\left(E|_\Sigma\right)$ be an adapted boundary operator to D with $\sigma_{A_0}(x,\xi)=\sigma_D\left(x,\vec{N}\right)^{-1}\circ\sigma_D(x,\xi)$. Then define

$$A:=\frac{1}{2}\left(A_0+A_0^\dagger\right),$$

which is formally self-adjoint by construction. It remains to show that the principal symbol of A is given by (5.5). For that,

$$\begin{aligned}\sigma_A(x,\xi)&=\frac{1}{2}\sigma_{A_0}(x,\xi)+\frac{1}{2}\sigma_{A_0^\dagger}(x,\xi)\\&=\frac{1}{2}\sigma_{A_0}(x,\xi)-\frac{1}{2}\sigma_{A_0}(x,\xi)^*\\&=\frac{1}{2}\sigma_{A_0}(x,\xi)+\frac{1}{2}\sigma_{A_0}(x,\xi)\\&=\sigma_{A_0}(x,\xi)\\&=\sigma_D\left(x,\vec{N}\right)^{-1}\circ\sigma_D(x,\xi)\,,\end{aligned}\qquad\square$$

where the second equality follows from Proposition 3.16.

5.7.1 The formally self-adjoint case

Let us now suppose that, in addition to D being Dirac-type, that $E=F$ and that D is formally self-adjoint. In this case, since $\sigma_D(x,\xi)^*=-\sigma_D(x,\xi)$, the Clifford relations reduce to

$$\sigma_D(x,\xi)\sigma_D(x,\eta)+\sigma_D(x,\eta)\sigma_D(x,\xi)=2g(\xi,\eta)\,\mathrm{id}_{E_x}\,. \tag{5.6}$$

Proposition 5.33. *Suppose that $E=F$, $D\in\mathrm{Diff}_1(E)$ elliptic and formally self-adjoint. Then, there exists an adapted boundary operator A that is self-adjoint with principal symbol*

$$\sigma_A(x,\xi)=\sigma_D(x,\vec{N})^{-1}\circ\sigma_D(x,\xi)$$

that satisfies the following:

I) The anti-commutativity relation

$$\sigma_D\left(x,\vec{N}\right)\circ A=-A\circ\sigma_D\left(x,\vec{N}\right)$$

holds, and it is unique up to the addition of $S\in\mathrm{SymEnd}(E)$ anti-commuting with $\sigma_D\left(x,\vec{N}\right)$.

II) $\sigma_D\left(x,\vec{N}\right) : \mathrm{Eig}_A(\lambda) \to \mathrm{Eig}_A(-\lambda)$ *is an isomorphism.*

III) $\mathrm{spec}(A)$ *is symmetric and* $\eta_A(s) = 0$ *for all* $s \in \mathbb{C}$.

Proof. a) Ad I).

First we establish the uniqueness statement. For that, let A_1, A_2 be adapted boundary operators satisfying (5.6). Then $A_1 - A_2 \in \mathrm{Diff}_0(E)$ because the have the same principal symbol. Clearly $B := A_1 - A_2$ anti-commutes with $\sigma_D(x,\vec{N})$ and $A_1 = A_2 + B$.

Now for existence. For that, let A_0 be any formally self-adjoint adapted boundary operator with principal symbol $\sigma_{A_0}(x,\xi) = \sigma_D(x,\vec{N})^{-1} \circ \sigma_D(x,\xi)$. Define

$$S := \frac{1}{2}A_0 + \frac{1}{2}\sigma_D\left(\cdot,\vec{N}\right)A_0\sigma_D\left(\cdot,\vec{N}\right)^*,$$

and note

$$S^* = \frac{1}{2}A_0^* + \frac{1}{2}\sigma_D\left(\cdot,\vec{N}\right)A_0^*\sigma_D\left(\cdot,\vec{N}\right)^* = S\,.$$

It is readily verified that $\sigma_S(x,\xi) = 0$ using Clifford relations for $\sigma_D(x,\cdot)$. This shows that $S \in \mathrm{Diff}_0(E)$.

Define

$$A := A_0 - S = \frac{1}{2}A_0 - \frac{1}{2}\sigma_D\left(\cdot,\vec{N}\right)A_0\sigma_D\left(\cdot,\vec{N}\right)^*.$$

Then,

$$A^* = A_0^* - S^* = A_0 - S = A\,.$$

A routine calculation then yields $\sigma_D\left(x,\vec{N}\right)A + A\sigma_D\left(x,\vec{N}\right) = 0$.

b) Ad II). $\varphi \in \mathrm{Eig}_A(\lambda)$, then

$$A\sigma_D\left(x,\vec{N}\right)\varphi = \sigma_D\left(x,\vec{N}\right)A\varphi = -\lambda\varphi\,,$$

since $\sigma_D\left(x,\vec{N}\right)^{-1} = -\sigma_D\left(x,\vec{N}\right)$, this is a bijection.

c) Ad III). We know that $\mathrm{spec}(A)$ is discrete and from II), $\lambda \in \mathrm{spec}(A) \Leftrightarrow -\lambda \in \mathrm{spec}(A)$. By definition,

$$\eta_A(s) = \sum_{\lambda \neq 0} \mathrm{sgn}(\lambda)|\lambda|^{-s} = 0\,.$$

$\square$

5.7.2 Dirac bundles and operators

In this section, following the terminology laid out by Lawson and Michelson in [35], we present a large class of bundles and associated operators that are often determined by geometry. This class of operator are stable in that, through twisting with auxiliary bundles, we remain in this class.

Let (M, g) be a Riemannian manifold. By ΔM we denote the Clifford algebra $\Lambda M = \Lambda T^*M$ with product

$$\eta \vartriangle \omega = \eta \wedge \omega - \eta \llcorner \omega \,,$$

where we recall

$$g(\xi \llcorner \eta, \varrho) = g(\eta, \xi \wedge \varrho)\,.$$

Definition 5.34 (Dirac bundle). $\left(\$, h^{\$}, \nabla\!\!\!\!/\right) \to (M, g,)$ is called a *Dirac bundle* if the following conditions are satisfied:

(I) $\varrho : \mathrm{C}^\infty(M, \Delta M) \to \mathrm{C}^\infty\big(M, \mathrm{End}(\$)\big)$ is an algebra homomorphism.

(II) $h^{\$}(\varrho(\eta)u, \varrho(\eta)v) = h^{\$}(u, v)$ for all $\eta \in \Lambda^1 M$ with $|\eta|_g = 1$.

(III) $\nabla\!\!\!\!/_v(\varrho(\eta)u) = \varrho(\nabla_v \eta)u + \varrho(\eta)\nabla\!\!\!\!/_v u$ for all $\eta \in \Lambda^1 M$.

Remark 5.35.

1. This definition is taken from [35]. However, there, the map ϱ acts on vectors rather than forms. Through the musical isomorphisms induced by the metric g, the passage between the two worlds are immediate.
2. Note that by (II), we have $\ker\Big(\varrho|_{\Lambda^1 M}\Big) = 0$.
3. Usually we write $v \cdot \varphi$ in place of $\varrho(v)\varphi$.

Definition 5.36 (Dirac operator). For $x \in M$ and (e_i) an orthonormal frame around x and (e^i) the dual co-frame, define

$$\big(D\!\!\!\!/\,u\big)(x) = \sum_{i=1}^n \varrho_x\big(\mathrm{e}^i\big)\big(\nabla\!\!\!\!/_{e_i} u\big)(x) = \sum_{i=1}^n \big(\mathrm{e}^i \cdot \nabla\!\!\!\!/_{e_i} u\big)(x)\,.$$

Remark 5.37. From the properties of the map ϱ and since $\Lambda^1 = T^*M$, we can see $\varrho|_{T^*M} \in \mathrm{C}^\infty(M, TM \otimes \$^* \otimes \$)$. Now, $\nabla\!\!\!\!/ u \in \mathrm{C}^\infty(M, T^*M \otimes \$)$ and since

$$(TM \otimes \$^*) = (T^*M \otimes \$)^*\,,$$

the operator $D\!\!\!\!/$ can be written more succinctly as

$$D\!\!\!\!/ = \varrho \circ \nabla\!\!\!\!/\,.$$

This is a particular advantage of defining ϱ on T^*M rather than TM.

Proposition 5.38. $\sigma_{\not{D}}(x,\xi) = \varrho(\xi) = \xi\,\cdot$, *and* $\not{D}$ *is formally self-adjoint and of Dirac-type.*

Proof. This is immediate from 5.34 (II). We leave its verification as an exercise. □

Remark 5.39. In fact, the inspiration for Dirac-type operators arises from this particular case of Dirac operators on Dirac bundles, where the Dirac-type nature of the operator is immediate.

In what is to follow, we write $dt^{-1} = \vec{N}^{-1}$.

Proposition 5.40. *Let* (M,g) *compact and* g *product near the boundary. That is, there is an* $r > 0$ *s.t.* $U_r \cong Z_r$ *and* $g(t,x) = dt \otimes dt + g_{\partial M}(x)$ *inside* U_r*. Furthermore, let* $(\not{S}, h^{\not{S}}, \not{\nabla}) \to M$ *be a Dirac bundle. Let*

$$Au := dt^{-1} \cdot e^i \cdot \not{\nabla}_{e_i} u\,.$$

Then, the following hold.

I) *The operator* A *is an adapted boundary operator with principal symbol* $\sigma_A(x,\xi) = dt^{-1} \cdot \xi\,\cdot$.

II) $dt \cdot A = -A \cdot dt$.

III) A *is formally self-adjoint.*

IV) *If* $h^{\not{S}}$ *and* $\not{\nabla}$ *are constant in the direction* $\vec{n}$*, the inward pointing normal, inside* U_r*, the operator* $\not{D}$ *takes the form*

$$\not{D} = dt \cdot (\partial_t + A)\,.$$

Proof. a) Ad I). It is clear from this expression that

$$\sigma_A(x,\xi) = dt^{-1} \cdot \xi\,\cdot\ ,$$

and therefore, it is an adapted boundary operator for $\not{D}$.

b) Ad II). First, we write

$$dt \cdot Au = dt \cdot \sum_{i=2}^{n} dt^{-1} \cdot \mathrm{e}^i \cdot \not{\nabla}_{e_i} u = \sum_{i=2}^{n} \mathrm{e}^i \cdot \not{\nabla}_{e_i} u\,.$$

On the other hand, we obtain

$$\begin{aligned} A(dt \cdot u) &= \sum_{i=2}^{n} dt^{-1} \cdot \mathrm{e}^i \cdot \not{\nabla}_{e_i}(dt \cdot u) \\ &= \sum_{i=2}^{n} dt^{-1} \cdot \mathrm{e}^i \cdot \big((\not{\nabla}_{e_i} dt) \cdot u + dt^{-1} \cdot \mathrm{e}^i \cdot dt \cdot \nabla_{e_i} u\big) \end{aligned}$$

Now, fix a point $x \in \partial M$ and choose a synchronous frame for TM at x, where $e_1 = \partial_t$. Then, we obtain that

$$(\nabla_{e_i} dt)(x) = 0\,.$$

Moreover,

$$\mathrm{e}^i \cdot dt \cdot v = \big(\mathrm{e}^i \wedge dt\big) \cdot v = -dt \cdot \mathrm{e}^i \cdot v$$

and therefore

$$(A(dt \cdot u))(x) = \left(-\sum_{i=2}^{n} \mathrm{e}^i \cdot \nabla\!\!\!\!/\,_{e_i} u\right)(x) = -(dt \cdot Au)(x)\,.$$

But since $x \in \partial M$ was arbitrary, the conclusion follows.

c) Ad III). Within an orthonormal frame $\{e_i\}$, we compute:

$$\begin{aligned}
h(Au, v) &= h\left(dt^{-1} \cdot \sum_{i=2}^{n} \mathrm{e}^i \cdot \nabla\!\!\!\!/\,_{e_i} u, v\right) \\
&= h\left(\sum_{i=2}^{n} \mathrm{e}^i \cdot \nabla\!\!\!\!/\,_{e_i} u, dt \cdot v\right) \\
&= \sum_{i=2}^{n} h\big(\mathrm{e}^i \cdot \nabla\!\!\!\!/\,_{e_i} u, dt \cdot v\big) \\
&= \sum_{i=2}^{n} h\Big(\nabla\!\!\!\!/\,_{e_i} u, \big(\mathrm{e}^i\big)^{-1} \cdot dt \cdot v\Big) \\
&= -\sum_{i=2}^{n} h\Big(u, \nabla\!\!\!\!/\,_{e_i}\Big(\big(\mathrm{e}^i\big)^{-1} \cdot dt \cdot v\Big)\Big) + R \\
&= \sum_{i=2}^{n} h\big(u, \nabla\!\!\!\!/\,_{e_i}\big(\mathrm{e}^i \cdot dt \cdot v\big)\big) + R \\
&= \sum_{i=2}^{n} h\big(u, \nabla\!\!\!\!/\,_{e_i}\big(\mathrm{e}^i \wedge dt - \mathrm{e}^i \llcorner dt\big)v + \big(\mathrm{e}^i \cdot dt\big) \cdot \nabla\!\!\!\!/\,_{e_i} v\big) + R\,,
\end{aligned}$$

where the penultimate equality is justified by $\mathrm{e}^i \vartriangle \mathrm{e}^i = \mathrm{e}^i \wedge \mathrm{e}^i - \mathrm{e}^i \llcorner \mathrm{e}^i = -1$. In the last term we can observe $\mathrm{e}^i \llcorner dt = 0$ and on choosing (e_i) with $e_1 = \partial_t$ synchronous at x, we absorb the remainder term R to an expression of the form $\mathrm{div}(X)$ for some $X \in \mathrm{C}^\infty(M, TM)$, and therefore,

$$\begin{aligned}
h(Au, v) &= \sum_{i=2}^{n} h\big(u, \mathrm{e}^i \cdot dt \cdot \nabla\!\!\!\!/\,_{e_i} v\big) + \mathrm{div}(X) \\
&= \sum_{i=2}^{n} h\big(u, -dt \cdot \mathrm{e}^i \cdot \nabla\!\!\!\!/\,_{e_i} v\big) + \mathrm{div}(X) \\
&= \sum_{i=2}^{n} h\big(u, dt^{-1} \cdot \mathrm{e}^i \cdot \nabla\!\!\!\!/\,_{e_i} v\big) + \mathrm{div}(X) \\
&= h(u, Av) + \mathrm{div}(X)\,.
\end{aligned}$$

Since

$$\int_{\partial M} \mathrm{div}(X)\ \mathrm{d}\nu = 0\,,$$

we obtain $A = A^\dagger$.

d) Ad IV). Let (e_i) be an orthonormal frame for TM near a point $x = (t, y) \in U_r$ with $e_1 = \vec{n} = \partial_t$. Then

$$\begin{aligned}\big(\not{D}u\big)(x) &= \big(dt \cdot \not{\nabla}_{\partial_t} u\big)(x) + \sum_{i=2}^{n} \big(\mathrm{e}^i \cdot \not{\nabla}_{e_i} u\big)(x) \\ &= dt \cdot \left(\big(\not{\nabla}_{\partial t} u\big)(x) + \sum_{i=2}^{n} \big(dt^{-1} \cdot \mathrm{e}^i \cdot \not{\nabla}_{e_i} u\big)(u) \right).\end{aligned}$$

We examine the first term. For that, let (s_α) be a frame for $\not{S}$ near $x = (t, y)$. Then,

$$\big(\not{\nabla}_{\partial_t} u\big)(x) = \big(\not{\nabla}_{\partial t} u^\alpha\big)(t_0, x_0) s_\alpha(t_0, x_0) + u^\alpha(t_0, x_0)\big(\not{\nabla}_{\partial_t} s^\alpha\big)(t_0, x_0)\,.$$

Since we have assumed that both $h^{\not{S}}$ and $\not{\nabla}$ are constant in the $\vec{n} = \partial_t$ direction, we have that

$$\nabla_{\partial_t} s^\alpha = 0$$

and therefore,

$$\big(\not{\nabla}_{\partial_t} u\big)(x) = (\partial_t u)(x)\,.$$

Therefore, we obtain that $\not{D}$ is of product form as promised. □

Corollary 5.41. *For (M, g) compact, g product near the boundary and with $(h^{\not{S}}, \not{\nabla})$ being constant in the normal direction near the boundary, the eta-function $\eta_A(s) = 0$ for $s \in \mathbb{C}$.*

Example 5.42. 1. Let $\not{S} = \Lambda M = \Lambda T^*M$ and $\not{\nabla} = \nabla^{\Lambda M}$ induced from the Levi-Civita connection ∇^g of g. Then

$$\varrho(\eta)\omega = \eta \wedge \omega - \eta \llcorner \omega\,,$$

and the associated Dirac operator is the Hodge-Dirac operator

$$\not{D} = D_{\mathrm{H}} = d + d_g^*\,.$$

If g is product near the boundary, then $\nabla^{\Lambda M}$ and the induced metric h on ΛM are also constant in the normal direction.

2. Let M be Spin and fix a Spin structure ξ. Let $\not{S} = \not{\Delta} M$, the Spin bundle with respect to ξ and g. On considering the canonical irreducible representations (one in odd, two in even dimensions) leads to $\varrho(\eta)\omega = \eta \cdot \omega$. Then the induced $\not{D}$ is a Spin-Dirac operator.

3. Let (D, ∇^E, h^E) be a Hermitian bundle with metric connection, then $\widetilde{\$} := \$ \otimes E$ produces a new Dirac bundle by defining
$$\varrho(\eta)(\varphi \otimes u) = \varrho(\eta)\varphi \otimes u\,.$$
We get a canonical connection $\widetilde{\$}$ and an induced Dirac operator $\not{D}_E$.

When $\$ = \not{\Delta} M$, then $\not{D}_E$ is typically called a *twisted Dirac operator* or *twisted Spin-Dirac operator*.

Exercise 5.43. Verify that the structures defined in Example 5.42 3. indeed produce a new Dirac bundle.

5.8 Geometric applications

The Atiyah-Patodi-Singer index theorem has numerous geometric applications, with the quintessential example being its consequences in the Spin case and to the so-called signature operator. We give an outline of these applications in this section. However, first, we must make a slight detour through genera and various characteristic differential forms, necessary to capture the geometric features.

5.8.1 Genera and forms

Definition 5.44. A polynomial map $P : \mathrm{Mat}(N, \mathbb{C}) \to \mathbb{C}$ is said to be *invariant* if
$$P\big(T^{-1}XT\big) = P(X)$$
for all $X \in \mathrm{Mat}(N, \mathbb{C})$ and for all $T \in \mathrm{GL}(N, \mathbb{C})$.

Let $(E, \nabla) \to M$ be a $\mathbb{C}$-vector bundle, and recall that the curvature of ∇ is
$$R(X,Y)v = \nabla_X\nabla_Y v - \nabla_Y\nabla_X v - \nabla_{[X,Y]}v\,.$$

Definition 5.45 (Connection one-form and curvature two-form). Let (u_j) be a frame of E inside $U \subset M$. Then the connection one-form $\omega_i^j \in \mathrm{C}^\infty(U, \Lambda^1 M)$ is given by
$$\nabla_X u_i = \sum_{i=1}^N \omega_i^j(X) u_j\,.$$
The curvature 2-form is $\Omega_i^j \in \mathrm{C}^\infty(U, \Lambda^2 M)$ with
$$R(X,Y)u_i = \sum_{j=1}^N \Omega_i^j(X,Y)u_i\,.$$

Note that since $R(X,Y) = -R(Y,X)$, Ω_i^j is is indeed a local 2-forms and (Ω_i^j) is a matrix of local 2-forms.

Indeed, the connection one-form and curvature two-forms are related by the readily verified expression

$$\Omega_i^j = d\omega_i^j + \sum_k \omega_k^j \wedge \omega_i^k .$$

Definition 5.46 (Even forms). Define

$$\Lambda^{\mathrm{ev}} M := \bigoplus_{k \in \mathbb{N}} \Lambda^{2k} M$$

Lemma 5.47. *$(\Lambda^{\mathrm{ev}} M, \wedge)$ is a commutative algebra.*

Proof. For forms $\alpha \in \Lambda^k M$ and $\beta \in \Lambda^l M$, we have that

$$\alpha \wedge \beta = (-1)^{k+l} \beta \wedge \alpha .$$

On $\Lambda^{\mathrm{ev}} M$ the forms are all even, the conclusion follows. □

Proposition 5.48. *An invariant polynomial map $P : \mathrm{Mat}(N, \mathbb{C}) \to \mathbb{C}$ defines a*

$$P\big((\Omega_i^j)\big) \in \Lambda^{\mathrm{ev}} M .$$

We shall not go into the guts of this statement, but the intuition is that scalar multiplication is simply replaced by the wedge-product. This is a well-defined operation by Lemma 5.47, which guarantees that $(\Lambda^{\mathrm{ev}} M, \wedge)$ is a commutative algebra.

Example 5.49. The following is the quintessential polynomial map that we will work with. Let

$$P(X) = \det(X) = \sum_{\sigma \in S_N} \mathrm{sgn}(\sigma) \prod_{i=1}^{N} X_{i,\sigma(i)} .$$

Then the induced action on a curvature 2-form written out locally as Ω_i^j is given by

$$P\big((\Omega_i^j)\big) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \bigwedge\nolimits_{i=1}^{N} \Omega_i^{\sigma(i)} .$$

Proposition 5.50. *If (η_i) and $(\tilde{\eta}_i)$ are two local frames in $U \cap \tilde{U}$. Then for any invariant polynomial map P,*

$$P\big((\Omega_i^j)\big) = P\Big(\big(\tilde{\Omega}_i^j\big)\Big) \in \mathrm{C}^\infty\Big(U \cap \tilde{U}, \Lambda^{\mathrm{ev}} M\Big) .$$

Proof. The transformation between (Ω^i_j) and $(\tilde{\Omega}^j_i)$ is precisely given by an invertible T such that

$$(\tilde{\Omega}^i_j) = T^{-1}(\Omega^i_j)T\,.$$

By the invariant condition, the conclusion follows. □

Exercise 5.51. Verify that there exists an invertible T such that $(\tilde{\Omega}^i_j) = T^{-1}(\Omega^i_j)T$.

Corollary 5.52. *Letting Ω be a curvature 2-form, we obtain*

$$P(\Omega) \in \mathrm{C}^\infty(M, \Lambda^{\mathrm{ev}}M)$$

for any invariant polynomial map P.

Remark 5.53. This shows that, as opposed to the frame-dependent object Ω^j_i, we are able to extract a global object $P(\Omega)$ due to the invariant nature of P.

Recall that

$$0 \to \mathrm{C}^\infty\big(M, \Lambda^0 T^*M\big) \xrightarrow{d} \mathrm{C}^\infty\big(M, \Lambda^1 M\big) \to \cdots \to \mathrm{C}^\infty(M, \Lambda^n M)\,.$$

Since $d^2 = 0$, this is a co-chain complex.

