$$\mathcal{W}_\beta(\Omega, f, \tau) \geq 2\tau a_\beta(\Omega) - c(n,\Omega)\left(1 + \log(1+\tau)\right).$$

$$a_\beta(\Omega) := \inf\left\{ \int_\Omega |\nabla\varphi|^2 + \int_{\partial\Omega} \beta\varphi^2 , \int_\Omega \varphi^2 = 1 \right\}.$$

Have "nearly" proved that if $\beta > 0$, then $a_\beta(\Omega) > 0$.
and hence, $\nu_\beta(\Omega) := \inf_{\tau>0} \mu_\beta(\Omega,\tau)$

$$= \inf_{\tau>0} \inf_f \left\{ \mathcal{W}_\beta(\Omega,f,\tau), \int_\Omega u = 1 \right\} > -\infty$$

"nearly" – modulo showing that $\exists !$ min φ_a for $a_\beta(\Omega)$.

Without sign conditions on β, estimate RHS further from below.

$$2\int_\Omega |\nabla\varphi|^2 + 2\int_{\partial\Omega} \beta\varphi^2$$

$$\geq 2\int_\Omega |\nabla\varphi|^2 - 2\sup_{\partial\Omega}|\beta| \int_{\partial\Omega} \varphi^2 .$$

$\partial\Omega \in C^{0,1}$, ~~and~~ Ω bdd, trace embedding in L^1:

$$\int_{\partial\Omega} \varphi^2 \leq c_2(\Omega)\left(\int_\Omega |\nabla\varphi^2| + \varphi^2\right)$$

$$= c_2(\Omega)\int_\Omega 2|\varphi||\nabla\varphi| + \varphi^2 .$$

$$\leq \varepsilon \int_\Omega |\nabla\varphi|^2 + \frac{c_2(\Omega)^2}{\varepsilon}\int_\Omega \varphi^2 + c_2(\Omega)\int_\Omega \varphi^2 .$$

$$= \varepsilon \int_\Omega |\nabla \varphi|^2 + C_2(\Omega)\left(1 + \frac{C_2(\Omega)}{\varepsilon}\right) \quad \text{since } \int_\Omega \varphi^2 = 1.$$

$$2\int_\Omega |\nabla\varphi|^2 + 2\int_{\partial\Omega} \beta\varphi^2 \geq 2\int_\Omega |\nabla\varphi|^2 - 2\sup_{\partial\Omega}|\beta| \int_{\partial\Omega}\varphi^2 .$$

$$\geq 2\int_\Omega |\nabla\varphi|^2 - 2\left(\sup_{\partial\Omega}|\beta|\right)\cdot\varepsilon\int_\Omega |\nabla\varphi|^2$$

$$- 2\left(\sup_{\partial\Omega}|\beta|\right) C_2(\Omega)\left(1 + \frac{C_2(\Omega)}{\varepsilon}\right).$$

w.l.o.g, $f \neq 0$, ε s.t. $\varepsilon\, 2 \sup_{\partial\Omega}|\beta| = 1$.

$$\Rightarrow \quad \geq \int_\Omega |\nabla\varphi|^2 - 2\sup_{\partial\Omega}|\beta|\, C_2(\Omega)\left(1 + 2\sup_{\partial\Omega}|\beta|\, C_2(\Omega)\right).$$

But now with a τ, one gets

$$2\tau\int_\Omega |\nabla\varphi|^2 + 2\tau\int_{\partial\Omega}\beta\varphi^2 \geq -c\left(1 + \sup_{\partial\Omega}|\beta|^2 + C_2(\Omega)^2\right)\cdot\tau$$

So,

$$W_\beta(\Omega, f, \tau) \geq 2\tau\, a_\beta(\Omega) - c(n,\Omega)\left(1 + \log(1+\tau)\right).$$

$$\geq \tau\int_\Omega |\nabla\varphi|^2 - c(n,\Omega,\tau,\beta), \qquad \int_\Omega \varphi^2 = 1.$$

$$\Rightarrow \quad \mu_\beta(\Omega,\tau) = \inf_f \left\{ W_\beta(\Omega, f, \tau) : \int_\Omega u = 1 \right\} > -\infty .$$

Let (φ_j). $\left(\Leftrightarrow f_j,\ \varphi_j^2 = \frac{e^{-f_j}}{(4\pi\tau)^{\frac{n+1}{2}}} = u(f_j)\right)$ be a minimising sequence for $\mu_\beta(\Omega,\tau)$.

