Klaus Ecker - Lec 5 31/05/2018

Recall: $\mathcal{W}_\beta(\Omega, f, \tau) := \int_\Omega [\tau|\nabla f|^2 + f - (n+1)]u + 2\tau \int_{\partial\Omega} \beta u$.

$\tau > 0$, $\underbrace{f : \bar\Omega \to \mathbb{R}}_{\text{"admissible"}}$, $\Omega \subset \mathbb{R}^{n+1}$, $\underbrace{\beta : \partial\Omega \to \mathbb{R}}_{\text{"reasonable"}}$.

$u = \dfrac{e^{-f}}{(4\pi\tau)^{\frac{n+1}{2}}}$, $\mu_\beta(\Omega, \tau) = \inf\{\mathcal{W}_\beta(\Omega, f, \tau) : \int_\Omega u = 1\}$.

$\nu_\beta(\Omega) :=$

Have seen: $\mu_\beta(\Omega, \tau) > -\infty$ if Ω satisfies the usual Gagliardo - Nirenberg inequality in $\mathbb{R}^{n+1}$.

(holds if Ω not necessarily,

$$\left(\int_\Omega |w|^{\frac{n+1}{n}}\right)^{n/n+1} \le c_S(\Omega) \left(\int_\Omega |\nabla w| + |w|\right) \quad w \in W^{1,1}(\Omega)$$

and $\sup_{\partial\Omega} |\beta| < \infty$, $\partial\Omega \in C^{0,1}$.

Under these conditions $\exists$ minimiser (for simplicity called f), uniqueness unknown unless $\Omega = B_R$ and $\tau = R^2/2n$.

Prop Let $\tau > 0$ be fixed, Ω, β "suitable" (w.l.o.g. $\beta \in C^\infty$, $\partial\Omega \in C^2$)

Smooth fct. $f : \bar\Omega \to \mathbb{R}$ minimises $\mu_\beta(\Omega, \tau)$.

(uniqueness not clear for this). Iff

(I) $W_\tau(f) := \tau(2\Delta f - |\nabla f|^2) + f - (n+1) = \mu_\beta(\Omega, \tau)$.

Linearisation: $\Delta\eta - \nabla f\cdot\nabla\eta =: L_f$.

$$\int_\Omega L_f v\cdot w\, u\,dx = \int_\Omega v\cdot L_f w\, u\,dx \quad , v,w \text{ sat. } \nabla w\cdot\nu = \nabla v\cdot\nu = 0. \text{ (Neumann)}.$$

(II) $\nabla f\cdot\nu = \beta$ on $\partial\Omega$.

(III) $\int_\Omega u = 1$.

($\Leftarrow$) previous lecture:

If $\nabla f\cdot\nu = \beta$, then $\int_\Omega \Delta u = -\int_{\partial\Omega}\beta u$.

($\Rightarrow$) first variation of $W_\beta(\Omega,f,\tau)$ in f (Ω,τ fixed). w.r.t. constrain $\int_\Omega u = 1$.

$$\delta f = \eta \quad \left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f_\varepsilon,\ f_\varepsilon(x),\ \varepsilon\in(-\alpha,\alpha),\ f_0(x)=f(x)\right)$$

$\delta u = -u\eta$, λ Lagrange mult:

$$\delta\left(W_\beta(\Omega,f,\tau) \cancel{-\delta\int_{\partial\Omega}} + \lambda\int_\Omega u\right)$$

$$= \delta\int_\Omega\left[\tau|\nabla f|^2 + f - (n+1)\right]u + \lambda\delta\int_\Omega u + 2\tau\delta\int_{\partial\Omega}\beta u.$$

we do not require $\int_\Omega u(f_\varepsilon) = 1\ \forall\varepsilon\in(-\alpha,\alpha)$.

$$\Rightarrow \int_\Omega\left[(2\tau\ \nabla f\cdot\nabla\eta + \eta)u - \left[\tau|\nabla f|^2 + f - (n+1)\right]u\,\eta\right.$$
$$-\lambda\int_\Omega u\eta$$

$$-2\tau\int_\Omega \beta u\eta = 0 \text{ if } f \text{ is a min } f\ \mu_\beta(\Omega,\tau).\ \forall\eta\in C^1(\bar\Omega)$$

Rewrite and rearrange:

$$\int_\Omega \nabla f\cdot\nabla\eta\; u = -\int_\Omega \nabla u\cdot\nabla\eta = +\int_\Omega \Delta u\,\eta + \int_{\partial\Omega} \nabla f\cdot\nu\, u\,\eta .$$

$\nabla u = -u\nabla f$.

$$2\tau\int_\Omega \nabla f\cdot\nabla\eta\cdot u = 2\tau\int_\Omega \Delta u\,\eta + 2\tau\int_{\partial\Omega} \nabla f\cdot\nu\; u\,\eta .$$

$$= 2\tau\int_\Omega u\left(|\nabla f|^2 - \Delta f\right)\eta + 2\tau\int_{\partial\Omega} \nabla f\cdot\nu\, u\,\eta .$$

$$\Rightarrow \quad 0 = \int_\Omega 2\tau|\nabla f|^2 u\eta - 2\tau\,\Delta f\, u\eta - \tau|\nabla f|^2 u\eta$$

$$- f u\eta + (n+1)u\eta + 2\tau\int_{\partial\Omega}(\nabla f\cdot\nu - \beta)\eta u$$

$$-\lambda\int_\Omega u\eta \qquad \forall \eta\in C^1(\bar\Omega).$$

η arb $\Rightarrow$ $\nabla f\cdot\nu = \beta$ on $\partial\Omega$.

And $\displaystyle\int_\Omega \left(2\tau\,\Delta f - \tau|\nabla f|^2 + f - (n+1) + \lambda\right) u\,\eta = 0$.

$\forall \eta\in C_0^1(\Omega)$.

$\Rightarrow$ $W_\tau(f) + (\lambda - 1) = 0$ in Ω.

assuming f minimum.

Constraint $\displaystyle\int_\Omega u = 1$: $\displaystyle\mu_\beta(\Omega,\tau) = \mathcal{W}_\beta(\Omega, f, \tau) = \int_\Omega (1-\lambda)u$

$= (1-\lambda)$.

Need Lemma: $\nabla f\cdot\nu = \beta$ on $\partial\Omega$

$\Rightarrow$ $\displaystyle\mathcal{W}_\beta(\Omega, f, \tau) = \int_\Omega W_\tau(f)\, u$.