Notation 5.54. The form $\eta \in \mathrm{C}^\infty(M, \Lambda M)$ is called *closed* if $d\eta = 0$ and *exact* if $\eta = d\vartheta$ for some $\vartheta \in \mathrm{C}^\infty(M, \Lambda M)$.

Definition 5.55 (de Rham cohomology). Let $d_k := d|_{\mathrm{C}^\infty(M,\Lambda^k M)} : \mathrm{C}^\infty\big(M, \Lambda^k M\big) \to \mathrm{C}^\infty\big(M, \Lambda^{k+1} M\big)$ and define

$$\mathrm{H}^k_{\mathrm{dR}}(M) := {}^{\ker(d_k)}\!\big/_{\mathrm{ran}(d_{k-1})}\,.$$

Theorem 5.56. *For P an invariant polynomial, given a curvature 2-form Ω, $P(\Omega)$ is closed. Consequently,*

$$[P(\Omega)] \in \bigoplus_{k\geq 0} \mathrm{H}^{2k}_{\mathrm{dR}}(M) =: \mathrm{H}^{\mathrm{ev}}_{\mathrm{dR}}(M)\,.$$

The class $[P(\Omega)]$ is independent of Ω.

Remark 5.57. On a manifold with boundary, it may be that $P(\Omega) \neq 0$ but $[P(\Omega)] = 0$. Suppose that $[P(\Omega)] = 0$. Then, $P(\Omega)$ is exact, i.e., $P(\Omega) = d\vartheta$ for some $\vartheta \in \mathrm{C}^\infty(M, \Lambda M)$. On integrating,

$$\int_M P(\Omega) = \int_M d\vartheta = \int_{\partial M} \theta\,,$$

using Stokes' Theorem. Therefore, integration of classes is not well-defined on a manifold with boundary, as it is the case on a boundaryless manifold. As a consequence, in our setting, it is important to keep track of the form, and not just the class as it is in the case with closed manifolds.

Definition 5.58. Let $P : \mathrm{Mat}(n, \mathbb{C}) \to \mathbb{C}$ an invariant polynomial map of degree k and Ω a curvature 2-form. Then

$$P(\Omega) \in \mathrm{C}^\infty\big(M, \Lambda^{2k} M\big)\,,$$

and this is called the *Chern-Weil form* of Ω via P.

$$[P(\Omega)] \in \mathrm{H}^{2k}_{\mathrm{dR}}\big(M, \Lambda^{2k} M\big)$$

is the associated Chern-Weil class.

Remark 5.59. The Chern-Weil class $[P(\Omega)]$ is independent of Ω. Therefore, it is often denoted by $P(E)$, emphasising its dependency on the bundle E.

Of particular geometric significance, particularly to index theory, are the following form and class constructed out of a particular polynomial map.

Definition 5.60 (Chern form/class). Let

$$c(A) = \det\left(I + \frac{1}{2\pi i} A\right).$$

Then, the form

$$c(\Omega) \in \Lambda^{\mathrm{ev}} M$$

is called the *total Chern form* and

$$c(E) := [c(\Omega)] \in \mathrm{H}^{\mathrm{ev}}_{\mathrm{dR}}(M, \mathbb{C})$$

is the *total Chern class* of E.

Remark 5.61. Since $c(\Omega) \in \Lambda^{\mathrm{ev}} M$, we can write

$$c(\Omega) = 1 + c_1(\Omega) + \ldots, +c_n(\Omega)\,,$$

where
$$c_k(\Omega) \in \mathrm{C}^\infty\big(M, \Lambda^{2k} M\big)\,.$$

Lemma 5.62. *Let Ω be a connection 2-form and Ω^* the 2-from corresponding to the induced connection*

$$\nabla^* : \mathrm{C}^\infty(M, E^*) \to \mathrm{C}^\infty(M, T^*M \otimes E^*)$$

satisfying
$$d(\varphi^*(\psi))(X) = (\nabla^*_X \varphi^*)(\psi) + \varphi^*(\nabla_X \psi)\,.$$

Then,
$$c_k(\Omega^*) = (-1)^k c_k(\Omega)\,.$$

Proof. Exercise. □

Lemma 5.63. *If (∇, h) is a compatible connection and metric on $E \to M$, then $c(\Omega)$ is real-valued.*

Proof. Let $\Omega = \big(\Omega^i_j\big)$ be written with respect to an orthonormal frame. By compatibility, $\Omega = \big(\Omega^i_j\big)$ is skew-Hermitian. Therefore,

$$\begin{aligned}\overline{c(\Omega)} &= \overline{\det\left(1 + \frac{1}{2\pi i}\Omega\right)} = \det\left(1 - \frac{1}{2\pi i}\overline{\Omega}\right) = \det\left(\left(1 - \frac{1}{2\pi i}\overline{\Omega}\right)^T\right) \\ &= \det\left(1 - \frac{1}{2\pi i}\Omega^*\right) = \det\left(1 + \frac{1}{2\pi i}\Omega\right) = c(\Omega)\,.\end{aligned}$$
□

5.8.2 Real bundles, Pontryagin forms and classes

So far, we have considered the Chern-Weil constructions for complex bundles. In applications, it is of vital importance to consider the real case. The real case can be accessed through complexification and the real nature of the bundle results in extra structural features of the associated constructions.

To proceed, let $V \to M$ be a real bundle and $E := V \otimes_{\mathbb{R}} \mathbb{C}$ its complexification.

Lemma 5.64. *Let (∇, h) be a compatible connection and metric on V and let $\nabla^{\mathbb{C}}$ and $h^{\mathbb{C}}$ denote the canonical extensions to $V \otimes_{\mathbb{R}} \mathbb{C}$. Then for the induced 2-form $\Omega_{\mathbb{C}}$ we have*
$$c_k(\Omega_{\mathbb{C}}) = c_k(\Omega^*_{\mathbb{C}})\,.$$

Proof. Since Ω is real, by the compatibility of the metric and connection, in an orthonormal frame, Ω can be seen to be skew-symmetric. The complexification $\Omega_{\mathbb{C}}$ retains the skew-symmetry, improving on the skew-hermitian condition, which we saw is valid in the complex case. Then, a direct calculation yields the conclusion.□

Corollary 5.65. *For (∇, h) as in the lemma, $c_k(\Omega_{\mathbb{C}}) = 0$ if k is odd.*

Proof. From Lemma 5.62, we have that $c_k(\Omega_{\mathbb{C}}^*) = (-1)^k c_k(\Omega_{\mathbb{C}})$. Combining this with Lemma 5.64, we obtain

$$c_k(\Omega_{\mathbb{C}}) = c_k(\Omega_{\mathbb{C}}^*) = (-1)^k c_k(\Omega_{\mathbb{C}})\,.$$

Clearly, this can only be satisfied if $c_k(\Omega_{\mathbb{C}}) = 0$ for k odd. □

These observations then allow us to define the Chern-Weil analogue in the real setting.

Definition 5.66 (Pontryagin forms/classes). For $(V, \nabla, h) \to M$ real vector bundle with (∇, h) compatible, define the k-th *Pontryagin form*

$$P_k(\Omega) := (-1)^k c_{2k}(\Omega_{\mathbb{C}}) \in \mathrm{C}^\infty\big(M, \Lambda^{4k} M\big)$$

and the *total Pontryagin form*

$$P(\Omega) := \sum_{k=0}^{\infty} P_0(\Omega) \in \mathrm{C}^\infty\big(M, \Lambda^{4*} M\big)\,.$$

Similarly,

$$P_k(V) := [P_k(\Omega)] \in \mathrm{H}^{4k}_{\mathrm{dR}}(M) \qquad \text{and} \qquad P(V) := [P(\Omega)] \in \mathrm{H}^{4*}_{\mathrm{dR}}(M)\,.$$

Remark 5.67. Note that as a consequence of Lemma 5.63, all of the objects in the definition are real-valued.

Definition 5.68. Let $f(x) := \sum_{k=0}^{\infty} f_k x^k \in \mathbb{R}[\![x]\!]$ be a formal power series. Define

$$\Lambda_f : \mathrm{C}^\infty(M, \Lambda^{\mathrm{ev}} M) \to \mathrm{C}^\infty(M, \Lambda^{\mathrm{ev}} M)$$

by

$$\Lambda_f\big|_{\mathrm{C}^\infty\left(M, \Lambda^{2k} M\right)} := (-1)^{k+1} k f_k \,\mathrm{id}_{\mathrm{C}^\infty\left(M, \Lambda^{2k} M\right)}\,.$$

Recall that $\log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \dots$. For a fixed degree form, only finitely many terms occur in the expansion. Consider

$$\begin{aligned}\log(P(\Omega)) &= \log(1 + P_1(\Omega) + P_2(\Omega) + \dots) \\ &= P_1(\Omega) + P_2(\Omega) - \frac{P_1(\Omega)^2}{2} + \dots\end{aligned}$$

Definition 5.69. For $f(x) := \sum_{k=0}^{\infty} f_k x^k \in \mathbb{R}[\![x]\!]$, define

$$\mathbb{P}_f(\Omega) := \exp(\Lambda_{\log \circ f}(\log(P(\Omega)))) \in \mathrm{C}^\infty\left(M, \Lambda^{4*}M\right),$$

this is called the *multiplicative characteristic form.*

5.8.3 Geometric applications in even dimensions

In the last section, we saw that the Chern-Weil forms are valued in $\Lambda^{\mathrm{ev}}M$ and the Pontryagin forms land in $\Lambda^{4*}M$. Later on, we will see that it is of importance to be able to integrate the top order form to relate geometric quantities to boundary conditions. Therefore, from now on, unless otherwise stated, we assume that (M, g) is compact with even dimension $n = 2m$. Clearly, $\dim(\partial M) = 2m - 1$ and we continue to assume that $g = dt \otimes dt + g_{\partial M}$ (i.e. is product) in a neighbourhood of ∂M.

The Spin case

The first applications we consider is in the context of Spin geometry. Therefore, in addition to the assumptions we have already made on M, we suppose that it is Spin. More precisely, we assume M admits a Spin structure. In particular, this means that M is orientable.

Since M is even dimensional, we have two nontrivial canonical irreducible representations $\varrho^\pm$. The Spin bundle then splits as

$$\not{\Delta}M = \not{\Delta}^+ M \oplus \not{\Delta}^- M$$

to the *positive* and *negative half-spinors.*

Moreover, $\dim\left(\not{\Delta}M\right) = 2^{\frac{n}{2}} = 2^m$ and $\dim\left(\not{\Delta}^\pm M\right) = 2^{\frac{n-2}{2}} = 2^{m-1}$. The Spin-Dirac operator decomposes, respecting the splitting into the positive and negative half-spinors as

$$\not{D} = \begin{pmatrix} 0 & \not{D}^- \\ \not{D}^+ & 0 \end{pmatrix} \qquad \text{with} \qquad \not{D}^\pm : \mathrm{C}^\infty\left(M, \not{\Delta}^\pm M\right) \to \mathrm{C}^\infty\left(\not{\Delta}^\mp M\right).$$

Since g is product near the boundary, we have that $\not{D} = \sigma_0(\partial_t + A)$ near ∂M with A a self-adjoint adapted boundary operator. A simple calculation then yields

$$A = \begin{pmatrix} A^+ & 0 \\ 0 & A^- \end{pmatrix},$$

where $A^{\pm}$ are adapted operators for $D^{\pm}$ respective. The symbol also splits as

$$\sigma_0 = \begin{pmatrix} 0 & \sigma_0^- \\ \sigma_0^+ & 0 \end{pmatrix}$$

and we obtain that

$$\not{D}^+ = \sigma_0^+ \left(\partial_t + A^+\right).$$

Lemma 5.70. *$\not{\Delta}^+ M|_{\partial M} \to \partial M$ can be identified with $\not{\Delta}\partial M \to \partial M$ and A^+ with the spin-Dirac operator on ∂M.*

Definition 5.71. Let

$$\hat{a}(x) := \frac{\frac{\sqrt{x}}{2}}{\sinh\left(\frac{\sqrt{x}}{2}\right)} = 1 - \frac{x}{24} + \frac{7x^2}{5760} + \dots .$$

We define he *A-hat* or *A-roof* form associated with a curvature 2-form Ω to be:

$$\hat{A}(\Omega) := \mathbb{P}_{\hat{a}}(\Omega)$$

Remark 5.72. Note that

$$\log(\hat{a}(x)) = \frac{-x}{24} + \frac{x^2}{2880} + \dots .$$

Therefore,

$$\begin{aligned} \hat{A}(\Omega) =& \exp[\Lambda_{\log \circ \hat{a}}(\log P(\Omega))] \\ =& 1 - \frac{P_1(\Omega)}{24} + \frac{7P_1(\Omega)^2 - 4P_2(\Omega)}{5760} + \dots . \end{aligned}$$

This is the crucial object that is required for us to give a geometric interpretation of the APS theorem in the Spin manifold context.

Theorem 5.73. *(M, g) is a Riemannian spin manifold with boundary of dimension $n = 2m$ and $g = dt \otimes dt + g_{\partial M}$ near ∂M. Then*

$$\int_M \alpha_0(x)\, \mathrm{d}\mu_g(x) = \int_M \hat{A}(\Omega)$$

and

$$\operatorname{ind}\left(\not{D}^+_{APS}\right) = \int_M \hat{A}(\Omega) - \frac{\dim(\ker(A^+)) + \eta_{A^+}(0)}{2}.$$

Remark 5.74. 1. Note that, since it only makes sense to integrate an n-form on an n-manifold, we are implicitly writing

$$\int_M \hat{A}(\Omega) := \int_M \hat{A}_n(\Omega)\,.$$

2. Let us compare this with the Atiyah-Singer index theorem on (M', g) with $\partial M' = \varnothing$. In this context, the index theorem reads

$$\operatorname{ind}\left(\not{D}^+\right) = \int_M \hat{A}(TM)\,,$$

where

$$\hat{A}(TM) = \left[\hat{A}(\Omega)\right].$$

This object is topological.

Therefore, in the absence of boundary, the index theorem says that

$$\text{analytical index} = \text{topological index}\,.$$

Boundary no longer permits this expression, due to the reasons we gave in Remark 5.57. Therefore, in the boundary situation, the analytic index is described by a geometric integral in the way of the $\hat{A}$ form of Ω and a boundary term in the way of $\eta_A(0)$.

Exercise 5.75. Let (E, ∇, h) be a hermitian bundle with (∇, h) compatible. The twisted bundle splits $\not{\Delta} M \otimes E = \not{\Delta}^+ M \otimes E \oplus \not{\Delta}^- M \otimes E$ and $\not{D}_E = \begin{pmatrix} 0 & \not{D}_E^- \\ \not{D}_E^+ & 0 \end{pmatrix}$.

Taking this exercise for granted, we have the following straightforward extension of the previous theorem. Note that here, the integral over $c(\Omega) \wedge \hat{A}(\omega)$ is again by selecting the n-th order term.

Theorem 5.76. *Let (E, ∇, h) be a hermitian bundle with (∇, h) compatible and constant in the normal direction in a neighbourhood of ∂M. Then, for the twisted spin-Dirac operator $\not{D}_E$ on $\not{\Delta} M \otimes E$, we have*

$$\operatorname{ind}\left(\not{D}^+_{E,\mathrm{APS}}\right) = \int_M c(\Omega) \wedge \hat{A}(\Omega) - \frac{\dim\left(\ker\left(A_E^+\right)\right) + \eta_{A_E^+}(0)}{2}\,.$$

With the aid of Theorem 5.73, we are able to obtain the following consequences in various dimensions where the $\hat{A}$-form vanishes. Let us remind ourselves that $n = 2m$.

1. Suppose that m is odd. In other words, $n = 2(k+1) = 4k+2 \equiv 2 \mod 4$. Then
$$\hat{A}_n(\Omega) = \hat{A}_{4k+2}(\Omega) \notin \mathrm{C}^\infty\big(M, \Lambda^{4*}M\big)\,,$$
and so $\hat{A}_n(\Omega) = 0$.

2. Now consider the case $m \equiv 1 \mod 4$. That is $m = 4k+1$ or $n = 2k+2$. Then $\dim(\partial M) = n-1 = 8k+1$. The operator A^+ can be identified with the Spin-Dirac operator on $\not{\Delta}\partial M$, and moreover, we know that
$$A^+ = iB\,,$$
where B is real skew-adjoint. Therefore $\operatorname{spec}(A^+)$ is symmetric (with multiplicity) which yields $\eta_{A^+} = 0$. Also $\hat{A}_n(\Omega) = \hat{A}_{8k+2}(\Omega) = 0$, so the index theorem yields
$$\dim\big(\ker\big(A^+\big)\big) = -2\operatorname{ind}\Big(\not{D}^+_{\mathrm{APS}}\Big)$$
In fact, through methods beyond our discussion here, $\dim(\ker(A^+)) \mod 2$ is an invariant of ∂M independent of the metric. Therefore, this yields an analytic proof that it vanishes for ∂M Spin.

3. It remains to consider $m \equiv 3 \mod 4$, which occurs if and only if $m = 4k+3$ or $n = 8k+6$. Through similar considerations as before, we have that
$$\hat{A}_n(\Omega) = \hat{A}_{8k+6}(\Omega) = 0\,.$$
From the classification of the Clifford algebras we know that the Spin bundle in this dimension has a *quaternionic structure* Q (particularly an isomorphism on spinors) satisfying the anti-commutativity relation
$$QA^+ = -A^+Q\,.$$
Using Q in place of $\sigma_D(x, \vec{N})$ in the proofs of II) and III) in Proposition 5.33, we obtain that $\operatorname{spec}(A^+)$ is symmetric (counting multiplicities) and therefore, $\eta_{A^+} = 0$. In this case, the index theorem says
$$-2\operatorname{ind}\Big(\not{D}^+_{\mathrm{APS}}\Big) = \dim\big(\ker\big(A^+\big)\big)\,,$$
or that $\dim(\ker(A^+))$ is even dimensional.

The Signature theorem

In this section, we return back to a more general setting. Let (M, g) be an Riemannian manifold with boundary ∂M. The additional structure we require is that M is *orientable*. Note then that the volume density μ_g is, in fact, an n-form. For the moment, we do not assume that M is compact.

First, we define the following metric on differential forms.

Definition 5.77. For $\alpha, \beta \in \Lambda^k M$ with $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_k$ and $\beta = \beta_1 \wedge \cdots \wedge \beta_k$ pure products of covectors, define $g_*(\alpha, \beta) = \det\Big((g(\alpha_i, \beta_j))_{i,j=1}^n\Big)$. Then extend to the whole of ΛM via linearity.

Definition 5.78 (Hodge-star). Define $* : \Lambda M \to \Lambda M$ as the operator satisfying

$$\alpha \wedge *\beta = g_*(\alpha, \beta)\mu_g$$

for all $\alpha, \beta \in \Lambda^k M$, extended by linearity to $* : \Lambda M \to \Lambda M$.

Lemma 5.79. *The Hodge-star operator $* : \Lambda^k M \to \Lambda^{n-k} M$ is an isometry that satisfies*

$$**|_{\Lambda^k M} = (-1)^{n(n-k)} \operatorname{id}_{\Lambda^k M} .$$

For the purposes of this section, the Hodge-star operator gives us with the ability to describe the formal adjoint d_g^* in the following manner.

Lemma 5.80. *The operator d_g^* can be written as*

$$d_g^*|_{\Lambda^k M} = (-1)^{n(n-1)+1} *d* = (-1)^k *^{-1} d* .$$

We omit the proofs of these lemmata as they are standard facts. They are both readily verified.

Notation 5.81. Recall that a bounded invertible map T on a Banach space is called an *involution* if $T^{-1} = T$.

Definition 5.82. Let $\dim(M) = n = 2m$. Then define an involution $\Upsilon : \Lambda M \to \Lambda M$ by

$$\Upsilon\varphi := \imath^{p(p-1)+m} *\varphi$$

for $\varphi \in \Lambda^p M$.

Remark 5.83. Note that in the definition of the involution, we have that $m = \frac{n}{2}$ appearing in the exponent. Therefore, the assumption that $n = 2m$ is central to our discussion from here on.

Exercise 5.84. Show that $\Upsilon : \Lambda M \to \Lambda M$ has spectrum $\operatorname{spec}(\Upsilon) = \{\pm 1\}$ and $\operatorname{Eig}_\Upsilon(-1) \cong \operatorname{Eig}_\Upsilon(1)$.

As a consequence of Exercise 5.84, Υ decomposes ΛM as a sum of its eigenspaces. Therefore, define

$$\Lambda^{\pm} M := \mathrm{Eig}_{\Upsilon}(\pm 1)\,.$$

From this, it is immediate that

$$\Lambda M = \Lambda^{+} M \oplus \Lambda^{-} M\,.$$

Lemma 5.85. *The involution Υ anti-commutes with the Hodge-Dirac operator $D_H = d + d_g^*$. That is,*

$$\Upsilon D_H = \Upsilon\big(d + d_g^*\big) = -D_H \Upsilon\,.$$

Proof. This follows from Lemma 5.80 for expressing d_g^* in terms of d and $*$. We leave the verification of this as an exercise. ☐

Corollary 5.86. *We have*

$$D_H|_{\mathrm{C}^\infty(M,\Lambda^+ M)} : \mathrm{C}^\infty\big(M, \Lambda^+ M\big) \to \mathrm{C}^\infty\big(M, \Lambda^- M\big)\,.$$

As a consequence of these considerations, we are now finally able to define the following operator. It is the protagonist of this subsection.

Definition 5.87. On (M, g) Riemannian manifold of dimension $n = 2m$, let

$$S := D_{\mathrm{H}}|_{\mathrm{C}^\infty(M,\Lambda^+ M)}$$

to be the signature operator.

Proposition 5.88. *We have that $S \in \mathrm{Diff}_1(\Lambda^+ M, \Lambda^- M)$ is elliptic.*

Although our discussion up to this point only required even dimensions, in what is to come, we assume that the dimension is divisible by 4. We begin with the following observation.

Proposition 5.89. *Let (M, g) be a Riemannian manifold of dimension $n = 4k$. Suppose that g is product near ∂M. Then, the following hold.*

I) The bundle $\Lambda^+ M|_{\partial M}$ can be identified with $\Lambda \partial M$.

II) The signature operator decomposes as

$$S = \sigma(\partial_t + A)\,,$$

where A is the self-adjoint adapted boundary operator for S with

$$A|_{\Lambda^p \partial M} = (-1)^{k+p+1}(\varepsilon_p *_{\partial M} d_{\partial M} - d_{\partial M} *_{\partial M}) .$$

Here, $*_{\partial M} : \Lambda \partial M \to \Lambda \partial M$ is the Hodge-star operator on $\Lambda \partial M$, $d_{\partial M}$ is the exterior derivative on $\Lambda \partial M$, and $\varepsilon_p = 1$ for a $2p$ form and $\varepsilon_p = -1$ for $(2p-1)$-forms of $\Lambda \partial M$.

III) The operator A commutes with $\varphi \mapsto (-1)^p *_{\partial M} \varphi$ on $\Lambda^p \partial M$ preserving parity. Therefore, we obtain the splitting

$$A = \begin{pmatrix} A^{\mathrm{ev}} & 0 \\ 0 & A^{\mathrm{odd}} \end{pmatrix} : \mathrm{C}^\infty\big(\partial M, \Lambda^{\mathrm{ev}} \partial M \oplus \Lambda^{\mathrm{odd}} \partial M\big) \\ \to \mathrm{C}^\infty\big(\partial M, \Lambda^{\mathrm{ev}} \partial M \oplus \Lambda^{\mathrm{odd}} \partial M\big) .$$

Proof sketch. The expression for A simply comes from the definition of S through a calculation. Also note that the expression for A requires $\dim(M) = 4k$, and this is what allows us to assert the preservation of parity and commutativity with $\varphi \mapsto (-1)^p * \varphi$. Consequently, we obtain the decomposition of the operator. □

Corollary 5.90. *The even part A^{ev} is obtained through composing A^{odd} with an isomorphism and therefore,*

$$\eta_A(s) = 2\eta_{A^{\mathrm{ev}}}(s) = 2\eta_{A^{\mathrm{odd}}}(s) .$$

Proof. Since $A^{\mathrm{ev}} = A^{\mathrm{odd}} \circ \Phi$, we have that $\operatorname{spec}(A^{\mathrm{ev}}) = \operatorname{spec}(A^{\mathrm{odd}})$ up to multiplicity. Therefore, the conclusion follows. □

In order to describe a geometric consequence via the index theorem, we require a polynomial of the curvature.

Definition 5.91 (Hirzebruch L-polynomial). Let $\ell(x)$ be the formal power series of

$$\frac{\sqrt{x}}{\tanh(\sqrt{x})} .$$

Then, for a curvature 2-form Ω, define

$$L(\Omega) := \mathbb{P}_\ell(\Omega) = \exp(\Lambda_{\log \circ \ell}(\log(P(\Omega)))) \in \mathrm{C}^\infty\big(M, \Lambda^{4*} M\big) .$$

This is called the *Hirzebruch L-polynomial* of Ω.

This term then appears as a geometric contribution in the application of the APS theorem to yield the following.

Theorem 5.92 (Local index theorem for the Singature operator). *On an oriented Riemannian manifold of* $\dim(M) = 4k$*, with* g *product near* ∂M*,*

$$\operatorname{ind}(S_{\mathrm{APS}}) = \operatorname{ind}\big(S_{B_{\mathrm{APS}}(A)}\big) = \int_M L(\Omega) - (\dim(\ker(A^{\mathrm{ev}})) + \eta_{A^{\mathrm{ev}}}(0))\,.$$

Corollary 5.93 (Signature theorem). *The quantity*

$$\operatorname{ind}(S_{\mathrm{APS}}) + \dim(\ker(A^{\mathrm{ev}})) = \operatorname{sign}(M)\,,$$

where $\operatorname{sign}(M)$ *is the signature of a non-degenerate quadratic form in the cohomology* $\mathrm{H}^{2k}(M)$*. Therefore,*

$$\operatorname{sign}(M) = \int_M L(\Omega) - \eta_{A^{\mathrm{ev}}}(0)\,.$$

Remark 5.94. This is a truly remarkable formula. The signature

$$\operatorname{sign}(M)$$

is a topological invariant while

$$\int_M L(\Omega)$$

is a differential geometric object. The remaining quantity

$$\eta_{A^{\mathrm{ev}}}(0)$$

is a spectral invariant. That is, if we scale g, the object $\eta_{A^{\mathrm{a}}}(s)$, a priori, is also altered. However, only the the non-zero eigenvalues move and the kernel remains fixed. Therefore, $\eta_{A^{\mathrm{ev}}}(0)$ remains invariant. It is remarkable that three objects, one measuring topology, one geometry and one an aspect of the boundary appear in a single formula.

6 Elliptically regular boundary conditions

In Chapter 5, we saw that significance of the APS boundary condition in its applicability to geometry and topology. We have already seen that this boundary condition is elliptically regular. Motivated by this, and through applications that we will present in this chapter, we will obtain a deep structural understanding of all such elliptically regular boundary conditions. Our discussion can be carried out in a very general context. Therefore, on returning to our assumptions earlier, unless we explicitly state otherwise, let us fix the following background assumptions on the objects appearing in this Chapter.

(I) (M, μ) measured manifold with ∂M compact.

(II) $\left(E, h^E\right), \left(F, h^F\right) \to M$ Hermitian bundles.

(III) $D : \mathrm{C}^\infty(M, E) \to \mathrm{C}^\infty(M, F)$ elliptic.

(IV) D and $D^\dagger$ are complete (i.e. compactly supported smooth sections permitted to kiss the boundary are dense in the maximal domain).

Let us further recall some terminology from earlier in this text. Applying Definition 3.75 to the first-order case, we see that a boundary condition $B \subset \check{\mathrm{H}}(D)$ (i.e. closed subspace of $\check{\mathrm{H}}(D)$) is *elliptically regular* if

$$B \subset \mathrm{H}^{\frac{1}{2}}(\partial M, E) \qquad \text{and} \qquad B^\dagger \subset \mathrm{H}^{\frac{1}{2}}(\partial M, F)$$

where $B^\dagger := \left\{ v|_{\partial M} \;\middle|\; v \in \mathrm{dom}(D^*_B) \right\}$.

Similarly, in Definition 3.78, we called a boundary condition B Fredholm if D_B is a Fredholm operator.

The definition we have just recalled of an elliptically regular boundary condition is of a qualitative nature. These boundary conditions have a very intricate and rich structure. The goal of this section is to obtain an extremely powerful and useful characterisation of such boundary conditions which reveals these structural features.

We have already seen that such boundary conditions are plentiful and arise naturally. Explicitly, given any inward pointing vectorfield T on ∂M and any adapted boundary operator A with

$$\sigma_A(x, \xi) = \sigma_D(x, dt)^{-1} \circ \sigma_D(x, \xi)\,,$$

in Definition 5.9, we defined the APS boundary condition

$$B_{\mathrm{APS}}(A) = \chi^-(A_a)\mathrm{H}^{\frac{1}{2}}(\partial M, E)\,.$$

Here, $a = \frac{1}{2}\min\{-\lambda_{-1}, \lambda_1\}$ where $\lambda_j \in \mathrm{Re}(\mathrm{spec}(A))$.

In Proposition 5.10, we proved that $B_{\mathrm{APS}}(A)$ is elliptically regular. In its proof, we conveniently used the fact that $\check{\mathrm{H}}(D) = \check{\mathrm{H}}_{A_a}(D_0)$, allowing us to compute via using the boundary condition to describe the Czech space, the space of all possible boundary conditions.

Motivated by this, we consider a class of projectors which are the basis of much of our considerations of this chapter.

Definition 6.1 (Boundary decomposing projector). We say that $\mathcal{P}_+$ is *boundary decomposing* for D if

(I) $\mathcal{P}_+ : \mathrm{H}^\alpha(\partial M, E) \to \mathrm{H}^\alpha(\partial M, E)$ is a projector for $\alpha \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

(II) $\mathcal{P}_+ : \check{\mathrm{H}}(D) \to \check{\mathrm{H}}(D)$ projector and $\mathcal{P}_- := (I - \mathcal{P}_+) : \check{\mathrm{H}}(D) \to \mathrm{H}^{\frac{1}{2}}(\partial M, E)$.

(III) $\|u\|_{\check{\mathrm{H}}(D)} \simeq \|\mathcal{P}_- u\|_{\mathrm{H}^{\frac{1}{2}}(\partial M, E)} + \|\mathcal{P}_+ u\|_{\mathrm{H}^{-\frac{1}{2}}(\partial M, E)}$.

Remark 6.2. Condition (III) is actually not required, it is implied by (II) since $\mathcal{P}_-$ lands in $\mathrm{H}^{\frac{1}{2}}(\partial M, E)$. Therefore, by the open mapping theorem, the norms $\mathrm{H}^{\frac{1}{2}}(\partial M, E)$ and $\mathrm{H}^{-\frac{1}{2}}(\partial M, E)$ are equivalent on $\mathcal{P}_-\check{\mathrm{H}}(D)$.

Example 6.3. 1. $\mathcal{P}_+ = \chi^+(A)$ for A invertible bisectorial adapted boundary operator for D.

2. When M is compact, let $\mathcal{P}_{\mathcal{C}} : \mathrm{H}^{\frac{1}{2}}(\partial M, E) \to \mathrm{H}^{-\frac{1}{2}}(\partial M, E)$ be a Calderón projector. That is, a projector such that

$$\mathcal{C} = \mathcal{P}_{\mathcal{C}}\mathrm{H}^{-\frac{1}{2}}(\partial M, E) = \left\{ u|_{\partial M} \;\middle|\; u \in \ker(D_{\max}) \right\}.$$

Then $\mathcal{P}_+ := \mathcal{P}_{\mathcal{C}}$ is boundary decomposing.

Motivated by our analysis of $\check{\mathrm{H}}_A(D_0)$ and its accompanying space $\hat{\mathrm{H}}_A(D)$, we define the following.

Definition 6.4. Let

$$\hat{\mathrm{H}}_{\mathcal{P}_+(D)} := \mathcal{P}_+^* \mathrm{H}^{\frac{1}{2}}(\partial M, E) \oplus \mathcal{P}_-^* \mathrm{H}^{-\frac{1}{2}}(\partial M, E)$$

with

$$\|u\|^2_{\hat{\mathrm{H}}_{\mathcal{P}_+}(D)} := \left\|\mathcal{P}_+^* u\right\|^2_{\mathrm{H}^{\frac{1}{2}}(\partial M, E)} + \left\|\mathcal{P}_-^* u\right\|^2_{\mathrm{H}^{-\frac{1}{2}}(\partial M, E)}.$$

Notation 6.5. If $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2$ is a Banach space, let $\mathcal{P}_{B_1, B_2}$ be the projection to B_1 along B_2 and define $\mathcal{B}_1^* := \mathcal{P}^*_{\mathcal{B}_1, \mathcal{B}_2}\mathcal{B}^*$ and $\mathcal{B}_2^* := \mathcal{P}^*_{\mathcal{B}_2, \mathcal{B}_2}\mathcal{B}^*$.

Definition 6.6 (Graphical L^2-decomposition). Let $B \subset \check{\mathrm{H}}(D)$ be a boundary condition and $\mathcal{P}_+$ a boundary decomposing projector. Suppose that:

(I) There exist mutually complementary subspaces $W_\pm$ and $V_\pm$ of $L^2(\partial M, E)$ such that
$$W_\pm \oplus V_\pm = \mathcal{P}_\pm \mathrm{L}^2(\partial M, E)\,.$$

(II) $W_\pm, W_\pm^* \subset \mathrm{H}^{\frac{1}{2}}(\partial M, E)$ and are finite dimensional.

(III) There exists a bounded linear map
$$g : V_- \to V_+$$
s.t.
$$g\Big(V_- \cap \mathrm{H}^{\frac{1}{2}}(\partial M, E)\Big) \subset V_+ \cap \mathrm{H}^{\frac{1}{2}}(\partial M, E)\,,$$
$$g^*\Big(V_+^* \cap \mathrm{H}^{\frac{1}{2}}(\partial M, E)\Big) \subset V_-^* \cap \mathrm{H}^{\frac{1}{2}}(\partial M, E)\,,$$
and
$$\begin{aligned} B &= \mathrm{graph}\Big(g|_{\mathrm{H}^{\frac{1}{2}}(\partial M, E)}\Big) \oplus W_+ \\ &= \Big\{v + gv \,\Big|\, v \in V\_ \cap \mathrm{H}^{\frac{1}{2}}(\partial M, E)\Big\} \oplus W_+\,. \end{aligned}$$

Then we say that B is L^2-*graphically decomposable* w.r.t. $\mathcal{P}_+$.

The fundamental theorem of this chapter is the following.

Theorem 6.7 (Equivalence of ellpitic regularity and L^2-graphical decomposition). *Let $\mathcal{P}_+$ be a boundary decomposing projector. Then, a boundary condition $B \subset \check{\mathrm{H}}(D)$ is elliptically regular if and only if B is L^2-graphically decomposable w.r.t. $\mathcal{P}_+$. In this case,*
$$\sigma_0^*\big(B^\dagger\big) = B^{\perp, \hat{\mathrm{H}}_{\mathcal{P}_+}(D)} = \Big\{u - g^* u \,\Big|\, u \in V_+^* \cap \mathrm{H}^{\frac{1}{2}}(\partial M, E)\Big\}\,.$$

The proof of this theorem is somewhat complicated, consisting of many smaller intricate pieces. We will present the proof later. First, we will consider some examples and an extremely important application of this decomposition - the relative index theorem à la Gromov-Lawson.

Example 6.8. 1. Let A be an adapted boundary operator for D and $a = \frac{1}{2}\min\{-\lambda_{-1}, \lambda_1\}$, $\lambda_j \in \mathbb{R}(\mathrm{spec}(A))$, as we have already used in defining the APS boundary condition with respect to A. Then $A_a = A - a$ is invertible bisectorial, and $\mathcal{P}_+ = \chi^+(A)$ is a boundary decomposing projector. Let
$$V_\pm := \chi^\pm(A_a)\mathrm{L}^2(\partial M, E) \qquad \text{and} \qquad W_\pm := \{0\}$$

and $g : V_- \to V_+$ be given by $g(v) = 0$. Then

$$
\begin{aligned}
B &:= \operatorname{graph}\Big(g|_{\mathrm{H}^{\frac{1}{2}}}\Big) \oplus W_+ = \Big\{ v + 0v \;\Big|\; v \in V_- \cap \mathrm{H}^{\frac{1}{2}}(\partial M, E) \Big\} \\
&= \chi^-(A_a)\mathrm{L}^2(\partial M, E) \cap \mathrm{H}^{\frac{1}{2}}(\partial M, E) = \chi^-(A_a)\mathrm{H}^{\frac{1}{2}}(\partial M, E) = B_{\mathrm{APS}}(A)\,.
\end{aligned}
$$

2. In Example 1., we can choose $r \neq a$. It is easily seen that the difference between $\chi^{\pm}(A_r)\mathrm{L}^2(\partial M, E)$ and $\chi^{\pm}(A_a)\mathrm{L}^2(\partial M, E)$ are a finite dimensional space of smooth sections corresponding to the span of generalised eigenspaces sandwiched between the spectral cuts at r and a. Therefore, to describe $B_{\mathrm{APS}}(A)$ with respect to $P_+ := \chi^+(A_r)$, the spaces W_+ and W_- need to be chosen appropriately. In this case, they will contained in smooth sections.

3. Suppose now that M compact, and let $\mathcal{P}_+ := \mathcal{P}_{\mathcal{C}}$ be a Calderón projector. Then with the choice of

$$
V_{\pm} := \mathcal{P}_{\pm}\big(\mathrm{L}^2(\partial M, E)\big) \qquad \text{and} \qquad W_{\pm} := \{0\}
$$

with $g : V_- \to V_+$ given by $g(v) := 0$, we obtain that

$$
B := \operatorname{graph}(g) \cap \mathrm{H}^{\frac{1}{2}}(\partial M, E) \oplus W_+ = (I - \mathcal{P}_{\mathcal{C}})\mathrm{H}^{\frac{1}{2}}(\partial M, E)\,.
$$

From Theorem 6.7, B is elliptically regular. Note that by construction, $B \subset \mathrm{H}^{\frac{1}{2}}(\partial M, E)$. Therefore, the nontrivial conclusion we obtain from invoking Theorem 6.7 is that $B^{\dagger} \subset \mathrm{H}^{\frac{1}{2}}(\partial M, F)$.

Note that $\ker(D_B) = \ker(D_{\min})$ since $\mathcal{C} = \Big\{ u|_{\partial M} \;\Big|\; u \in \ker(D_{\max}) \Big\}$ and $(I - \mathcal{P}_{\mathcal{C}})\mathcal{C} = 0$.

4. In 3., let us modify the construction by choosing $W_+ \subset \mathcal{C} \cap \mathrm{H}^{\frac{1}{2}}(\partial M, E)$ finite dimensional, ensuring $W_+^* \subset \mathrm{H}^{\frac{1}{2}}(\partial M, E)$ and then defining

$$
B := (I - \mathcal{P}_{\mathcal{C}})B \oplus W_+\,.
$$

Then,

$$
\ker(D_B) = \ker\big(D_{W_+}\big) = \Big\{ u \in \ker(D_{\max}) \;\Big|\; u|_{\partial M} \in W_+ \Big\}\,.
$$

Remark 6.9. Example 4. is of significance to boundary value problems. We have previously discussed that, from a modern perspective, it is useful to divorce the ‘boundary value’ part from the ‘problem’ aspect. However, from 6.8 4., at least historically, it is clear how these aspects might be seen hand-in-hand.

In 6.8 4., what we have is a boundary condition B, with prescribed solution space W_+, yielding D_B. I.e.,

$$
Du = 0 \qquad \text{with} \qquad u|_{\partial M} \in W_+\,.
$$

The advantage of the operator D_B over D_{W+} is that the former is a Fredholm operator while the latter is not.

Early on in the theory of boundary value problems, particularly from the work of Seeley in [**Seely66**], Calderón projectors were the only boundary decomposing projectors available, and therefore, it is tempting to think of boundary value problems without separating these into two aspects. We see, however, from a global analysis point of view, as in 6.8 1., the spectral projectors of associated adapted boundary operators yield an alternative geometry for $\check{\mathrm{H}}(D)$, better suited to another class of problems.

The real power of Theorem 6.7 is that it allows us to decompose an elliptically regular boundary condition from any imposed geometry on $\check{\mathrm{H}}(D)$, i.e. for any boundary decomposing projector.

6.1 The relative index theorem

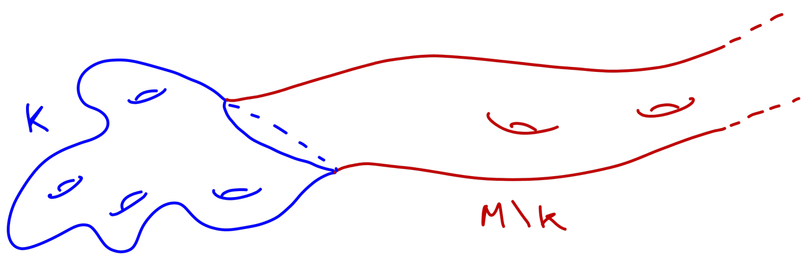
A significant application of Theorem 6.7 is is relative index theorem. Introduced by Gromov and Lawson in their seminal paper [23] on the study of positive scalar curvature. This is a topic that has recently seen a revival and is currently an active area of research.

In application, the relative index theorem needs to be applied to the non-compact setting, even though the guiding questions are confined to the study of positive scalar curvature on closed manifolds. Given a closed manifold, a noncompact version is created with this compact manifold as the boundary, and the so-called relative index of this non-compact manifold is then used to understand the space of positive curvature metrics on the compact boundary. Since we are forced to deal with the index of operators on non-compact manifolds, we require the operators to be Fredholm. Therefore, in addition to the usual requirement of completeness, we require further control of the operator at infinity. This leads us to the following definition.

Definition 6.10. $D \in \mathrm{Diff}_1(E, F)$ elliptic is *coercive at infinity* if there exists a compact set $K \subset M$ and $C < \infty$ s.t.

$$\|u\|_{\mathrm{L}^2(M,E)} \leq C\|Du\|_{\mathrm{L}^2(M,F)}$$

for all $u \in \mathrm{C}^\infty_{\mathrm{c}}(M, E)$ with $\mathrm{spt}\, u \subset M \setminus K$.



Example 6.11. 1. For M compact, choosing $K = M$, every elliptic D is coercive at infinity.

2. Suppose that (M, g) is Riemannian Spin manifold with compact boundary ∂M and $D = \not{D}$ is the Spin-Dirac operator. Then, the Weitzenböck formula reads

$$\not{D}^\dagger \not{D} u = \not{\nabla}^\dagger \not{\nabla} u + \frac{1}{4} \operatorname{scal}_g(u)\,,$$

where scal_g is the scalar curvature of g.

Suppose that $\operatorname{scal}_g \geq \kappa > 0$ outside $K' \subset M$ and set $K := K' \cup \overline{B(\partial M, \varepsilon)}$. Let $u \in \mathrm{C}^\infty_{\mathrm{c}}(M, E)$ with $\operatorname{spt}(u) \subset M \setminus K$. Then

$$\begin{aligned}\left\|\not{D}u\right\|^2 = \left\langle \not{D}^\dagger \not{D} u, u\right\rangle = \left\langle \not{\nabla}^\dagger \not{\nabla} u, u\right\rangle + \frac{1}{4}\langle \operatorname{scal}_g(u), u\rangle \\ \geq \left\|\not{\nabla} u\right\|^2 + \frac{1}{4}\kappa\|u\|^2 \geq \frac{\kappa}{4}\|u\|^2\,.\end{aligned}$$

3. More generally, if (M, g) is Riemannian (not necessarily Spin) with compact boundary and $D : \mathrm{C}^\infty(M, E) \to \mathrm{C}^\infty(M, F)$ is a Dirac-type operator, then a Weitzenböck formula

$$D^\dagger D = \nabla^\dagger \nabla + \mathcal{K}$$

holds. Here, $\mathcal{K}$ is a symmetric endomorphism field, which is typically related to curvature. If this endomorphism is bounded blow by κ on a compact subset $K \subset M$, meaning that

$$h^E(x)[\mathcal{K}(x)u, u] \geq \kappa |u|^2_{h^E(x)}\,,$$

then mirroring the argument in Exercise 2., we obtain that D is coercive at infinity.

Lemma 6.12. *D is coercive at infinity iff there does not exist a sequence $(u_n) \subset \mathrm{C}^\infty_{\mathrm{cc}}(M, E)$ satisfying*

(I) $\|u_n\|_{\mathrm{L}^2(M,E)} = 1$.

(II) $\lim_{n\to\infty} \|Du_n\|_{\mathrm{L}^2(M,F)} = 0$.

(III) For all $\tilde{K} \subset M$ compact there is a (large) number $N_{\tilde{K}}$ s.t. for all $n \geq N_{\tilde{K}}$ we have $\operatorname{spt}(u_n) \cap \tilde{K} = \varnothing$.

Proof. We prove the statement by the equivalent contrapositive. That is, we show that D is not coercive at infinity if and only if such a sequence as stated exists.

First, let us assume that D is not coercive at infinity. Let $K_1 \subset K_2 \subset \cdots \subset M$ be an exhaustion of M with K_j compact. By our assumption that D is not coercive at infinity, for each $j \in \mathbb{N}$ there is a $v_j \in \mathrm{C}^\infty_{\mathrm{c}}(M, E)$ such that $\operatorname{spt}(v_j) \cap K_j = \varnothing$ and

$$\|v_j\| \geq j\|Dv_j\|\,.$$

Let $u_j := v_j/\|v_j\|$ so that $\|u_j\| = 1$. This shows (I) and (III). Moreover,

$$\|Du_j\| \leq \frac{1}{j}$$

and so $\|Du_j\| \to 0$ as $j \to \infty$. This establishes (III).

To prove the converse, suppose such a sequence exists. Then, it is easy to see that Definition 6.10 is violated. □

Proposition 6.13. *If B is semi-elliptically regular, i.e. B is a boundary condition and $B \subset \mathrm{H}^{\frac{1}{2}}(\partial M, E)$. Then D is coercive at infinity iff D_B has finite dimensional kernel and closed range.*

Proof. a) Let D be coercive at infinity, and let u_n be a bounded sequence in $\mathrm{dom}(D_B)$ s.t. $Du_n \to v$. We prove that, possibly on passing to a subsequence, there exists $u \in \mathrm{dom}(D_B)$ such that $Du = v$.

Let $K \subset M$ a compact subset from the definition, and let $\chi \in \mathrm{C}^\infty_\mathrm{c}(M, [0,1])$ with $\chi = 1$ on K and $K' := \mathrm{spt}(\chi)$. Similarly, let $\tilde\chi \in \mathrm{C}^\infty_\mathrm{c}(M, [0,1])$ with $\tilde\chi = 1$ on K'.

Since B is elliptically regular, from Proposition hyperref[itm:ellBCfromBoundary3]5.1 (III), we obtain that $\mathrm{dom}(D_B) \subset \mathrm{H}^1_{\mathrm{loc}}(M, E)$. Therefore, fixing a connection ∇ on E,

$$\begin{aligned} \|u\|_{\mathrm{H}^1(K',E)} &\leq \|\nabla(\tilde\chi u)\|_{\mathrm{L}^2(M,E)} + \|\tilde\chi u\|_{\mathrm{L}^2(M,E)} \\ &\lesssim \|D_B(\tilde\chi u)\|_{\mathrm{L}^2(M,F)} + \|\tilde\chi u\|_{\mathrm{L}^2(M,E)} \end{aligned} \tag{6.1}$$

for all $u \in \mathrm{dom}(D_B)$. Now, recall that

$$D_B(\chi u) = \chi D_B u + \sigma_D(\cdot, d\chi)u\,,$$

and $\mathrm{spt}\,\sigma_D(\cdot, d\chi) \subset K'$. Therefore,

$$\begin{aligned} \|\chi(u_m - u_n)\|_{\mathrm{L}^2(M,E)} &\leq \|\chi(u_m - u_n)\|_{\mathrm{H}^1(K',E)} \\ &\leq \|D(\tilde\chi\chi(u_m - u_n))\|_{\mathrm{L}^2(M,F)} + \|\tilde\chi\chi(u_m - u_n)\|_{\mathrm{L}^2(M,E)} \\ &\lesssim \|\chi D_B(u_m - u_n)\|_{\mathrm{L}^2(K',F)} + \|\sigma_D(\cdot, d\chi)(u_m - u_n)\|_{\mathrm{L}^2(K',E)} \\ &\qquad + \|\chi(u_m - u_n)\|_{\mathrm{L}^2(K',E)} \\ &\lesssim \|D_B(u_m - u_n)\|_{\mathrm{L}^2(M,F)} + \|\chi(u_m - u_n)\|_{\mathrm{L}^2(K',E)} \\ &\qquad + \|\chi(u_m - u_n)\|_{\mathrm{L}^2(K',E)} \\ &\lesssim \|D_B(u_m - u_n)\|_{\mathrm{L}^2(M,F)} + \|\chi(u_m - u_n)\|_{\mathrm{L}^2(K',E)}\,, \end{aligned}$$

where we used (6.1) in the second inequality and that $\tilde\chi = 1$ on $\mathrm{spt}\,\chi = K'$.

By hypothesis, $\|D_B(u_m - u_n)\|_{\mathrm{L}^2(M,F)} \to 0$ as $m, n \to \infty$. It remains to obtain convergence of the term $\|\chi(u_m - u_n)\|_{\mathrm{L}^2(K',E)}$. Note that, since K' is compact, $\mathrm{H}^1(K', E)$ embeds compactly into $\mathrm{L}^2(K', E)$. Since $\chi u_n \in \mathrm{H}^1(K', E)$, we can pass to a subsequence and obtain $u_{K'} \in \mathrm{L}^2(K', E)$ such that $u_n \to u_{K'}$. Therefore, on passing to a subsequence, we obtain that $\|(\chi u_m - u_n)\|_{\mathrm{L}^2(K',E)} \to 0$ as $m, n \to \infty$.

We now fix this subsequence, and we slightly abuse denoting it also by u_n. We compute

$$\begin{aligned}\|u_m - u_n\|_{\mathrm{L}^2(M,E)} &\leq \|\chi(u_m - u_n)\|_{\mathrm{L}^2(M,E)} + \|(1-\chi)(u_m - u_n)\|_{\mathrm{L}^2(M,E)} \\ &\lesssim \|\chi(u_m - u_n)\|_{\mathrm{L}^2(K',E)} + \|D((1-\chi)(u_m - u_n))\|_{\mathrm{L}^2(M,F)} \\ &\leq \|\chi(u_m - u_n)\|_{\mathrm{L}^2(K',E)} + \|\sigma_D(\cdot, d\chi)(u_m - u_n)\|_{\mathrm{L}^2(M,E)} \\ &\qquad + \|(1-\chi)D(u_m - u_n)\|_{\mathrm{L}^2(M,F)} \\ &\leq \|\chi(u_m - u_n)\|_{\mathrm{L}^2(K',E)} + \|D(u_m - u_n)\|_{\mathrm{L}^2(M,F)}\,,\end{aligned}$$

where in the second inequality, we used the coercive at infinity property since $\mathrm{spt}(1-\chi)(u_n - u_m) \subset M \setminus K$. From our earlier estimate, along with the fact that $Du_n \to v$, we obtain that the right hand side, and hence $\|u_m - u_n\|_{\mathrm{L}^2(M,E)} \to 0$ as $m, n \to \infty$. Therefore, we obtain $u \in \mathrm{L}^2(M,E)$ s.t. $u_m \to u$, $Du_m \to v$ and since D_B is closed, we conclude $u \in \mathrm{dom}(D_B)$ and $v = D_B u$.

b) We prove the remaining implication by contraposition. For that, suppose D is not coercive. Then by Lemma 6.12, we obtain $u_n \in \mathrm{C}^\infty_{\mathrm{cc}}(M,E)$ which, on taking larger and larger compact subsets K_j, have $\mathrm{spt}\big(u_{n_j}\big) \cap K_j = \varnothing$. So, $\big\langle u_{n_j}, f\big\rangle \to 0$ for all $f \in \mathrm{C}^\infty_{\mathrm{cc}}(M,E)$. Since $\|u_n\| = 1$, we have no convergent subsequence in $\mathrm{dom}(D_B)$ and in particular $\mathrm{L}^2(M,E)$. But $\big\|D_B u_{n_j}\big\| \to 0$ and if D_B had finite dimensional kernel and closed range, then every bounded sequence $\|v_n\| \leq C$ with $D_B v_n \to w$ has a convergent subsequence. Clearly this is a contradiction and so we conclude that D_B either has infinite kernel or the range is not closed. ☐

Corollary 6.14. *If $D, D^\dagger$ are coercive at infinity, and B is elliptically regular, then D_B is Fredholm and*

$$\mathrm{ind}(D_B) = \dim(\ker(D_B)) - \dim\Big(\ker\Big(D^\dagger_{B^\dagger}\Big)\Big) \in \mathbb{Z}\,.$$

Proof. Proposition 6.13 tells us that D_B is a Fredholm operator. On application of Lemma 5.4, we obtain

$$\mathrm{ind}(D_B) = \dim(\ker(D_B)) - \dim(\ker(D_B^*))\,.$$

From Corollary 4.125, we have that $D^\dagger_{B^\dagger} = D^*_B$. ☐

6.1.1 Deformations of boundary conditions

A virtue of Fredholm operators is that their index is stable under continuous deformations. In the context of boundary conditions, it is useful to understand how continuous deformations of boundary conditions yield to an appropriate continuous deformations of the operator itself. We formalise this in the following definition.

Definition 6.15. A family of boundary conditions $B_s \subset \check{\mathrm{H}}(D)$ for $s \in [0,1]$ is a continuous deformation from B_0 to B_1, if there exist isomorphisms

$$\varphi_s : B_0 \to B_s \qquad \text{with} \qquad \varphi_0 = \mathrm{id}$$

with $s \to \varphi_s \in \mathscr{B}\big(B_0, \check{\mathrm{H}}(D)\big)$ continuous.

Remark 6.16. The continuity condition can be more compactly written as $\varphi_\cdot \in \mathrm{C}^0\big([0,1], \mathscr{B}\big(B_0, \check{\mathrm{H}}(D)\big)\big)$.

Exercise 6.17. Suppose that B_s are elliptically regular for all $s \in [0,1]$. Then, $\varphi_s \in \mathscr{B}\Big(B_0, \mathrm{H}^{\frac{1}{2}}(\partial M, E)\Big)$.

The way in which continuous deformations of boundary conditions yield continuous deformations of operators is precisely captured in the following proposition.

Proposition 6.18. *Suppose that $s \mapsto \varphi_s$ is a continuous deformation of boundary conditions. Then there exists a $\Phi_s \in \mathscr{B}(\mathrm{dom}(D_{B_0}), \mathrm{dom}(D_{\max}))$ continuous in s s.t. $\Phi_s \,\mathrm{dom}(D_{B_0}) = \mathrm{dom}(D_{B_s})$.*

Proof. Recall that ${}^{\mathrm{dom}(D_{\max})}\!\big/_{\mathrm{dom}(D_{\min})} \cong \check{\mathrm{H}}(D)$. Since $\mathrm{dom}(D_{B_s}) \subset \mathrm{dom}(D_{\max})$ is a closed subspace and ${}^{\mathrm{dom}(D_{B_s})}\!\big/_{\mathrm{dom}(D_{\min})} \subset {}^{\mathrm{dom}(D_{\max})}\!\big/_{\mathrm{dom}(D_{\min})}$ we obtain that

$${}^{\mathrm{dom}(D_{B_s})}\!\big/_{\mathrm{dom}(D_{\min})} \cong B_s$$

in the sense of Banach spaces with the constant in the isomorphism independent of B_s.

Now,

$$\begin{aligned}
\mathrm{dom}(D_{B_s}) &\cong {}^{\mathrm{dom}(D_{B_s})}\!\big/_{\mathrm{dom}(D_{\min})} \oplus \mathrm{dom}(D_{\min}) \\
&\cong B_s \oplus \mathrm{dom}(D_{\min}) \\
&\cong \varphi_s(B_0) \oplus \mathrm{dom}(D_{\min}) \\
&\cong B_0 \oplus \mathrm{dom}(D_{\min}) \\
&\cong {}^{\mathrm{dom}(D_{B_0})}\!\big/_{\mathrm{dom}(D_{\min})} \oplus \mathrm{dom}(D_{\min}) \\
&\cong \mathrm{dom}(D_{B_0}) \oplus \mathrm{dom}(D_{\min})\,.
\end{aligned}$$

In the fourth isomorphism, $s \mapsto \varphi_s$ is continuous and determines $\Phi_s : \mathrm{dom}(D_{B_0}) \to \mathrm{dom}(D_{B_s})$ continuously. The conclusion follows. $\square$

Exercise 6.19. Describe the map Φ_s explicitly from ϕ_s.

Corollary 6.20. *Let $\mathcal{P}_+$ be a boundary decomposing projector. Let B be an elliptically regular boundary condition, and write*

$$B = \operatorname{graph}\left(g|_{\mathrm{H}^{\frac{1}{2}}}\right) \oplus W_+$$

via Theorem 6.7 with respect to $\mathcal{P}_+$. Define

$$\varphi_s : B_0 = V_- \oplus W_+ \to B_s := \operatorname{graph}\left(sg|_{\mathrm{H}^{\frac{1}{2}}}\right) \oplus W_+,$$
$$\varphi_s(v + w_+) := v + sgv + w_+ \,,$$

where we recall $V_\pm \oplus W_\pm = \mathcal{P}_\pm \mathrm{L}^2(\partial M, E)$ from Definition 6.6.

Then $s \mapsto \varphi_s : B_0 \to \check{\mathrm{H}}(D)$ is a continuous deformation of boundary conditions and

$$\operatorname{ind}(D_{B_0}) = \operatorname{ind}(D_{B_s}) = \operatorname{ind}(D_B)\,.$$

Proof. It is immediate from construction that $s \mapsto \varphi_s \in \mathrm{C}^0([0,1], \mathscr{B}(B_0, \check{\mathrm{H}}(D))) = \mathrm{C}^0([0,1], \mathscr{B}(B_0, \mathrm{H}^{\frac{1}{2}}(\partial M, E))$ and it is an isomorphism for each s. Also, $\varphi_0 = \mathrm{id}$ and therefore, it is a continuous deformation of boundary conditions.

Let $\Phi_s : \operatorname{dom}(D_{B_0}) \to \operatorname{dom}(D_{B_s})$ be the induced isomorphism from Prop 6.18. Then, we $D_{B_s} \circ \Phi_s : \operatorname{dom}(D_{B_0}) \to \mathrm{L}^2(M, E)$ is bounded, continuous in s, and therefore,

$$\operatorname{ind}(D_{B_s} \circ \Phi_s) = \operatorname{ind}(B_0)$$

since the index is invariant under a continuous deformation. But since Φ_s is an isomorphism, the dimension of the kernel and cokernel remains unchanged, so therefore $\operatorname{ind}(D_{B_s}) = \operatorname{ind}(D_{B_s} \circ \Phi_s)$. □

A consequence of this corollary is that we are able to relate $\operatorname{ind}(D_B)$ to $\operatorname{ind}\left(D_{\mathcal{P}_- \mathrm{H}^{\frac{1}{2}}(\partial M, E)}\right)$. If we choose A bisectorial invertible and let $\mathcal{P}_+ = \chi^+(A)$, then $\mathcal{P}_-\left(\mathrm{H}^{\frac{1}{2}}\right) = B_{\mathrm{APS}}(A)$. Therefore, relating these two indices means in effect that we are relating the index of a general boundary condition B to a generalised APS boundary condition given a boundary decomposing projector $\mathcal{P}_+$.

We begin with the following useful technical lemma.

Lemma 6.21. *Let $X \subset Y \subset \mathcal{H}$ be closed subspaces of a Hilbert space $\mathcal{H}$. Then*

$$\mathcal{H}/X \cong Y/X \oplus \mathcal{H}/Y\,.$$

Proof. Since $\mathcal{H}$ is a Hilbert space, we find complementary subspaces (i.e., by taking orthogonal complements) s.t.

$$Y = X \oplus X' \text{ and } \mathcal{H} = Y \oplus Y'\,.$$

Therefore,

$$\mathcal{H}/X \cong (Y \oplus Y')/X \cong (X \oplus X' \oplus Y')/X \cong X' \oplus Y' \cong Y/X \oplus \mathcal{H}/Y\,. \qquad \square$$

Recall that we say a boundary condition B is Fredholm if D_B is a Fredholm operator.

Proposition 6.22. *Let $B_1 \subset B_2$ be Fredholm boundary conditions. Then, $\dim(B_2/B_1) < \infty$ and*

$$\operatorname{ind}(D_{B_2}) = \operatorname{ind}(D_{B_1}) + \dim\left(B_2/B_1\right).$$

Proof. The condition $B_1 \subset B_2$ is equivalent to $D_{B_1} \subset D_{B_2}$. Therefore, $\operatorname{dom}(D_{B_1}) \subset \operatorname{dom}(D_{B_2})$, $\ker(D_{B_1}) \subset \ker(D_{B_2})$ and $\operatorname{ran}(D_{B_1}) \subset \operatorname{ran}(D_{B_2})$. Note that the ranges are closed because B_i are Fredholm boundary conditions.

Since these are Hilbert spaces, we find orthogonal complements

$$\ker(D_{B_2}) = \ker(D_{B_1}) \oplus^{\perp} K \ \text{ and } \ \operatorname{dom}(D_{B_i}) = \ker(D_{B_i}) \oplus^{\perp} R_i\,.$$

Note that $R_i \cong \operatorname{ran}(D_{B_i})$ via $D : R_i \to \operatorname{ran}(D_{B_i})$ and therefore, $R_1 \subset R_2$ let

$$R_2 = R_1 \oplus^{\perp} R\,.$$

Therefore,

$$\operatorname{dom}(D_{B_2}) = \ker(D_{B_2}) \oplus R_2 = \ker(D_{B_1}) \oplus K \oplus R_1 \oplus R = \operatorname{dom}(D_{B_1}) \oplus K \oplus R\,.$$

Now

$$B_2/B_1 \cong \operatorname{dom}(D_{B_2})/\operatorname{dom}(D_{B_1}) \cong K \oplus R\,,$$

where the first isomorphism is readily verified, and the second follows from our construction above.

We prove that B_2/B_1 is finite dimensional. It suffices to prove that K and R are finite dimensional. First, we note that K is finite dimensional, since $\ker(D_{B_2})$ is finite dimensional by Fredholmness of B_2.

To show R is finite dimensional, note

$$\begin{aligned}\operatorname{coker}(D_{B_1}) &\cong \mathrm{L}^2(M,E)/\operatorname{ran}(D_{B_1}) \cong \mathrm{L}^2(M,E)/\operatorname{ran}(D_{B_2}) \oplus \operatorname{ran}(D_{B_2})/\operatorname{ran}(D_{B_1}) \\ &\cong \operatorname{coker}(D_{B_2}) \oplus R_2/R_1 \cong \operatorname{coker}(D_{B_2}) \oplus R\,,\end{aligned}$$

where the second isomorphism uses Lemma 6.21. By the Fredholmness of B_1, $\operatorname{coker}(D_{B_1})$ is finite dimensional and hence R is finite dimensional.

Now we prove the index formula in the conclusion. We simply calculate:

$$\begin{aligned}\operatorname{ind}(D_{B_2}) &= \dim(\ker(D_{B_2})) - \dim(\operatorname{coker}(D_{B_2})) \\ &= \dim(\ker(D_{B_1})) + \dim(K) - \dim(\operatorname{coker}(D_{B_1})) + \dim(R) \\ &= \operatorname{ind}(D_{B_1}) + \dim(K \oplus R) \\ &= \operatorname{ind}(D_{B_1}) + \dim\left(B_2/B_1\right). \qquad \square\end{aligned}$$

Theorem 6.23. *Let $\mathcal{P}_+$ be a boundary decomposing projector, B elliptically regular and let $W_\pm$ be the subspaces arising from the graphical decomposition of B with respect to $\mathcal{P}_+$ in Definition 6.6. Assume further that $D, D^\dagger$ are coercive at infinity. Then letting $B_- := \mathcal{P}_- \mathrm{H}^{\frac{1}{2}}(\partial M, E)$, we have that*

$$\operatorname{ind}(D_B) = \operatorname{ind}\big(D_{B_-}\big) + \dim(W_+) - \dim(W_-)\,,$$

Proof. Using Theorem 6.7, we write

$$B = \operatorname{graph}(g|_{\mathrm{H}^{\frac{1}{2}}}) \oplus W_+$$

where $\mathcal{P}_\pm \check{\mathrm{H}}(D) = V_\pm \oplus W_\pm$. Let $B_0 := V_- \oplus W_+$ as in Corollary 6.20 and from there, we obtain

$$\operatorname{ind}(D_B) = \operatorname{ind}(D_{B_0})\,.$$

Now let us consider $B_- \oplus W_+$, which is an elliptically regular boundary condition since $W_+ \subset \mathrm{H}^{\frac{1}{2}}$ is finite dimensional and $B_- = \mathcal{P}_- i\mathrm{H}^{\frac{1}{2}}(\partial M, E)$. Since $B_- \oplus W_+ \supset B_0$ and

$$B_- \oplus W_+ \big/ B_0 = (V_- \oplus W_- \oplus W_+) \big/ (V_- \oplus W_+) \cong W_-\,,$$

using Proposition 6.22,

$$\operatorname{ind}\big(D_{B_- \oplus W_+}\big) = \operatorname{ind}(D_{B_0}) + \dim(W_-)\,.$$

Also, $B_- \subset B_- \oplus W_+$ and therefore,

$$\operatorname{ind}\big(D_{B_- \oplus W_+}\big) = \operatorname{ind}(B_-) + \dim(W_+)\,.$$

Combining these two equations, we get

$$\operatorname{ind}(D_{B_0}) + \dim(W_-) = \operatorname{ind}\big(D_{B_-}\big) + \dim(W_+)$$

which is the formula appearing in the conclusion. □

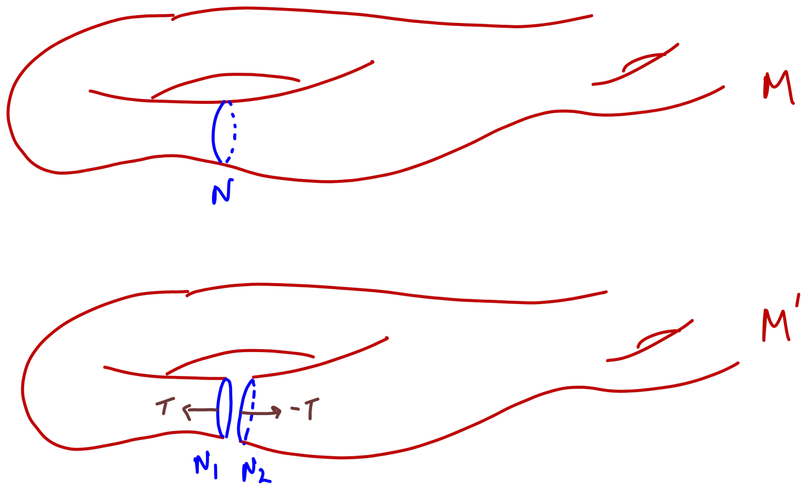
Remark 6.24. This allows us to understand and compute the index of a general elliptically regular boundary condition via $\mathcal{P}_- \mathrm{H}^{\frac{1}{2}}(\partial M, E)$ of any boundary decomposing projector $\mathcal{P}_+$ of our choice. We will see later, that the spaces W_+ and W_- can be explicitly described as $W_+ = B \cap \mathcal{P}_+\Big(\mathrm{H}^{-\frac{1}{2}}(\partial M, E)\Big)$ and $W_- \cong B^\perp \cap \mathcal{P}_-^*\Big(\mathrm{H}^{-\frac{1}{2}}(\partial M, E)\Big)$.

6.1.2 Splittings and decompositions

In this subsection, we consider how the index of an operator on a boundaryless manifold can be related to an induced operator on this manifold by cutting along a hypersurface to produce a manifold with boundary.To fix notation, let M be a connected manifold with $\partial M = \varnothing$. Let $N \subset M$ be a two sided compact hypersurface in M (i.e. N has a trivial normal bundle). Then we can cut M along N, to obtain a manifold

$$M' := (M \setminus N) \cup (N_1 \sqcup N_2)\,,$$

where $N_1 = N$, $N_2 = -N$ (i.e. with opposite orientation) and with $\partial M' = N_1 \sqcup N_2$.



Remark 6.25. As the image shows, cutting along N need not necessarily force M' to be disconnected.

Given a density μ on M and bundles $E, F \to M$, there are the naturally and canonically induced objects μ', E', F' via pullback to M'. If $D \in \mathrm{Diff}_1(E, F)$, then we obtain $D' \in \mathrm{Diff}_1(E', F')$.

Proposition 6.26. *We have*

$$\mathrm{L}^2(\partial M', E) = \mathrm{L}^2(N_1, E) \oplus^{\perp} \mathrm{L}^2(N_2, E) = \mathrm{L}^2(N, E) \oplus^{\perp} \mathrm{L}^2(N, E)\,.$$

Suppose that A_0 is an adapted boundary operator on $N_1 = N$. Then, $-A_0$ is an adapted boundary operator on N_2 and $A = A_0 \oplus -A_0$ is an adapted boundary operator on $\partial M' = N_1 \sqcup N_2$. Moreover,

$$\check{\mathrm{H}}(D') = \check{\mathrm{H}}_A(D) = \Big(\chi^-(A_0)\mathrm{H}^{\frac{1}{2}}(N, E) \oplus \chi^+(A_0)\mathrm{H}^{-\frac{1}{2}}(N, E)\Big) \\ \oplus \Big(\chi^+(A_0)\mathrm{H}^{\frac{1}{2}}(N, E) \oplus \chi^-(A_0)\mathrm{H}^{-\frac{1}{2}}(N, E)\Big)\,.$$

Proof. We leave the majority of these assertions as an exercise, only noting that

$$\check{\mathrm{H}}_A(D) = \Big(\chi^-(A_0)\mathrm{H}^{\frac{1}{2}}(N_1, E) \oplus \chi^+(A_0)\mathrm{H}^{-\frac{1}{2}}(N_2, E)\Big) \\ \oplus \Big(\chi^-(-A_0)\mathrm{H}^{\frac{1}{2}}(N, E) \oplus \chi^+(-A_0)\mathrm{H}^{-\frac{1}{2}}(N, E)\Big)\,,$$

from which the formula for $\check{\mathrm{H}}(D)$ follows. ◻

Lemma 6.27. *$D, D^\dagger$ are complete and coercive at infinity iff D' and $(D')^\dagger$ are.*

Proof. This follows from Definition 6.10 using that N is compact. We leave the verification of this assertion as an exercise. □

Definition 6.28 (Matching condition). The boundary condition

$$B_{\mathrm{M}} := \left\{ (u, u) \;\middle|\; u \in \mathrm{H}^{\frac{1}{2}}(N, E) \right\}$$

is called the *matching condition.*

The following is of paramount importance. It also demonstrates the usefulness and power of the graphical decomposition.

Lemma 6.29. *B_{M} is elliptically regular.*

Proof. Fix A_0, an invertible bisectorial adapted boundary operator. In light of Proposition 6.26, define:

$$\begin{aligned} V_- &:= \chi^-(A_0)\mathrm{L}^2(N, E) \oplus \chi^+(A_0)\mathrm{L}^2(N, E)\,, \\ V_+ &:= \chi^+(A_0)\mathrm{L}^2(N, E) \oplus \chi^-(A_0)\mathrm{L}^2(N, E)\,, \\ W_\pm &:= \{0\}\,. \end{aligned}$$

Moreover, define $g : V_- \to V_+$ by

$$g = \begin{pmatrix} & \mathrm{id} \\ \mathrm{id} & \end{pmatrix}.$$

Then,

$$\begin{aligned} \operatorname{graph}\left(g|_{\mathrm{H}^{\frac{1}{2}}}\right) &= \left\{ \begin{pmatrix} u \\ v \end{pmatrix} + g\begin{pmatrix} u \\ v \end{pmatrix} \;\middle|\; u, v \in \mathrm{H}^{\frac{1}{2}}(N, E) \right\} \\ &= \left\{ \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} v \\ u \end{pmatrix} \;\middle|\; u, v \in \mathrm{H}^{\frac{1}{2}}(N, E) \right\} \\ &= \left\{ \begin{pmatrix} u + v \\ u + v \end{pmatrix} \;\middle|\; u, v \in \mathrm{H}^{\frac{1}{2}}(N, E) \right\} \\ &= B_{\mathrm{M}}\,. \end{aligned}$$

By Theorem 6.7, we have that B_{M} is elliptically regular. □

Lemma 6.30. *Let A_0 be an invertible bisectorial adapted boundary operator on N, which means that $A := A_0 \oplus (-A_0)$ is an invertible bisectorial adapted boundary operator on $\partial M = N_1 \sqcup N_2$. Then*

$$B_{\mathrm{APS}}(A) = \mathrm{H}^{\frac{1}{2}}(N, E)$$

and

$$\operatorname{ind}\big(D'_{B_{\mathrm{M}}}\big) = \operatorname{ind}\big(D_{B_{\mathrm{APS}}(A)}\big)\,.$$

Proof. We have

$$\begin{aligned} B_{\mathrm{APS}}(A) &= \chi^-(A)\mathrm{H}^{\frac{1}{2}}(\partial M, E) \\ &= \chi^-(A_0 \oplus (-A_0))\mathrm{H}^{\frac{1}{2}}(N_1 \sqcup N_2, E) \\ &= \chi^-(A_0)\mathrm{H}^{\frac{1}{2}}(N_1, E) \oplus \chi^-(-A_0)\mathrm{H}^{\frac{1}{2}}(N_2, E) \\ &= \chi^-(A_0)\mathrm{H}^{\frac{1}{2}}(N, E) \oplus \chi^+(A_0)\mathrm{H}^{\frac{1}{2}}(N, E) \\ &= \mathrm{H}^{\frac{1}{2}}(N, E)\,. \end{aligned}$$

Now, for the choices of $V_\pm$ used in Lemma 6.29, namely Let $V_\pm := \chi^\pm(A_0)\mathrm{L}^2(N_1, E) \oplus \chi^\mp(A_0)\mathrm{L}^2(N_2, E)$ and for g as defined there, define

$$B^s_{\mathrm{M}} := \operatorname{graph}\Big(sg|_{\mathrm{H}^{\frac{1}{2}}}\Big)\,.$$

Clearly this is a continuous deformation of B_{M} to $B^0_{\mathrm{M}} = B_{\mathrm{APS}}(A)$ and so by Corollary 6.20, the conclusion follows. □

Theorem 6.31 (Splitting theorem). *Let M, μ, E, F and M', μ', E', F' be as above, and $D \in \mathrm{Diff}_1(E, F)$ elliptic. Suppose that D, D' are complete and coercive at infinity.*

Suppose that (on identifying $\mathrm{H}^{\frac{1}{2}}(N_i, E)$ with $\mathrm{H}^{\frac{1}{2}}(N, E)$) that

$$B_1 \oplus B_2 = \mathrm{H}^{\frac{1}{2}}(N, E)\,.$$

Then

$$\operatorname{ind}(D) = \operatorname{ind}\big(D'_{B_1 \oplus B_2}\big)\,.$$

Proof. Let B_{M} be the matching condition as defined in Definition 6.28. On identifying the pullback sections, say via a map Φ from E to E' , we get

$$\operatorname{dom}(D_{B_{\mathrm{M}}} \circ \Phi) = \operatorname{dom}(D)\,.$$

Moreover,

$$\operatorname{ind}(D_{B_{\mathrm{M}}}) = \operatorname{ind}(D)\,.$$

From Lemma 6.30

$$\operatorname{ind}\big(D'_{B_{\mathrm{M}}}\big) = \operatorname{ind}\Big(D'_{\mathrm{H}^{\frac{1}{2}}(N,E)}\Big) = \operatorname{ind}\big(D'_{B_1 \oplus B_2}\big)\,.$$ □

Example 6.32. 1. On fixing an invertible bisectorial adapted boundary operator A_0 on N, the canonical choices are $B_1 = \chi^-(A_0)\mathrm{H}^{\frac{1}{2}}(N,E)$ and $B_2 = \chi^+(A_0)\mathrm{H}^{\frac{1}{2}}(N,E)$. Clearly, $B_1 \oplus B_2 = \mathrm{H}^{\frac{1}{2}}(N,E) = B_{\mathrm{APS}}(A)$.

2. If D admits a self-adjoint adapted boundary operator on N, we can let $B_1 \subset \mathrm{H}^{\frac{1}{2}}(N,E)$ be elliptically regular on N_1 and $B_2 := B_1^{\perp,\mathrm{L}^2} \cap \mathrm{H}^{\frac{1}{2}}(N,E)$ considered as a boundary condition on N_2.

   Now choose A_0 invertible self-adjoint adapted boundary operator, by subtracting a small number from the self-adjoint boundary adapted operator we assumed exists. Set

$$\begin{aligned} V_- &:= V_{-,1} \oplus V_{-,2}\,, \\ V_+ &:= V_{+,1} \oplus V_{+,2}\,, \\ V_{\pm 1} &:= \chi^{\pm}(A_0)\mathrm{L}^2(N,E) \\ V_{\pm,2} &:= \chi^{\pm}(-A_0)\mathrm{L}^2(N,E) = \chi^{\mp}(A_0)\mathrm{L}^2(N,E)\,. \end{aligned}$$

   Write

$$\begin{aligned} B_1 &= W_{+,1} \oplus \mathrm{graph}(g_1 : V_{-,1} \to V_{+,1}) \cap \mathrm{H}^{\frac{1}{2}}(N,E)\,, \\ B_2 &= W_{+,2} \oplus \mathrm{graph}(g_2 : V_{-,2} \to V_{+,2}) \cap \mathrm{H}^{\frac{1}{2}}(N,E)\,. \end{aligned}$$

   But since $B_1 \perp B_2$ in L^2, we have that $V_{\pm,2} = V_{\mp 1}$, $W_{\pm,2} = W_{\mp,1}$ and $g_2 = -g_1^*$.

   The adjoint boundary condition for $B_1 \oplus B_2$ is $B_2 \oplus B_1$. Therefore, $B_1 \oplus B_2 = \mathrm{H}^{\frac{1}{2}}(N,E)$ and so Theorem 6.31 applies.

The splitting theorem applies in a particular and useful way when N separates M, leading to a decomposition theorem. That is, suppose $M = M' \cup M''$ where $N = \partial M' = \partial M''$. In this case, we obtain induced objects E', E'', F', F'' et cetera, on M' and M'' respectively,

Corollary 6.33 (Decomposition). *Let A_0 be an invertible bisectorial adapted boundary operator on N pointing into $\partial M'$. Assume $D, D^\dagger$ are coercive at infinity. Then on letting $B_1 = \chi^-(A_0)\mathrm{H}^{\frac{1}{2}}(N,E) = \chi^-(A_0)\mathrm{H}^{\frac{1}{2}}(\partial M',E)$ and $B_2 = \chi^+(A_0)\mathrm{H}^{\frac{1}{2}}(N,E) = \chi^-(-A_0)\mathrm{H}^{\frac{1}{2}}(\partial M'',E)$, we obtain that*

$$\mathrm{ind}(D) = \mathrm{ind}\big(D'_{B_1}\big) + \mathrm{ind}\big(D''_{B_2}\big)\,.$$

Proof. Exercise, follows from $D = D' \oplus D''$. □

6.1.3 The Φ-relative index and relative index theorems

Relative index theorems are index formulas measuring the difference of the indices of two operators living on two distinct manifolds, which can be identified outside of a compact set. These theorems were originally proved by Gromov-Lawson in [23], motivated by the study of positive scalar curvature metrics on closed manifolds. Their method was to construct a noncompact manifold with their original closed manifold as a boundary, apply the relative index theorem to the Spin-Dirac operator, and use the Weitzenböck identity where scalar curvature appears to study the space of positive scalar curvature metrics.

The methods used by Gromov-Lawson are quite different to those that we present here. Our perspective, using boundary value problems, emerges from the work of Bär-Ballmann in [10], where they generalised the theorems of [23] to Dirac-type operators. In our exposition, we will drop this assumption altogether to prove these theorems for general first-order elliptic operators.

In the previous subsections, we have developed the technical tools necessary to prove the relative index theorems. However, the statement of these theorems require us to relate two manifolds, bundles and operators outside a closed subset. We formalise this notion in the following definition.

Definition 6.34. Let M_1, M_2 be manifolds and $\left(E_i, h^{E_i}\right), \left(F_i, h^{F_i}\right) \to M_i$ Hermitian vector bundles. Let $D_i \in \mathrm{Diff}_1(E_i, F_i)$ and $K_i \subset M_i$ a closed subset. Then we say that D_1 and D_2 agree outside K_1, K_2 if they are related by vector bundle isometries $E_1|_{M_1 \setminus K_1} \cong E_2|_{M_2 \setminus K_2}$ and $F_1|_{M_1 \setminus K_1} \cong F_2|_{M_2 \setminus K_2}$. Explicitly, we require the following.

(I) There is a diffeomorphism $f : M_1 \setminus K_1 \to M_2 \setminus K_2$.

(II) There exist vector bundle isometries

$$I_E : E_1|_{M_1 \setminus K_1} \to E_2|_{M_2 \setminus K_2} \qquad \text{and} \qquad I_F : F_1|_{M_1 \setminus K_1} \to F_2|_{M_2 \setminus K_2}$$

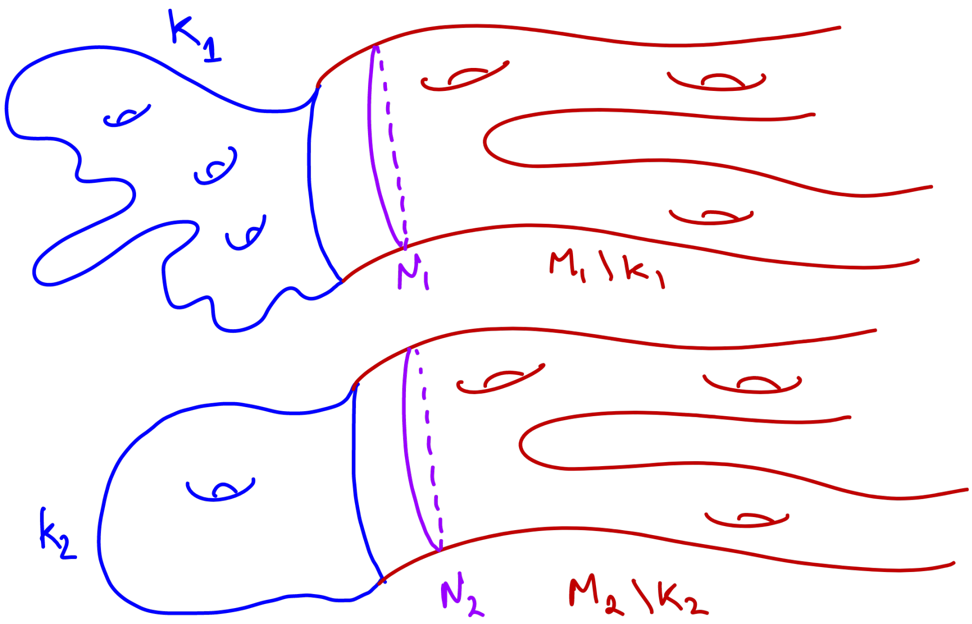
over f. That is, I_E and I_F are fibrewise linear isometries s.t. the following diagrams commute:

$$\begin{array}{ccc} E_1|_{M_1\setminus K_1} & \xrightarrow{I_E} & E_2|_{M_2\setminus K_2} \\ \downarrow & & \downarrow \\ M_1 \setminus K_1 & \xrightarrow{f} & M_2 \setminus K_2 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} F_1|_{M_1\setminus K_1} & \xrightarrow{I_F} & F_2|_{M_2\setminus K_2} \\ \downarrow & & \downarrow \\ M_1 \setminus K_1 & \xrightarrow{f} & M_2 \setminus K_2 \end{array} .$$

(III) The operators D_1 and D_2 are related by I_E, I_F and f. Explicitly,

$$I_F \circ D_1 u \circ f^{-1} = D_2\left(I_E \circ u \circ f^{-1}\right)$$

for all $u \in \mathrm{C}^\infty(M_1 \setminus K_1, E_1)$.



Theorem 6.35 (Φ-relative index theorem). *Let M_i, E_i, D_i, $\partial M_i = \varnothing$ with compact $K_i \subset M_i$ such that D_1, D_2 agree outside of K_1, K_2 as in in Definition 6.34. Additionally, let μ_i be densities on M_i such that $\mu_1 = f^*\mu_2$ on $M_1 \setminus K_1$ and assume that D_1, D_2 complete, elliptic and coercive at infinity.*

Suppose that there exist a compact hypersurfaces N_1 separating $M_1 = M_1' \cup M_1''$ with $\partial M_1' = \partial M_1'' = N_1$ and $K_1 \subset \mathring{M}_1'$. Then, $N_2 := f(N_1)$ separates $M_2 = M_2' \cup M_2''$ and $K_2 \subset \mathring{M}_2'$. Denote the induced operators on M_i' and M_i'' from D_i by D_i' and D_i'' respectively.

Fix an invertible bisectorial adapted boundary operator A to D_1' on M_1', i.e. using a transversal vector field pointing into M_1'. Let $B_1 := \chi^-(A)\mathrm{H}^{\frac{1}{2}}(\partial M_1, E_1)$ and B_2 be identified with B_1 under (I_E, I_F, f). Then D_i, D_{i,B_i}' are Fredholm operators and

$$\mathrm{ind}(D_1) - \mathrm{ind}(D_2) = \mathrm{ind}\big(D_{1,B_1}'\big) - \mathrm{ind}\big(D_{2,B_2}'\big).$$

Proof. Since $M_1 \setminus K_1$ is diffeomorphic to $M_2 \setminus K_2$, it is clear that $N_2 = f(N_1)$ separates $M_2 = M_2' \cup M_2''$ and $K_2 \subset \mathring{M}_2'$.

By Proposition 6.26, since A is an adapted boundary operator on $\partial M_1'$, we obtain that $-A$ is an adapted boundary operator on $\partial M_1''$. On setting $B_1' := B_1$

$$B_1'' := \chi^+(A)\mathrm{H}^{\frac{1}{2}}(\partial M_1'', E) = \chi^+(A)\mathrm{H}^{\frac{1}{2}}(N_1, E)$$

from Corollary 6.33, we obtain

$$\mathrm{ind}(D_1) = \mathrm{ind}\Big(D_{1,B_1'}'\Big) + \mathrm{ind}\Big(D_{1,B_1''}''\Big).$$

Let $B_2' := B_2$ and B_2'' be the boundary condition B_2'' pulled across to $\partial M_2''$ via (I_E, I_F, f), we get

$$\mathrm{ind}(D_2) = \mathrm{ind}\Big(D_{2,B_2'}'\Big) + \mathrm{ind}\Big(D_{2,B_2''}''\Big).$$

Since D_1 and D_2 agree outside M_1' and M_2' we get

$$\operatorname{ind}\left(D''_{1,B''_1}\right) = \operatorname{ind}\left(D''_{2,B''_2}\right).$$

By taking the difference we obtain the conclusion. □

We now want to express the right hand side of this index theorem in terms of index densities for D_1 and D_2 respectively. For that, we need to embed the manifold M_i' inside a larger closed and fixed manifold, extending D_i appropriately.

Lemma 6.36. *Let M_1 and M_2 be two compact manifolds with boundary s.t. D_1, D_2 elliptic agree outside $K_i \subset \mathring{M}_i$. Then there are $\tilde{M}_i$ compact with $\partial \tilde{M}_i = \varnothing$ s.t. the following hold.*

(I) $M_i \subset \tilde{M}_i$,

(II) $E_i \subset \tilde{E}_i$, $h^{\tilde{E}_i}$ *smooth with* $h^{\tilde{E}_i}|_{M_i} = h^{E_i}$,

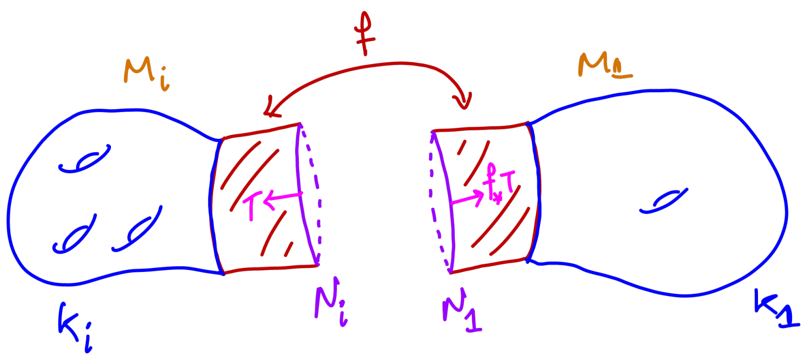
(III) $\tilde{D}_i$ *elliptic s.t.* D_i *and* $\tilde{D}_1$ *and* $\tilde{D}_2$ *agree outside* K_1, K_2,

(IV) $\tilde{D}_i|_{K_i} = D_i|_{K_i}$.

Proof. Take M_i and set $M_1^{\text{2nd}} := M_1$ as the second copy of M_1. We will glue this second copy M_1^{2nd} to M_i, regardless of whether $i = 1$ or $i = 2$. That is, define

$$\tilde{M}_i := M_i \cup_{U_i} M_1^{\text{2nd}}$$

identifying inside $U_i := M_i \setminus K_i$, which by hypothesis is identified with $M_1 \setminus K_1$ via a diffeomorphism.



By hypothesis, the D_i agree on U_i open and containing ∂M_i. In U_i, D_1 and D_2 agree through identification by (I_E, I_F, f). So on doubling, we keep the smooth coefficients and obtain $\tilde{D}_i$.

Similarly, $h^{\tilde{E}_i}$ have smooth coefficients also. Therefore, the conclusions as stated follow. □

Theorem 6.37 (Relative index theorem). *Let M_1, M_2 be measured manifolds without boundary, $D_i \in \mathrm{Diff}_1(E_i, F_i)$ elliptic and complete, and D_1, D_2 agree outside compact K_1, K_2, and $\mu_1 = f^*\mu_2$. Suppose further that there is a compact hypersurface $N_1 \subset M_1$ s.t. $M_1 = M_1' \cup M_1''$ with $\partial M_1' = \partial M_1'' = N_1$ with $K_1 \subset \mathring{M}_1$.*

Then D_1 is Fredholm iff D_2 is Fredholm and in that case

$$\mathrm{ind}(D_1) - \mathrm{ind}(D_2) = \int_{K_1} \alpha_{0,D_1}(x)\ \mathrm{d}\mu_1(x) - \int_{K_2} \alpha_{0,D_2}(y)\ \mathrm{d}\mu_2(y)\,,$$

where α_{0,D_i} is the local index density for D_i inside of M_i.

Proof. Since K_i are compact, applying Corollary 6.14 to a manifold without boundary gives that D_i are Fredholm iff D_i are coercive at infinity, say w.r.t. a set $\tilde{K}_i$ (c.f Definition 6.10). But $K_i \cup \tilde{K}_i$ is still compact and D_i are coercive at infinity w.r.t. $K_i \cup \tilde{K}_i$ and hence it is easy to see that D_1 is Fredholm iff D_2 is Fredholm since D_1, D_2 agree outside the compact set $K_i \cup \tilde{K}_i$.

Now from Theorem 6.35, we obtain $N_2 = f(N_1)$ separates $M_2 = M_2' \cup M_2''$ with $K_2 \subset \mathring{M}_2'$. Let D_i' and D_i'' be the induced operators on M_i' and M_i'' respectively. Choosing $B_1 := \chi^-(A)\mathrm{H}^{\frac{1}{2}}(N, E)$ for an invertible adapted boundary operator A on N_1, and with B_2 pulled back to $N_2 = f(N_1)$ via (I_E, I_F, f), we obtain from Theorem 6.35 that

$$\mathrm{ind}(D_1) - \mathrm{ind}(D_2) = \mathrm{ind}\big(D_{1,B_1}'\big) - \mathrm{ind}\big(D_{2,B_2}'\big)\,.$$

On application of Lemma 6.36, we obtain $\tilde{M}_i$ containing M_i' and $\tilde{D}_i$. But $\tilde{M}_i$ is closed, $\tilde{D}_i$ is elliptic on a closed manifold, so by Corollary 6.33,

$$\mathrm{ind}\Big(\tilde{D}_i\Big) = \mathrm{ind}\Big(\tilde{D}_{i,B_i}'\Big) + \mathrm{ind}\Big(\tilde{D}_{i,B_i''}''\Big)\,,$$

where $B_1'' := \chi^+(A)\mathrm{H}^{\frac{1}{2}}(N, E)$ and B_2'' is B_1'' identified on $\tilde{M}_2''$ through $(\tilde{I}_E, \tilde{I}_F, \tilde{f})$ given by Lemma 6.36.

By construction, $\tilde{D}_1$ and $\tilde{D}_2$ agree outside of K_1, K_2. In particular, $\tilde{D}_1$ and $\tilde{D}_2$ agree on $\tilde{M}_1''$ and $\tilde{M}_2''$. Therefore

$$\mathrm{ind}\Big(\tilde{D}_{1,B_1''}''\Big) = \mathrm{ind}\Big(\tilde{D}_{2,B_2''}''\Big)\,,$$

and

$$\mathrm{ind}(D_{1,B_1}) - \mathrm{ind}\big(D_{2,B_2'}\big) = \mathrm{ind}\Big(\tilde{D}_1\Big) - \mathrm{ind}\Big(\tilde{D}_2\Big)\,.$$

Since $\tilde{M}_i$ are closed and $\tilde{D}_i$ are first-order elliptic, we can apply Atiyah-Singer index theorem to obtain

$$\begin{aligned}\mathrm{ind}\Big(\tilde{D}_i\Big) &= \int_{\tilde{M}_i} \alpha_{0,\tilde{D}_i}(x)\ \mathrm{d}\mu_i(x) \\ &= \int_{K_i} \alpha_{0,\tilde{D}_i}(x)\ \mathrm{d}\mu_i(x) + \int_{\tilde{M}_i\setminus K_i} \alpha_{0,\tilde{D}_i}(x)\ \mathrm{d}\mu_i(x)\,,\end{aligned}$$

where $\alpha_{0,\tilde{D}_i}$ is the constant term in asymptotic expansion of $\operatorname{tr}(\mathrm{e}^{-\tau\tilde{D}_i^*\tilde{D}_i} - \mathrm{e}^{-\tau\tilde{D}_i\tilde{D}_i^*})$.

The operators D_1, D_2 agree outside K_1, K_2 and so certainly $\tilde{D}_1, \tilde{D}_2$ agree outside K_1, K_2. Moreover, since we assume $\mu_2 = f^*\mu_1$,

$$\int_{\tilde{M}_1\setminus K_1} \alpha_{0,\tilde{D}_1}(x)\, \mathrm{d}\mu_1(x) = \int_{\tilde{M}_2\setminus K_2} \alpha_{0,\tilde{D}_2}(y)\, \mathrm{d}\mu_2(y)\,.$$

Also, $\tilde{D}_i = D_i$ on K_i, so

$$\alpha_{0,\tilde{D}_i}(x) = \alpha_{0,D_i}(x)$$

for $x \in K_i$. Hence, we conclude

$$\operatorname{ind}\big(\tilde{D}_1\big) - \operatorname{ind}\big(\tilde{D}_2\big) = \int_{K_1} \alpha_{0,D_1}(x)\, \mathrm{d}\mu_1(x) - \int_{K_2} \alpha_{0,D_2}(y)\, \mathrm{d}\mu_2(y)\,. \qquad \square$$

Remark 6.38. 1. This theorem demonstrates the enormous applicability of the heat kernel proof of the Atiyah-Singer index theorem. Since the index densities are pointwise, we are able to localise and exploit cancellations. The topological approach to the Atiyah-Singer index theorem would not lend itself to such a calculation as we have done here.

2. If we were to dispense with using Lemma 6.36, we would need to directly compute $\operatorname{ind}\big(D'_{i,B_i}\big)$. In the situation where $A_1 := A$ is self-adjoint, we can accomplish this using the Atiyah-Patodi-Singer index theorem, Theorem 5.27. By pulling across A to M'_2, to obtain A_2 we have that $\dim\ker A_2 = \dim\ker A_1$ and $\eta_{A_1}(s) = \eta_{A_2}(s)$. Therefore, these terms cancel, and we obtained the stated relative index formula. The advantage of using the Atiyah-Singer index theorem is that we do not need to make any extra assumptions on D_i, as we would need to for using the Atiyah-Patodi-Singer index theorem.

3. We have asserted multiple times that the L^2-graphical decomposition is at the heart of the proof of these relative index theorems. It is always true, without any additional assumptions, that

$$\operatorname{ind}(D) = \operatorname{ind}\big(D'_{B_{\mathrm{M}}}\big)\,.$$

However, in order to compute the index on the right, we needed to deform B_{M} to the APS condition with respect to $A = A_0 \oplus (-A_0)$. In order to do so, we needed to assert that B_{M} is an elliptically regular boundary condition and then represent it in a way that lends itself to deformation. It is in this seemingly trivial point where the L^2-graphical decomposition becomes of paramount importance.

This is, in fact, a deep and significant achievement. Classically, boundary conditions that were considered were *elliptic*, which traditionally meant that they were obtained as f

$$B = \mathcal{P}\mathrm{H}^{\frac{1}{2}}(\partial M, E)$$

where $\mathcal{P}$ is a pseudo-differential projector of order 0 satisfying a certain ellipticity condition with respect to a boundary decomposing projector. Such an operator is pseudo-local, meaning that it does not increase the support of a section by 'too much'. However, it is clear that a matching condition B_{M} cannot be written in this way. The matching can happen in a region where the boundaries are as far away as we like them to be, and through matching, whatever happens at one boundary is immediately reflected in the other. It was the genius of Bär-Ballmann in [10] to understand ellipticity, which we call elliptic regularity, from the equivalent graphical decomposition perspective. Through this powerful tool, the central issue lying at the heart of these deep relative index theorems are almost trivially resolved.

6.2 Proof of Theorem 6.7

We now return to the matter of proving Theorem 6.7, which states that the L^2-graphical decomposition for B, as defined in Definition 6.6 is equivalent to the elliptic regularity of B. The proof of this theorem contains many small parts, and it is advantageous to organise it in a way which makes the conceptual ideas of the proof clear. First, we prove this following abstract lemma regarding the behaviour of projectors on Hilbert spaces which are densely contained in each other. To motivate this lemma, we note that in application, we take $\mathcal{H}_0 = \mathrm{H}^{\frac{1}{2}}(\partial M, E)$, $\mathcal{H} = \mathrm{L}^2(\partial M, E)$ and $\mathcal{H}_1 = \mathrm{H}^{-\frac{1}{2}}(\partial M, E)$.

Lemma 6.39. *Let $\mathcal{H}_0 \subset \mathcal{H} \subset \mathcal{H}_1$, be densely and continuously embedded Hilbert spaces. Suppose that $\langle \mathcal{H}_0, \mathcal{H}_1 \rangle$ (a perfect pairing) with $\langle \cdot, \cdot \rangle|_{\mathcal{H}_0} = \langle \cdot, \cdot \rangle_{\mathcal{H}}$. Moreover, suppose that $\mathcal{H} = W \oplus \mathcal{H}_W$ with finite dimensional $W \subset \mathcal{H}_0$. Let $\mathcal{P}_{\mathcal{H}_W, W} : \mathcal{H} \to \mathcal{H}$ be the projection to $\mathcal{H}_W$ along W. Then we have the following.*

I) $\mathcal{P}_{\mathcal{H}_W, W}|_{\mathcal{H}_0} : \mathcal{H}_0 \to \mathcal{H}_0$ is bounded and $\mathcal{P}_{\mathcal{H}_W, W}\mathcal{H}_0 = \mathcal{H}_W \cap \mathcal{H}_0$.

II) $\mathcal{P}_{\mathcal{H}_W, W}|_{\mathcal{H}_0}^* : \mathcal{H}_1 \to \mathcal{H}_1$ (adjoint with respect to $\langle \mathcal{H}_0, \mathcal{H}_1 \rangle$) satisfies

$$(\mathcal{H}_W)_1^* := \mathcal{P}_{\mathcal{H}_W, W}|_{\mathcal{H}_0}^* \mathcal{H}_1 = W^{\perp, \langle \mathcal{H}_0, \mathcal{H}_1 \rangle}$$
$$\text{and} \qquad (W)_1^* := \mathcal{P}_{W, \mathcal{H}_W}|_{\mathcal{H}_0}^* \mathcal{H}_1 = (\mathcal{H}_W \cap \mathcal{H}_0)^{\perp, \langle \mathcal{H}_0, \mathcal{H}_1 \rangle}.$$

III) If further $(W^*)_1 \subset \mathcal{H}_0$, then with $\mathcal{H}_W^* = \mathcal{P}_{\mathcal{H}_W, W}^* \mathcal{H}$ (the adjoint projector in $\mathcal{H}$), we have that $\mathcal{H}_W^* \cap \mathcal{H}_0$ is dense in $(\mathcal{H}_W^*)_1$ and

$$\mathcal{P}_{\mathcal{H}_W, W}|_{\mathcal{H}_0}^* = \overline{\mathcal{P}_{\mathcal{H}_W, W}}^{\mathcal{H}_1} = \overline{\mathcal{P}_{\mathcal{H}_W, W}^*|_{\mathcal{H}_0}}^{\mathcal{H}_1}.$$

Proof. a) Ad I).

Fix $u \in \mathcal{H}_0$ and write $u = \mathcal{P}_{\mathcal{H}_W,W}u + \mathcal{P}_{W,\mathcal{H}_W}u$. Then,

$$\mathcal{P}_{\mathcal{H}_W,W}u = u - \mathcal{P}_{W,\mathcal{H}_W}u \in \mathcal{H}_0 + \mathcal{H}_0 \subset \mathcal{H}_0 \,.$$

By definition, $\mathcal{P}_{\mathcal{H}_W,W}u \in \mathcal{H}_W$, and so, $\mathcal{P}_{\mathcal{H}_W,W}u \in \mathcal{H}_W \cap \mathcal{H}_0$. This proves $\mathcal{P}_{\mathcal{H}_W,W}\mathcal{H}_0 = \mathcal{H}_W \cap \mathcal{H}_0$.

Now we show $\mathcal{P}_{\mathcal{H}_W,W}|_{\mathcal{H}_0} \in \mathscr{B}(\mathcal{H}_0)$. For that, first note that since W is finite dimensional, $\|\mathcal{P}_{W,\mathcal{H}_W}u\|_{\mathcal{H}_0} \simeq \|\mathcal{P}_{W,\mathcal{H}_W}u\|_{\mathcal{H}}$. Therefore,

$$\|\mathcal{P}_{\mathcal{H}_W,W}u\|_{\mathcal{H}_0} \lesssim \|u\|_{\mathcal{H}_0} + \|\mathcal{P}_{W,\mathcal{H}_W}u\|_{\mathcal{H}_0} \lesssim \|u\|_{\mathcal{H}_0} \,.$$

b) Ad II).

Since we are only considering Hilbert spaces, by Proposition 2.62, $\mathcal{P}_{\mathcal{H}_W,W}|_{\mathcal{H}_0}^{*} : \mathcal{H}_1 \to \mathcal{H}_1$ is a bounded projector. Now,

$$\begin{aligned} u \in W^{\perp,\langle \mathcal{H}_0,\mathcal{H}_1\rangle} \quad &\Leftrightarrow \quad \forall w \in W \colon 0 = \langle u, w\rangle \\ &\Leftrightarrow \quad \forall h_0 \in \mathcal{H}_0 \colon 0 = \left\langle u, \mathcal{P}_{W,\mathcal{H}_W}|_{\mathcal{H}_0} h_0 \right\rangle \\ &\Leftrightarrow \quad \forall h_0 \in \mathcal{H}_0 \colon 0 = \left\langle \mathcal{P}_{W,\mathcal{H}_W}|_{\mathcal{H}_0}^{*} u, h_0 \right\rangle \\ &\Leftrightarrow \quad u \in \ker\left(\mathcal{P}_{W,\mathcal{H}_W}|_{\mathcal{H}_0}^{*}\right) = \mathcal{P}_{\mathcal{H}_W,W}|_{\mathcal{H}}^{*}\mathcal{H}_1 \,, \end{aligned}$$

where the second equivalence follows from $W \subset \mathcal{H}_0$ and the last from the fact that $\mathrm{id}_{\mathcal{H}_1} = \mathcal{P}_{W,\mathcal{H}_W}|_{\mathcal{H}_0}^{*} + \mathcal{P}_{\mathcal{H}_W,W}|_{\mathcal{H}_0}^{*}$.

A similar argument shows $(\mathcal{H}_W \cap \mathcal{H}_0)^{\perp,\langle \mathcal{H}_0,\mathcal{H}_1\rangle} = \mathcal{P}_{\mathcal{H}_W,W}|_{\mathcal{H}_0}^{*}\mathcal{H}_1$.

c) Ad III).

Fix $u \in \mathcal{H}_1$ and by the density hypothesis, fix a sequence $u_n \in \mathcal{H}_0$ such that $u_n \to u$. Write

$$u_n = \mathcal{P}^{*}_{\mathcal{H}_W,W}|_{\mathcal{H}_0}u_n + \mathcal{P}^{*}_{W,\mathcal{H}_W}|_{\mathcal{H}_0}u_n \,.$$

It suffices to prove $\mathcal{P}^{*}_{\mathcal{H}_W,W}|_{\mathcal{H}_0}u_n \to u$ and that $\mathcal{P}^{*}_{\mathcal{H}_W,W}|_{\mathcal{H}_0}u_n \in \mathcal{H}_0$, where the latter utilises the fact that $(W)_1^{*} \subset \mathcal{H}_0$. We leave this as an exercise. $\square$

From here on, unless otherwise stated, we fix a boundary decomposing projector $\mathcal{P}_+$.

Notation 6.40. For a subspace $V \subset \bigcup_{\alpha\in\mathbb{R}} \mathrm{H}^{\alpha}(\partial M, E)$ define

$$\begin{aligned} V^{\beta} &:= \overline{V \cap \mathrm{H}^{\beta}(\partial M, E)}^{\|\cdot\|_{\mathrm{H}^{\beta}}} \,, \\ \check{V} &:= \overline{V \cap \check{\mathrm{H}}_{\mathcal{P}_+}(D)}^{\|\cdot\|_{\check{\mathrm{H}}_{\mathcal{P}_+}(D)}} \,, \\ \hat{V} &:= \overline{V \cap \hat{\mathrm{H}}_{\mathcal{P}_+}(D)}^{\|\cdot\|_{\hat{\mathrm{H}}_{\mathcal{P}_+}(D)}} \,. \end{aligned}$$

Recall that when we have a Banach space $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2$, and $\left\langle \mathcal{B}, \widetilde{\mathcal{B}} \right\rangle$ is reflexive, we write $\mathcal{B}_1^* := \mathcal{P}^*_{\mathcal{B}_1,\mathcal{B}_2} \widetilde{\mathcal{B}}$ where $\widetilde{\mathcal{B}} = \mathcal{B}_1^* \oplus \mathcal{B}_2^*$.

Proposition 6.41. *Suppose that* $V_\pm \oplus W_\pm = \mathcal{P}_\pm \mathrm{L}^2(\partial M, E)$ *with* $W_\pm, W_\pm^* \subset \mathrm{H}^{\frac{1}{2}}(\partial M, E)$ *finite dimensional. Then the following hold.*

I) $\mathcal{P}_{V_\pm, W_\pm \oplus \mathcal{P}_\mp \mathrm{L}^2}|_{\mathrm{H}^{\frac{1}{2}}} : \mathrm{H}^{\frac{1}{2}}(\partial M, E) \to \mathrm{H}^{\frac{1}{2}}(\partial M, E)$ *is a bounded projector.*

II) $\mathrm{H}^\alpha(\partial M, E) = V_-^\alpha \oplus W_- \oplus V_+^\alpha \oplus W_+ = \left(V_-^*\right)^\alpha \oplus W_-^* \oplus \left(V_+^*\right)^\alpha \oplus W_+^*$ *for* $\alpha \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$.

III) $V_\pm^{\frac{1}{2}}$ *is dense in* $V_\pm^{-\frac{1}{2}}$ *and* $\left(V_\pm^*\right)^{\frac{1}{2}}$ *is dense in* $\left(V_\pm^*\right)^{-\frac{1}{2}}$.

Proof. This follows from making the right choice for spaces $\mathcal{H}_0$, $\mathcal{H}$ and $\mathcal{H}_1$ in Lemma 6.39, noting that $\mathcal{P}_+|_{\mathrm{H}^\beta} : \mathrm{H}^\beta(\partial M, E) \to \mathrm{H}^\beta(\partial M, E)$ is a bounded projector for $\beta \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. The details are left as an exercise. $\square$

Corollary 6.42. *Under the hypothesis of Proposition 6.41, the* L^2 *inner product* $\langle \cdot, \cdot \rangle$ *extends to perfect pairings*

$$\left\langle \mathcal{P}_\pm^* \mathrm{H}^{\frac{1}{2}}(\partial M, E), \mathcal{P}_\pm \mathrm{H}^{-\frac{1}{2}}(\partial M, E) \right\rangle, \qquad \left\langle \mathcal{P}_\pm^* \mathrm{H}^{-\frac{1}{2}}(\partial M, E), \mathcal{P}_\pm \mathrm{H}^{\frac{1}{2}}(\partial M, E) \right\rangle,$$
$$\langle W_\pm^*, W_\pm \rangle, \qquad \langle V_\pm^*, V_\pm \rangle, \qquad \left\langle \left(V_\pm^*\right)^{-\frac{1}{2}}, V_\pm^{\frac{1}{2}} \right\rangle, \qquad \left\langle \left(V_\pm^*\right)^{\frac{1}{2}}, V_\pm^{-\frac{1}{2}} \right\rangle,$$
$$\left\langle \hat{\mathrm{H}}_{\mathcal{P}_+}(D), \check{\mathrm{H}}_{\mathcal{P}_+}(D) \right\rangle.$$

Proof. This is an immediate consequence of Proposition 2.63, combined with the assertions of Proposition 6.41. $\square$

As a consequence of this corollary, for the purposes of legibility in the coming sections, we make the following notational remark.

Notation 6.43. For $V \subset \mathrm{H}^\alpha(\partial M, E)$, we let $V^{\perp, \mathrm{H}^{-\alpha}}$ be the annihilator w.r.t. $\langle \mathrm{H}^\alpha(\partial M, E), \mathrm{H}^{-\alpha}(\partial M, E) \rangle$ induced from L^2. Similarly for $V \subset \check{\mathrm{H}}_{\mathcal{P}_+}(D)$ we write $V^{\perp, \hat{\mathrm{H}}_{\mathcal{P}_+}(D)}$ and for $V \subset \hat{\mathrm{H}}_{\mathcal{P}_+}(D)$ we write $V^{\perp, \check{\mathrm{H}}_{\mathcal{P}_+}(D)}$ to be the annihilators w.r.t. $\left\langle \check{\mathrm{H}}_{\mathcal{P}_+}(D), \hat{\mathrm{H}}_{\mathcal{P}_+}(D) \right\rangle$.

6.2.1 L^2-graphical decomposability implies elliptical regularity

Having developed some technical tools, the easier direction in the proof of Theorem 6.7 is that the L^2-graphical decomposition yields elliptic regularity for a boundary condition B.

Proof of Theorem 6.7, L^2-graphical decomposability implies ellpitic regularity. By assumption that B is L^2-graphically decomposable, we write

$$B = \mathrm{graph}\Big(g|_{\mathrm{H}^{\frac{1}{2}}}\Big) \oplus W_+ \subset \mathrm{H}^{\frac{1}{2}}(\partial M, E)\,.$$

To show that B is elliptically regular, we need to show that $B^\dagger \subset \mathrm{H}^{\frac{1}{2}}(\partial M, F)$. However, this is equivalent to proving that

$$B^{\perp, \hat{\mathrm{H}}_{\mathcal{P}_+}(D)} \subset \mathrm{H}^{\frac{1}{2}}(\partial M, E)\,.$$

We note that

$$\begin{aligned} B^{\perp, \hat{\mathrm{H}}_{\mathcal{P}_+}(D)} &= B^{\perp, \mathrm{H}^{-\frac{1}{2}}} \cap \hat{\mathrm{H}}_{\mathcal{P}_+}(D) \\ &= W_\pm^{\perp, \mathrm{H}^{-\frac{1}{2}}} \cap \mathrm{graph}(g|_{\mathrm{H}^{\frac{1}{2}}})^{\perp, \mathrm{H}^{-\frac{1}{2}}} \cap \hat{\mathrm{H}}_{\mathcal{P}_+}(D) \end{aligned}$$

and so we first compute $W_\pm^{\perp, \mathrm{H}^{-\frac{1}{2}}}$ and $\mathrm{graph}(g|_{\mathrm{H}^{\frac{1}{2}}})^{\perp, \mathrm{H}^{-\frac{1}{2}}}$.

Recall from Definition 6.6 that $\mathrm{L}^2(\partial M, E) = V_- \oplus W_- \oplus V_+ \oplus W_+$ and $g : V_- \to V_+$. From Proposition 6.41, $\mathrm{H}^{\frac{1}{2}}(\partial M, E) = V_-^{\frac{1}{2}} \oplus W_- \oplus V_+^{\frac{1}{2}} \oplus W_+$. Using Lemma 6.39, we obtain that

$$W_\pm^{\perp, \mathrm{H}^{-\frac{1}{2}}} = W_\mp^* \oplus \big(V_-^*\big)^{-\frac{1}{2}} \oplus \big(V_+^*\big)^{-\frac{1}{2}}\,.$$

Similarly,

$$\begin{aligned} \mathrm{graph}(g|_{\mathrm{H}^{\frac{1}{2}}})^{\perp, \mathrm{H}^{-\frac{1}{2}}} &= \Big\{ v + gv \,\Big|\, v \in V_-^{\frac{1}{2}} \Big\}^{\perp, \mathrm{H}^{-\frac{1}{2}}} \\ &= W_-^* \oplus W_+^* \oplus \Big\{ u - g^* u \,\Big|\, u \in \big(V_+^*\big)^{-\frac{1}{2}} \Big\}\,. \end{aligned}$$

Therefore,

$$B^{\perp, \mathrm{H}^{-\frac{1}{2}}} = W_+^{\perp, \mathrm{H}^{-\frac{1}{2}}} \cap \mathrm{graph}\Big(g|_{\mathrm{H}^{\frac{1}{2}}}\Big)^{\perp, \mathrm{H}^{-\frac{1}{2}}} = W_-^* \oplus \Big\{ u - g^* u \,\Big|\, u \in \big(V_+^*\big)^{-\frac{1}{2}} \Big\}$$

and hence

$$\begin{aligned} B^{\perp, \hat{\mathrm{H}}_{\mathcal{P}_+}(D)} &= B^{\perp, \mathrm{H}^{-\frac{1}{2}}} \cap \hat{\mathrm{H}}_{\mathcal{P}_+}(D) \\ &= B^{\perp, \mathrm{H}^{-\frac{1}{2}}} \cap \mathcal{P}_-^* \mathrm{H}^{-\frac{1}{2}}(\partial M, E) \oplus \mathcal{P}_+^* \mathrm{H}^{\frac{1}{2}}(\partial M, E) \\ &= W_-^* \oplus \Big\{ u - g^* u \,\Big|\, \big(V_+^*\big)^{\frac{1}{2}} \Big\}\,. \end{aligned}$$

Since by assumption $g^*\Big(\big(V_+^*\big)^{\frac{1}{2}}\Big) \subset \big(V_-^*\big)^{\frac{1}{2}} \subset \mathrm{H}^{\frac{1}{2}}(\partial M, E)$, we obtain that $B^{\perp, \hat{\mathrm{H}}_{\mathcal{P}_+}(D)} \subset \mathrm{H}^{\frac{1}{2}}(\partial M, E)$. ∎

6.2.2 Elliptical regularity implies L^2-graphical decomposability

Showing that an elliptically regular boundary condition has an L^2-graphical decomposition requires a lot more effort. The difficulty is that we have to construct the decomposition from having B. We will accomplish this in three steps. First, we define the space and prove that it has the required properties (c.f. Proposition 6.46). Then, we show that the spaces interact nicely with the boundary condition (c.f. Proposition 6.47) Lastly, we prove the theorem, where the main point is to construct the map g.

Throughout this subsection, we assume that B is elliptically regular. That is, we assume that B is a boundary condition and

$$B, B^{\perp, \hat{\mathrm{H}}_{\mathcal{P}_+}(D)} \subset \mathrm{H}^{\frac{1}{2}}(\partial M, E)\,.$$

For the first step, as an ansatz, we define the following spaces:

$$\begin{aligned} W_-^* &:= \mathcal{P}_-^* \mathrm{L}^2(\partial M, E) \cap B^{\perp, \hat{\mathrm{H}}_{\mathcal{P}_+}(D)}\,, \\ W_+ &:= \mathcal{P}_+ \mathrm{L}^2(\partial M, E) \cap B\,, \\ V_-^* &:= \mathcal{P}_-^* \mathrm{L}^2(\partial M, E) \cap \left(W_-^*\right)^{\perp, \mathrm{L}^2}\,, \\ V_+ &:= \mathcal{P}_+ \mathrm{L}^2(\partial M, E) \cap W_+^{\perp, \mathrm{L}^2}\,. \end{aligned} \tag{6.2}$$

Using these, define:

$$\begin{aligned} W_- &:= \mathcal{P}_- W_-^*\,, \\ W_+^* &:= \mathcal{P}_+^* W_+\,, \\ V_- &:= \mathcal{P}_- V_-^*\,, \\ V_+^* &:= \mathcal{P}_+^* V_+\,. \end{aligned} \tag{6.3}$$

First, we note a couple of abstract results that we will require in computations.

Lemma 6.44. *Consider Hilbert spaces $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 = \mathcal{H}_0 \oplus \mathcal{H}_1'$. Then*

$$\mathcal{P}_{\mathcal{H}_1', \mathcal{H}_0}|_{\mathcal{H}_1} : \mathcal{H}_1 \to \mathcal{H}_1'$$

is a Banach space isomorphism.

Proof. Exercise. □

Lemma 6.45. *Let $\mathcal{B}_1 \subset \mathcal{B}_2$ be Banach-spaces s.t. for all $u \in \mathcal{B}_1$ we have*

$$\|u\|_{\mathcal{B}_2} \lesssim \|u\|_{\mathcal{B}_1}\,.$$

If $\mathcal{B}_3 \subset \mathcal{B}_2$ is closed, then $\mathcal{B}_3 \cap \mathcal{B}_1$ is closed in $\mathcal{B}_1$.

Proof. Exercise. ☐

The first step in our proof is the following proposition, which gives us the properties required of the spaces above to satisfy Definition 6.6.

Proposition 6.46. *For the spaces defined in (6.2) and (6.3), the following hold.*

I) $\mathcal{P}_- B$ and $\mathcal{P}_+^* B^{\perp, \hat{\mathrm{H}}_{\mathcal{P}_+}(D)}$ are dense subspaces of $\mathrm{H}^{\frac{1}{2}}(\partial M, E)$.

II) $W_\pm, W_\pm^* \subset \mathrm{H}^{\frac{1}{2}}(\partial M, E)$ are finite dimensional.

III) All spaces we have defined above are closed in $\mathrm{L}^2(\partial M, E)$.

IV) $\mathcal{P}_\pm \mathrm{L}^2(\partial M, E) = V_\pm \oplus W_\pm$ and $\mathcal{P}_\pm^* \mathrm{L}^2(\partial M, E) = V_\pm^* \oplus W_\pm^*$.

Proof. First we note that $B \subset \check{\mathrm{H}}_{\mathcal{P}_+}(D)$ is a closed subspace since it is a boundary condition. Also, by hypothesis, $B \subset \mathrm{H}^{\frac{1}{2}}(\partial M, E)$. Now, recall that the inclusion $\mathrm{H}^{\frac{1}{2}}(\partial M, E) \hookrightarrow \check{\mathrm{H}}_{\mathcal{P}_+}(D)$ is continuous. Therefore, by Lemma 6.45, B is closed in $\mathrm{H}^{\frac{1}{2}}(\partial M, E)$.

Equivalently, this is saying that $\left(B, \|\cdot\|_{\mathrm{H}^{\frac{1}{2}}}\right)$ and $\left(B, \|\cdot\|_{\check{\mathrm{H}}_{\mathcal{P}_+}(D)}\right)$ are both Banach spaces. Moreover, we have that the inclusion $\left(B, \|\cdot\|_{\mathrm{H}^{\frac{1}{2}}}\right) \hookrightarrow \left(B, \|\cdot\|_{\check{\mathrm{H}}_{\mathcal{P}_+}(D)}\right)$ is a bounded bijection. Therefore, the open mapping theorem says that the inclusion is continuously invertible. This yields

$$\|u\|_{\mathrm{H}^{\frac{1}{2}}} \simeq \|u\|_{\check{\mathrm{H}}_{\mathcal{P}_+}(D)}$$

for all $u \in B$. Hence,

$$\|u\|_{\mathrm{H}^{\frac{1}{2}}} \simeq \|u\|_{\check{\mathrm{H}}_{\mathcal{P}_+}(D)} \simeq \|\mathcal{P}_- u\|_{\mathrm{H}^{\frac{1}{2}}} + \|\mathcal{P}_+ u\|_{\mathrm{H}^{-\frac{1}{2}}}.$$

Now

$$\mathcal{P}_+ : B \subset \mathrm{H}^{\frac{1}{2}}(\partial M, E) \to \mathrm{H}^{\frac{1}{2}}(\partial M, E) \xhookrightarrow{\text{compact}} \mathrm{H}^{-\frac{1}{2}}(\partial M, E).$$

That is, $\mathcal{P}_+|_B : B \to \mathrm{H}^{-\frac{1}{2}}(\partial M, E)$ is a compact map. By Lemma 5.5, we obtain $\mathrm{ran}\left(\mathcal{P}_-|_B\right)$ is closed in $\mathrm{H}^{\frac{1}{2}}(\partial M, E)$ and $\ker\left(\mathcal{P}_-|_B\right)$ is finite dimensional. But note that

$$\begin{aligned}\ker\left(\mathcal{P}_-|_B\right) &= \{u \in B \mid \mathcal{P}_- u = 0\} \\ &= \left\{u \in B \,\middle|\, u \in \mathcal{P}_+ \mathrm{L}^2(\partial M, E)\right\} \\ &= B \cap \mathcal{P}_+ \mathrm{L}^2(\partial M, E) = W_+.\end{aligned}$$

Similar calculation yields that $\mathcal{P}_+^* B^{\perp, \hat{\mathrm{H}}_{\mathcal{P}_+}(D)}$ is closed and W_+^* is finite dimensional. This proves I) and part of II).

To prove the remaining parts of II), we note

$$\begin{aligned}\mathrm{L}^2(\partial M, E) &= \mathcal{P}_- \mathrm{L}^2(\partial M, E) \oplus \mathcal{P}_+ \mathrm{L}^2(\partial M, E) \\ &= \mathcal{P}_-^* \mathrm{L}^2(\partial M, E) \oplus^\perp \mathcal{P}_+^* \mathrm{L}^2(\partial M, E).\end{aligned}$$

By previous Lemma 6.44, we have that $\mathcal{P}_+^* : \mathcal{P}_+\mathrm{L}^2(\partial M, E) \to \mathcal{P}_+^*\mathrm{L}^2(\partial M, E)$ is an isomorphism. Therefore, $W_+^* = \mathcal{P}_+ W_+$ is finite dimensional, and $W_+^* \subset \mathrm{H}^{\frac{1}{2}}(\partial M, E)$ since $\mathcal{P}_+ : \mathrm{H}^\beta \to \mathrm{H}^\beta$ for $\beta \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Similarly $W_- = \mathcal{P}_- W_-^* \subset \mathrm{H}^{\frac{1}{2}}(\partial M, E)$ is finite dimensional. This proves II).

We prove III) and IV). For that, first we prove that $\mathcal{P}_+\mathrm{L}^2(\partial M, E) = V_+ \oplus W_+$. First, $\mathrm{L}^2(\partial M, E) = W_+ \oplus^\perp W_+^\perp$. Fix $u \in \mathcal{P}_+\mathrm{L}^2(\partial M, E)$ and write $u = \mathcal{P}_{W_+, W_+^\perp} u + \mathcal{P}_{W_+^\perp, W_+} u$. Rearranging this, we find

$$u - \mathcal{P}_{W_+, W_+^\perp} u = \mathcal{P}_{W_+^\perp, W_+} u \,.$$

But by construction, $W_+ \subset \mathcal{P}_+\mathrm{L}^2(\partial M, E)$ and therefore,

$$\mathcal{P}_{W_+^\perp, W_+} u \in W_+^\perp \cap \mathcal{P}_+\mathrm{L}^2(\partial M, E) = V_+ \,.$$

This shows $\mathcal{P}_+\mathrm{L}^2(\partial M, E) = V_+ \oplus W_+$.

A similar computation yields $\mathcal{P}_-^*\mathrm{L}^2(\partial M, E) = V_-^* \oplus W_-^*$. Since the maps

$$\begin{aligned} \mathcal{P}_\pm &: \mathcal{P}_\pm^*\mathrm{L}^2(\partial M, E) \to \mathcal{P}_\pm\mathrm{L}^2(\partial M, E)\,, \\ \mathcal{P}_\pm^* &: \mathcal{P}_\pm\mathrm{L}^2(\partial M, E) \to \mathcal{P}_\pm^*\mathrm{L}^2(\partial M, E) \end{aligned}$$

are isomorphisms, we obtain that all the spaces in (6.2) and (6.3) are closed in $\mathrm{L}^2(\partial M, E)$ proving III) and also IV). ☐

The second part we require is the following proposition.

Proposition 6.47. *$\mathcal{P}_- B = V_-^{\frac{1}{2}}$ and $\mathcal{P}_+^* B^{\perp, \hat{\mathrm{H}}_{\mathcal{P}_+}(D)} = \left(V_+^*\right)^{\frac{1}{2}}$.*

Proof. We prove this in a number of steps.

a) *Claim:* $W_-^* = \mathcal{P}_-^*\mathrm{H}^{-\frac{1}{2}}(\partial M, E) \cap B^{\perp, \mathrm{H}^{-\frac{1}{2}}}$.

Direction '$\subset$' is clear, so we prove the reverse inclusion. Let $u \in \mathcal{P}_-^*\mathrm{H}^{-\frac{1}{2}}(\partial M, E) \cap B^{\perp, \mathrm{H}^{-\frac{1}{2}}}$. In particular, we have that $u \in \hat{\mathrm{H}}_{\mathcal{P}_+}(D)$. Therefore,

$$\begin{aligned} \mathcal{P}_-^*\mathrm{H}^{-\frac{1}{2}}(\partial M, E) \cap B^{\perp, \mathrm{H}^{-\frac{1}{2}}} &= \mathcal{P}_-^*\mathrm{H}^{-\frac{1}{2}}(\partial M, E) \cap B^{\perp, \mathrm{H}^{-\frac{1}{2}}} \cap \hat{\mathrm{H}}_{\mathcal{P}_+}(D) \\ &= \mathcal{P}_-^*\mathrm{H}^{-\frac{1}{2}}(\partial M, E) \cap B^{\perp, \hat{\mathrm{H}}_{\mathcal{P}_+}(D)} \\ &\subset \mathcal{P}_-^*\mathrm{H}^{\frac{1}{2}}(\partial M, E) \cap B^{\perp, \hat{\mathrm{H}}_{\mathcal{P}_+}(D)} \subset W_-^*\,, \end{aligned}$$

where the last set containment follows from the fact that, by the elliptic regularity of B, we have that $B^{\perp, \hat{\mathrm{H}}_{\mathcal{P}_+}(D)} \subset \mathrm{H}^{\frac{1}{2}}(\partial M, E)$.

b) *Claim:* $(\mathcal{P}_- B)^{\perp, \mathcal{P}_-^*\mathrm{H}^{-\frac{1}{2}}} = W_-^*$.

We simply compute:

$$
\begin{aligned}
w \in W_-^* \quad &\Leftrightarrow \quad w \in \mathcal{P}_-^* \mathrm{H}^{-\frac{1}{2}}(\partial M, E) \text{ and } \forall b \in B\colon \langle w, b \rangle = 0 \\
&\Leftrightarrow \quad w \in \mathcal{P}_-^* \mathrm{H}^{-\frac{1}{2}} \text{ and } \forall b \in B\colon \left\langle \mathcal{P}_-^* w, b \right\rangle = 0 \\
&\Leftrightarrow \quad w \in \mathcal{P}_-^* \mathrm{H}^{-\frac{1}{2}} \text{ and } \forall b \in B\colon \langle w, \mathcal{P}_- b \rangle = 0 \\
&\Leftrightarrow \quad w \in \mathcal{P}_-^* \mathrm{H}^{-\frac{1}{2}} \text{ and } \forall v \in \mathcal{P}_- B\colon \langle w, v \rangle = 0 \\
&\Leftrightarrow \quad w \in \mathcal{P}_-^* \mathrm{H}^{-\frac{1}{2}}(\partial M, E) \cap (\mathcal{P}_- B)^{\perp, \mathrm{H}^{-\frac{1}{2}}} \\
&\Leftrightarrow \quad w \in (\mathcal{P}_- B)^{\perp, \mathcal{P}_-^* \mathrm{H}^{-\frac{1}{2}}},
\end{aligned}
$$

where the last equivalence follows from a).

c) *Claim:* $\mathcal{P}_- B = \left(W_-^*\right)^{\perp, \mathcal{P}_- \mathrm{H}^{\frac{1}{2}}(\partial M, E)}$.

This is immediate since from Corollary 6.42, $\left\langle \mathcal{P}_-^* \mathrm{H}^{\frac{1}{2}}(\partial M, E), \mathcal{P}_- \mathrm{H}^{-\frac{1}{2}}(\partial M, E) \right\rangle$ is a perfect pairing.

d) *Claim:* $\left(W_-^*\right)^{\perp, \mathcal{P}_- \mathrm{H}^{\frac{1}{2}}} = \left(W_-^*\right)^{\perp, \mathrm{L}^2} \cap \mathcal{P}_- \mathrm{L}^2(\partial M, E) \cap \mathrm{H}^{\frac{1}{2}}(\partial M, E)$.

This is also readily verified.

e) *Claim:* $\left(W_-^*\right)^{\perp, \mathrm{L}^2} \cap \mathcal{P}_- \mathrm{L}^2(\partial M, E) = \mathcal{P}_- \Big(\left(W_-^*\right)^{\perp, \mathrm{L}^2} \cap \mathcal{P}_-^* \mathrm{L}^2(\partial M, E) \Big)$.

Containment '$\supset$'.

Let $u \in \mathcal{P}_- \Big(\left(W_-^*\right)^{\perp, \mathrm{L}^2} \cap \mathcal{P}_-^* \mathrm{L}^2(\partial M, E) \Big)$. Then $u = \mathcal{P}_- u'$ for some $u' \in \left(W_-^*\right)^{\perp, \mathrm{L}^2} \cap \mathcal{P}_-^* \mathrm{L}^2(\partial M, E)$. Let $w \in W_-^*$ and note

$$\langle w, u \rangle = \langle w, \mathcal{P}_- u' \rangle = \left\langle \mathcal{P}_-^* w, u' \right\rangle = \langle w, u' \rangle = 0\,.$$

Therefore, $u \in \left(W_-^*\right)^{\perp, \mathrm{L}^2}$. Furthermore, $u \in \mathcal{P}_- \mathrm{L}^2(\partial M, E)$ and so the containment '$\supset$' follows.

Containment '$\subset$'.

Let $u \in \left(W_-^*\right)^{\perp, \mathrm{L}^2} \cap \mathcal{P}_- \mathrm{L}^2(\partial M, E)$. Applying Lemma 6.44, we find a $u^* \in \mathcal{P}_-^* \mathrm{L}^2(\partial M, E)$ s.t. $u = \mathcal{P}_-^* u^*$. Then, fixing $w \in W_-^*$, we obtain

$$0 = \langle u, w \rangle = \langle \mathcal{P}_- u^*, w \rangle = \left\langle u^*, \mathcal{P}_-^* w \right\rangle = \langle u^*, w \rangle\,.$$

Therefore, so $u^* \in \left(W_-^*\right)^{\perp, \mathrm{L}^2}$ and also by hypothesis $u^* \in \mathcal{P}_-^* \mathrm{L}^2(\partial M, E)$. This shows '$\subset$'.

f) *Claim:* $\mathcal{P}_- B = V_-^{\frac{1}{2}}$.

We compute

$$\begin{aligned}
\mathcal{P}_-B &= \left(W_-^*\right)^{\perp,\mathcal{P}_-\mathrm{H}^{\frac12}} \\
&= \left(W_-^*\right)^{\perp,\mathrm{L}^2} \cap \mathcal{P}_-\mathrm{L}^2(\partial M,E)\cap \mathrm{H}^{\frac12}(\partial M,E) \\
&= \mathcal{P}_-\Big(\left(W_-^*\right)^{\perp,\mathrm{L}^2} \cap \mathcal{P}_-\mathrm{L}^2(\partial M,E)\Big)\cap \mathrm{H}^{\frac12}(\partial M,E) \\
&= \mathcal{P}_-V_-^* \cap \mathrm{H}^{\frac12}(\partial M,E) \\
&= V_- \cap \mathrm{H}^{\frac12}(\partial M,E) \\
&= V_-^{\frac12},
\end{aligned}$$

where the first equality is b), the second d), the third e) and the remaining steps are by definitions (6.2) and (6.3).

The claim $\mathcal{P}_+^* B^{\perp,\hat{\mathrm{H}}_{\mathcal{P}_+}(D)} = \left(V_+^*\right)^{\frac12}$ follows by mimicking this argument mutatis mutandis. □

Finally, we have the required ingredients to prove the remaining direction of Theorem 6.7.

Proof of Theorem 6.7, B elliptically regular implies L^2*-graphical decomposability .* We fix the spaces we have defined in (6.2) and (6.3). Proposition 6.46 then guarantees that these spaces satisfy the requirements in Definition 6.6.

To prove the stated assertion, it remains to construct $g : V_- \to V_+$ s.t. $g|_{\mathrm{H}^{\frac12}} V_-^{\frac12} \to V_+^{\frac12}$ with $g^*|_{\mathrm{H}^{\frac12}} : \left(V_+^*\right)^{\frac12} \to \left(V_-^*\right)^{\frac12}$ satisfying $B = \operatorname{graph}\left(g|_{\mathrm{H}^{\frac12}}\right) \oplus W_+$. We perform this construction in the following steps.

a) First we construct $g_0 : V_-^{\frac12} \to V_+^{\frac12}, h_0 : \left(V_+^*\right)^{\frac12} \to \left(V_-^*\right)^{\frac12}$ and $h_0 : (V_+^*)^{\frac12} \to (V_-^*)^{\frac12}$. For that, define

$$\begin{aligned}
X_- &:= \mathcal{P}_-|_{B\cap W_+^{\perp,\mathrm{L}^2}} : B\cap W_+^{\perp,\mathrm{L}^2} \to \mathcal{P}_-B = V_-^{\frac12}, \\
X_+^* &:= \mathcal{P}_+^*|_{B^{\perp,\hat{\mathrm{H}}_{\mathcal{P}_+}(D)}\cap\left(W_-^*\right)^{\perp,\mathrm{L}^2}} : B^{\perp,\hat{\mathrm{H}}_{\mathcal{P}_+}(D)} \cap \left(W_-^*\right)^{\perp,\mathrm{L}^2} \to \mathcal{P}_+^* B^{\perp,\hat{\mathrm{H}}_{\mathcal{P}_+}(D)}.
\end{aligned}$$

We show that X_- is a bounded isomorphism with bounded inverse (i.e. a Banach space isomorphism).

i) *Claim:* X_- is surjective.

Let $x \in \mathcal{P}_-B$ and write $x = \mathcal{P}_-b$ with $b \in B$. Then

$$x = \mathcal{P}_-b = \mathcal{P}_-\left(\mathcal{P}_{W_+^{\perp,\mathrm{L}^2},W_+} b + \mathcal{P}_{W_+,W_+^{\perp,\mathrm{L}^2}} b\right) = \mathcal{P}_-\mathcal{P}_{W_+^{\perp,\mathrm{L}^2},W_+} b,$$

where the ultimate equality follows from the fact that $W_+ \subset \ker(\mathcal{P}_-) = \mathcal{P}_+\mathrm{L}^2(\partial M,E)$. Also

$$\mathcal{P}_{W_+^{\perp,\mathrm{L}^2},W_+} b = b - \mathcal{P}_{W_+,W_+^{\perp,\mathrm{L}^2}} b \in B + B = B,$$

and therefore, $x = \mathcal{P}_- \mathcal{P}_{W_+^{\perp,\mathrm{L}^2}, W_+} b$. Since $\mathcal{P}_{W_+^{\perp,\mathrm{L}^2}, W_+} b \in B \cap W_+^{\perp,\mathrm{L}^2}$, the claim is proved.

ii) *Claim:* X_- is injective.

By linearity of X_-, it suffices to show $\ker(X_-) = 0$. So fix $u \in B \cap W_+^{\perp,\mathrm{L}^2}$ and assume that $X_- u = 0$. By definition

$$0 = X_- u = \mathcal{P}_- u$$

and so $u \in \mathcal{P}_+ \mathrm{L}^2(\partial M, E)$. Therefore,

$$u \in \mathcal{P}_+ \mathrm{L}^2(\partial M, E) \cap B \cap W_+^{\perp,\mathrm{L}^2} = W_+ \cap W_+^{\perp,\mathrm{L}^2} = = 0\,.$$

iii) *Claim:* X_- is bounded with bounded inverse.

$B \cap W_+^{\perp,\mathrm{L}^2}$ is closed in $\mathrm{H}^{\frac{1}{2}}(\partial M, E)$ and from Proposition 6.46 I), we know that $\mathcal{P}_- B$ is also closed $\mathrm{H}^{\frac{1}{2}}(\partial M, E)$. Moreover, since $\mathcal{P}_+$ is boundary decomposing, $\mathcal{P}_- : \mathrm{H}^{\beta}(\partial M, E) \to \mathrm{H}^{\beta}(\partial M, E)$ bounded for all $\beta \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Clearly, it follows that X_- is bounded. But we have shown that X_- is a bijection, and by the closedness of the above said spaces, it is a bounded linear bijection between Banach spaces. Therefore, the open mapping theorem guarantees that X_-^{-1} is bounded.

Similarly, $X_+^* : B^{\perp,\hat{\mathrm{H}}_{\mathcal{P}_+}(D)} \cap \left(W_-^*\right)^{\perp,\mathrm{L}^2} \to \mathcal{P}_+^* B^{\perp,\hat{\mathrm{H}}_{\mathcal{P}_+}(D)}$ is also a bounded map with bounded inverse, which is seen by mimicking this argument.

Define

$$\begin{aligned} g_0 &:= \mathcal{P}_{V_+, W_+ \oplus \mathcal{P}_- \mathrm{L}^2} \circ (X_-)^{-1} : V_-^{\frac{1}{2}} \to V_+^{\frac{1}{2}}\,, \\ h_0 &:= \mathcal{P}_{V_-^*, W_-^* \oplus \mathcal{P}_+^* \mathrm{L}^2} \circ \left(X_+^*\right)^{-1} : \left(V_+^*\right)^{\frac{1}{2}} \to \left(V_-^*\right)^{\frac{1}{2}}\,. \end{aligned}$$

Proposition 6.47 gives us that $V_-^{\frac{1}{2}} = \mathcal{P}_- B$ and $\left(V_+^*\right)^{\frac{1}{2}} = \mathcal{P}_+^* B^{\perp,\hat{\mathrm{H}}_{\mathcal{P}_+}(D)}$ and Proposition 6.41 yields $\mathcal{P}_{V_+, W_+ \oplus \mathcal{P}_- \mathrm{L}^2} : \mathrm{H}^{\frac{1}{2}}(\partial M, E) \to \mathrm{H}^{\frac{1}{2}}(\partial M, E)$. Therefore, the maps g_0 and h_0 are bounded in the induced subspace topology from $\mathrm{H}^{\frac{1}{2}}(\partial M, E)$.

b) Next, we show that $B = \mathrm{graph}(g_0) \oplus W_+$.

i) We leave it as an exercise to verify that $\mathrm{graph}(g_0) \cap W_+ = 0$.

ii) The direction ‘$\supset$’.

Clearly, $W_+ \subset B$ by construction. Let $v \in V_-^{\frac{1}{2}} = \mathcal{P}_- B$ and by a) write $u := (X_-)^{-1} v \in B \cap W_+^{\perp,\mathrm{L}^2}$. Then,

$$\begin{aligned} v + g_0 v &= v + \mathcal{P}_{V_+, W_+ \oplus \mathcal{P}_- \mathrm{L}^2} (X_-)^{-1} v \\ &= X_- u + \mathcal{P}_{V_+, W_+ \oplus \mathcal{P}_- \mathrm{L}^2} u \\ &= \mathcal{P}_- u + \mathcal{P}_{V_+, W_+ \oplus \mathcal{P}_- \mathrm{L}^2} u + \mathcal{P}_{W_+, V_+ \oplus \mathcal{P}_- \mathrm{L}^2} u \\ &= \mathcal{P}_- u + \mathcal{P}_{V_+, W_+ \oplus \mathcal{P}_- \mathrm{L}^2} u + \mathcal{P}_{W_+, V_+ \oplus \mathcal{P}_- \mathrm{L}^2} u - \mathcal{P}_{W_+, V_+ \oplus \mathcal{P}_- \mathrm{L}^2} u \\ &= \mathcal{P}_- u + \mathcal{P}_+ u - \mathcal{P}_{W_+, V_+ \oplus \mathcal{P}_- \mathrm{L}^2} u \\ &= u - \mathcal{P}_{W_+, V_+ \oplus \mathcal{P}_- \mathrm{L}^2} u \\ &\in B \cap W_+^{\perp,\mathrm{L}^2} \oplus W_+ \subset B\,, \end{aligned}$$

where the fifth equality follow from $\mathcal{P}_+ \mathrm{L}^2(\partial M, E) = V_+ \oplus W_+$.

iii) The direction '$\subset$'.

Let $b \in B$ and write $b = b_+ + b_{+,\perp}$, where $b_+ := \mathcal{P}_{W_+, W_+^{\perp,\mathrm{L}^2}} b$ and $b_{+,\perp} := \mathcal{P}_{W_+^{\perp,\mathrm{L}^2}, W_+} b$. Clearly, $b_{+,\perp} \in B \cap W_+^{\perp,\mathrm{L}^2}$ and on letting $v := X_- b_{+,\perp} \in \mathcal{P}_- B$,

$$\begin{aligned} b_{+,\perp} &= (X_-)^{-1} v \\ &= \mathcal{P}_- (X_-)^{-1} v + \mathcal{P}_{V_+, W_+ \oplus \mathcal{P}_- \mathrm{L}^2} (X_-)^{-1} v + \mathcal{P}_{W_+, V_+ \oplus \mathcal{P}_- \mathrm{L}^2} (X_-)^{-1} v \\ &= v + g_0 v + \mathcal{P}_{W_+, V_+ \oplus \mathcal{P}_- \mathrm{L}^2} b_{+,\perp}\,. \end{aligned}$$

However, $\mathcal{P}_{W_+, V_+ \oplus \mathcal{P}_- \mathrm{L}^2} b_{+,\perp} \in W_+ \subset B$ and so $b_{+,\perp} \in \mathrm{graph}(g_0) \oplus W_+$. Therefore,

$$b = b_+ + b_{+,\perp} \in W_+ + \mathrm{graph}(g_0) \oplus W_+ \subset \mathrm{graph}(g_0) \oplus W_+\,.$$

Similarly, $B^{\perp, \hat{\mathrm{H}}_{\mathcal{P}_+}(D)} = \mathrm{graph}(h_0) \oplus W_-^*$ by mimicking this argument.

c) The adjoint map of g_0 in $\mathrm{H}^{-\frac{1}{2}}(\partial M, E)$ satisfies $g_0^* : (V_+^*)^{-\frac{1}{2}} \to (V_-^*)^{-\frac{1}{2}}$ with $g_0^*|_{\mathrm{H}^{\frac{1}{2}}} = -h_0$. Similarly, the adjoint map of h_0 in $\mathrm{H}^{-\frac{1}{2}}(\partial M, E)$ satisfies $h_0 : (V_-^*)^{-\frac{1}{2}} \to (V_+^*)^{-\frac{1}{2}}$ with $h_0^*|_{\mathrm{H}^{\frac{1}{2}}} = -g_0$.

Recall from Corollary 6.42 that $\left\langle V_\pm^{\frac{1}{2}}, \left(V_\pm^*\right)^{-\frac{1}{2}} \right\rangle$ extends $\langle \mathrm{L}^2(\partial M, E), \mathrm{L}^2(\partial M, E)\rangle$. Consequently, we obtain that

$$g_0^* := \left(V_+^*\right)^{-\frac{1}{2}} \to \left(V_-^*\right)^{-\frac{1}{2}}$$

be the adjoint map with respect to this pairing.

By definition,

$$\left\langle V_-, V_+^* \right\rangle = \left\langle V_-^*, V_+ \right\rangle = \left\langle B, B^{\perp, \hat{\mathrm{H}}_{\mathcal{P}_+}(D)} \right\rangle = 0\,. \tag{6.4}$$

Now, fix $u \in \left(V_+^*\right)^{\frac{1}{2}}$ and $v \in V_-^{\frac{1}{2}}$. Then, using (6.4), we obtain

$$0 = \langle v + g_0 v, u + h_0 u\rangle = \langle g_0 v, u\rangle + \langle v, h_0 u\rangle\,.$$

That is,

$$\langle v, g_0^* u\rangle = \langle v, -h_0 u\rangle$$

for all $v \in V_-^{\frac{1}{2}}$. By previous Proposition 6.41, we know that $V_-^{\frac{1}{2}}$ is dense in V_- and $V_-^{-\frac{1}{2}}$. Therefore, we conclude that $g_0^* u = -h_0 u$ for all $u \in (V_+^*)^{\frac{1}{2}}$.

By considering the adjoint map to h_0, it is clear that the stated conclusion for h_0^* follows.

d) The map g_0 extends to $g : V_- \to V_+$ bounded.

We prove this this by resorting to complex interpolation (c.f. Section 3.9). Note that we know that

$$\mathrm{L}^2(\partial M, E) = \left[\mathrm{H}^{\frac{1}{2}}(\partial M, E), \mathrm{H}^{-\frac{1}{2}}(\partial M, E)\right]_{\vartheta=\frac{1}{2}}.$$

Moreover, we know

$$\mathcal{P}_{V_\pm^{\frac{1}{2}}, W_\pm \oplus \mathcal{P}_\mp \mathrm{H}^{\frac{1}{2}}} : \mathrm{H}^{\frac{1}{2}}(\partial M, E) \to \mathrm{H}^{\frac{1}{2}}(\partial M, E),$$
$$\mathcal{P}_{V_\pm^{-\frac{1}{2}}, W_\pm \oplus \mathcal{P}_\mp \mathrm{H}^{-\frac{1}{2}}} : \mathrm{H}^{-\frac{1}{2}}(\partial M, E) \to \mathrm{H}^{-\frac{1}{2}}(\partial M, E).$$

We use the Retraction-Coretraction theorem, Theorem 3.57. For that, define

$$\mathcal{B}_1 := \mathrm{H}^{\frac{1}{2}}(\partial M, E) \qquad \mathcal{B}_2 := \mathrm{H}^{-\frac{1}{2}}(\partial M, E)$$
$$\widetilde{\mathcal{B}_1} := V_\pm^{\frac{1}{2}} \qquad \widetilde{\mathcal{B}_2} := V_\pm^{-\frac{1}{2}}.$$

Define $R := \mathcal{P}_{V_\pm^{-\frac{1}{2}}, W_+ \oplus \mathcal{P}_\mp \mathrm{H}^{-\frac{1}{2}}}$ and $S := \mathrm{id}$. Indeed, since R is now a projection on $\widetilde{\mathcal{B}}_i$, the choice of S indeed makes R a retraction with coretraction S.

With this choice of spaces, Theorem 3.57 then asserts

$$S_\vartheta\left[V_\pm^{\frac{1}{2}}, V_\pm^{-\frac{1}{2}}\right] = \left(\mathcal{P}_{V_\pm^{-\frac{1}{2}}, W_+ \oplus \mathcal{P}_\mp \mathrm{H}^{-\frac{1}{2}}}\right)_\vartheta \left[\mathrm{H}^{\frac{1}{2}}(\partial M, E), \mathrm{H}^{-\frac{1}{2}}(\partial M, E)\right]_\vartheta.$$

However $S_\vartheta = \mathrm{id}$, and hence $S_\vartheta\left[V_\pm^{\frac{1}{2}}, V_\pm^{-\frac{1}{2}}\right] = \left[V_\pm^{\frac{1}{2}}, V_\pm^{-\frac{1}{2}}\right]_\vartheta$. Therefore, by choosing $\vartheta = \frac{1}{2}$, we deduce

$$V_\pm = \left[V_\pm^{\frac{1}{2}}, V_\pm^{-\frac{1}{2}}\right]_{\vartheta=\frac{1}{2}}.$$

By c), we conclude

$$h_0^* : V_-^{-\frac{1}{2}} \to V_+^{-\frac{1}{2}} \text{ and } h_0^*|_{\mathrm{H}^{\frac{1}{2}}} : V_-^{\frac{1}{2}} \to V_+^{\frac{1}{2}}.$$

By interpolation, obtain $h := (h_0^*)_{\vartheta=\frac{1}{2}} : V_- \to V_+$ boundedly. By construction, this extends $-g_0$ and therefore, $g := -h$ is the desired map. □

Remark 6.48. In b) iii), $\mathcal{P}_{W_+,V_+\oplus\mathcal{P}_-\mathrm{L}^2} b_{+,\perp}$ is not necessarily 0 even though $b_{+,\perp} \in W_+^{\perp,\mathrm{L}^2}$. This is due to the fact that $\mathcal{P}_{W_+,V_+\oplus\mathcal{P}_-\mathrm{L}^2} b_{+,\perp}$ does not project along $W_+^{\perp,\mathrm{L}^2}$. By Lemma 6.44, we certainly have $W_+^{\perp,\mathrm{L}^2} \cong V_+ \oplus \mathcal{P}_-\mathrm{L}^2(\partial M, E)$, but in the calculations here, what is required is equality.

This is one reason why the proof of this theorem is cumbersome. We are required to organise splittings of the relevant spaces in a very precise way so that we can obtain equalities than just isomorphisms.

6.3 Properties of elliptically regular boundary conditions

In this short section, we examine some salient features of elliptically regular boundary conditions. To that end, we start with the following definition.

Definition 6.49 (Semiregular ad regular boundary conditions). For $s \geq \frac{1}{2}$, we say an elliptically regular boundary condition B is $\left(s + \frac{1}{2}\right)$*-semiregular* if there is a boundary decomposing projector $\mathcal{P}_+$ s.t. $W_+ \subset \mathrm{H}^s(\partial M, E)$ and $g\left(V_-^s\right) \subset V_+^s$, where W_+ and g are the objects appearing in Definition 6.6 for the L^2-graphical decomposition of B:w w.r.t. $\mathcal{P}_+$. We say B is $\left(s + \frac{1}{2}\right)$*-regular*, if both B and $B^\dagger$ are $(s + \frac{1}{2})$-semiregular.

A map $\mathcal{P}$ is called a *classical pseudo-differential projector of order* 0 if locally, it can be written as an integral against a symbol, which is simply an endomorphism, and where the symbol enjoys a certain asymptotic expansion. The 'classical' part of this is precisely the asymptotic expansion. Pseudo-differential operators of order 0 act boundedly on all Sobolev spaces on a closed manifold. The quintessential example of a classical pseudo-differential operator of order 0 in our context are the spectral projections $\chi^+(A)$ associated an invertible bisectorial adapted boundary operator A.

These are traditionally well studied objects, and we shall refrain from exploring their technical underpinnings to greater lengths, as this is beyond the scope of this text. We mention these as they have historically been central to the study of boundary value problems. These projectors induce an important class of boundary conditions as identified in the following definition.

Definition 6.50. Let $\mathcal{P}$ be a classical pseudo-differential projector of order 0. Then

$$B_{\mathcal{P}} := \overline{\mathcal{P}\mathrm{H}^{\frac{1}{2}}(\partial M, E)}^{\|\cdot\|_{\check{\mathrm{H}}(D)}}$$

is called a *pseudo-local boundary condition.* If $\mathcal{E} \subset E|_{\partial M}$ is a smooth subbundle, then

$$B_{\mathcal{E}} := \overline{\mathrm{H}^{\frac{1}{2}}(\partial M, E)}^{\|\cdot\|_{\check{\mathrm{H}}(D)}}$$

is called a *local boundary condition.*

Remark 6.51. If $\mathcal{E}$ is a smooth subbundle, then it has an associated projection on each fibre, which varies smoothly. This then defines a classical pseudo-differential projector of order zero. Therefore, every local boundary condition is pseudo-local.

Theorem 6.52. *Let $B_{\mathcal{P}}$ be a pseudo-local boundary condition. Then the following are equivalent.*

(I) $B_{\mathcal{P}}$ is elliptically regular and $B_{\mathcal{P}} = \mathcal{P}\mathrm{H}^{\frac{1}{2}}(\partial M, E)$ (i.e. $\mathcal{P}\mathrm{H}^{\frac{1}{2}}(\partial M, E)$ is automatically closed in $\check{\mathrm{H}}(D)$).

(II) For some (equivalently all) invertible adapted boundary operators A,

$$\mathcal{P} - \chi^+(A) : \mathrm{L}^2(\partial M, E) \to \mathrm{L}^2(\partial M, E)$$

is elliptic (equivalently Fredholm).

(III) For all $x \in \partial M$ and $\xi \in T^*_x\partial M \setminus \{0\}$ the principal symbol

$$\sigma_{\mathcal{P}}(x, \xi) : E_x \to E_x$$

restricts to an isomorphism

$$\bigoplus_{\mathrm{Re}(\lambda)<0} \mathrm{Eig}_{\imath\sigma_A(x,\xi)}(\lambda) \cong \sigma_{\mathcal{P}}(x, \xi)E_x$$

and $\sigma_{\mathcal{P}^*}(x, \xi)$ restricts to an isomorphism

$$\bigoplus_{\mathrm{Re}(\lambda)<0} \mathrm{Eig}_{\imath\sigma_{A^*}(x,\xi)}(\lambda) \cong \sigma_{\mathcal{P}^*}(x, \xi)E_x\,.$$

Corollary 6.53. *Every pseudo-local elliptically regular $B_{\mathcal{P}}$ is ∞-regular. In particular, if $D_{B_{\mathcal{P}}}v \in \mathrm{C}^\infty(M, F)$, then $v \in \mathrm{C}^\infty(M, E)$ (i.e. smooth up to the boundary ∂M).*

Clearly, the APS boundary conditions associated with an adapted boundary operator are pseudo-local. We have seen in Chapter 5, these are the most significant boundary conditions from a point of view of global analysis and index theory. However, there are also interesting local boundary conditions which are elliptically regular for first-order operators, even if they do not lead to index formulae.

Example 6.54 (Absolute and relative boundary conditions). Let (M,g) be a Riemannian manifold, and $T = \vec{n}$ the inward pointing normal. Let $\tau = \vec{N} = \vec{n}^\flat$, the associated conormal. Define

$$E := F := \Lambda_{\mathbb{C}} T^*M = \Lambda T^*M \otimes \mathbb{C}\,.$$

The natural operator on this bundle is the Hodge-Dirac-operator

$$D_{\mathrm{H}} = d + d^*\,.$$

On the boundary, we have the following splitting

$$\Lambda^p T_x^* M|_{\partial M} = \Lambda^p T_x^* \partial M \oplus \tau(x) \wedge \Lambda^{p-1} T_x^* \partial M\,.$$

Therefore, we have that $u \in \Lambda^p T_x^* M|_{\partial M}$ can be written as

$$u = u^T + \tau(x) \wedge u^\perp\,.$$

Define $E' := \Lambda_{\mathbb{C}} \partial M = \Lambda \partial M \otimes \mathbb{C}$. Then, *absolute boundary conditions* for D_{H} is given by

$$B_{\mathrm{abs}} := \mathrm{H}^{\frac{1}{2}}(\partial M, E') = \left\{ u \in \mathrm{H}^{\frac{1}{2}}(\partial M, E) \,\middle|\, u^\perp = 0 \right\}.$$

Now, define $E'' := \tau \wedge \Lambda_{\mathbb{C}} \partial M$. Then, the *relative boundary conditions* for D_{H} is given by

$$B_{\mathrm{rel}} := \mathrm{H}^{\frac{1}{2}}(\partial M, E'') = \left\{ u \in \mathrm{H}^{\frac{1}{2}}(\partial M, E) \,\middle|\, u^T = 0 \right\}.$$

Both these boundary conditions are locally elliptically regular.

Example 6.55 (Chiral boundary conditions). Assume now that (M,g) is Riemannian Spin manifold. Then, the bundles in question are $E := F := \Delta\!\!\!/ M$ and $D\!\!\!/$ is the Spin-Dirac.

We say that a map $G : \Delta\!\!\!/ M|_{\partial M} \to \Delta\!\!\!/ M|_{\partial M}$ is a *boundary chirality operator* if

$$\begin{aligned} &G^2 = \mathrm{id}, && h(Gu, Gv) = h(u,v), \\ &\nabla\!\!\!/ G = 0, && \sigma_{D\!\!\!/}(x,\xi) \circ G = -G \circ \sigma_{D\!\!\!/}(x,\xi), \text{ and} \\ &\sigma_{D\!\!\!/}(x,\tau) \circ G = G \circ \sigma_{D\!\!\!/}(x,\tau) && \end{aligned}$$

for all $\xi \in T_x^* \partial M$. Fibrewise, G is an involution and therefore its spectrum consists of $\{\pm 1\}$. Hence, we obtain bundles

$$E_\pm := \mathrm{Eig}_G(\pm 1), \qquad \Delta\!\!\!/ M = E_+ \oplus E_-.$$

The local boundary conditions

$$B_\pm := \mathrm{H}^{\frac{1}{2}}(\partial M, E_\pm)$$

are the chiral boundary conditions. They are local elliptically regular boundary conditions.

Let us now look at some particular examples of boundary chirality operators.

1. If $\dim(M)$ is even then $G := \mu_{\mathbb{C}}$ complexified volume element is a boundary chirality operator. In this situation, the usual splittings to the subspaces $\not{\Delta}^{\pm}\partial M$ are precisely $\mathrm{Eig}_{\mu_{\mathbb{C}}}(\pm 1)$.

2. If $\dim(M)$ is odd and $G := \imath\tau$, then the space B_+ is called the *MIT Bag Condition*. This was originally conceived to model electrons contained in a bag. It illustrates that, despite the lack of an index formula, local elliptically regular boundary conditions for first-order operators may still be of physical relevance. It is this boundary condition which appears in the following illustration, also appearing on the cover of this text.

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Symbols

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