I.e., $\mathcal{E}_\beta(\Omega,\varphi_j,\tau) = \mathcal{W}_\beta(\Omega, f_j, \tau)$ and

$$\min\{\mathcal{E}_\beta(\Omega,\varphi_j,\tau) : \int_\Omega \varphi_j^2 = 1\} = \mu_\beta(\Omega,\tau).$$

I.e. $\mathcal{E}_\beta(\Omega,\varphi_j,\tau) \to \mu_\beta(\Omega,\tau) > -\infty$.

$\Rightarrow$ $\mathcal{W}_\beta(\Omega,f_j,\tau)$ ~~bdd~~ $= \mathcal{E}_\beta(\Omega,\varphi_j,\tau)$ bdd indep of j

$\Leftrightarrow$ $\tau \int_\Omega |\nabla\varphi_j|^2 \leq \tilde{C}(n,\Omega,\tau,\beta)$. and $\int_\Omega \varphi_j^2 = 1$.

$\Rightarrow$ $\int_\Omega |\nabla\varphi_j|^2 \leq C(n,\Omega,\tau,\beta)$. and $\int_\Omega \varphi_j^2 = 1$.

where $C = \frac{\tilde{C}}{\tau}$.

2. Functional Analysis

$\exists$ subsequence (again called φ_j) and $\varphi \in H^1(\Omega)$.

s.t. $\varphi_j \rightharpoonup \varphi$ $j\to\infty$.

$H^1(\Omega) \hookrightarrow L^2(\Omega)$ cpct $(\partial\Omega \in C^{0,1})$.

$\exists$ subseqn (φ_j again) $\varphi_j \to \varphi$ in $L^2(\Omega)$.

$\Rightarrow$ $1 = \int_\Omega \varphi_j^2 \Rightarrow \int_\Omega \varphi^2$.

$\varphi_j \xrightarrow[j\to\infty]{} \varphi$ in $L^2(\partial\Omega)$ (upto subsequence).

Since $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$.

$$\Rightarrow \int_{\partial\Omega} \beta \varphi_j^2 \longrightarrow \int_{\partial\Omega} \beta\varphi^2 .$$

Also $\liminf_{j\to\infty} \int_{\partial\Omega} |\nabla \varphi_j|^2 \geq \int_\Omega |\nabla\varphi|^2$.

Rothaus argument. $\int_\Omega \varphi_j^2 \log \varphi_j^2 \xrightarrow{j\to\infty} \int_\Omega \varphi^2 \log\varphi^2$.

$$\Rightarrow \liminf_{j\to\infty} \mathcal{W}_\beta(\Omega, f_j, \tau) = \liminf_{j\to\infty} \mathcal{E}_\beta(\Omega, \varphi_j, \tau)$$
$$\geq \mathcal{E}_\beta(\Omega,\varphi,\tau) = \mathcal{W}_\beta(\Omega, f, \tau).$$
where $\varphi^2 = \dfrac{e^{-f}}{(4\pi\tau)^{\frac{n+1}{2}}}$

Since $\int u = \int \varphi^2 = 1 \Rightarrow f$ is a minimizer of $\mu_\beta(\Omega,\tau)$.

Exercise: Much easier:

$\exists!\ \varphi_a$ with $\int_\Omega \varphi_a^2 = 1$ of $\int_\Omega |\nabla\varphi|^2 + \int_{\partial\Omega}\beta\varphi^2$ if $\beta \geq 0$.

Uniqueness of minimizer? – hard known., but one case:

Proposition: (a) Let $R>0$. Then $\mu_{\frac{n}{R}}\left(B_R, \frac{R^2}{2n}\right)$ ~~has~~

($\frac{n}{R}$ – mean curv., $\tau = \frac{R^2}{2n}$ time left before collapse.)

(Pf later) has f^R ~~given that~~ unique minimizer given by

$$f^R(x) := \frac{n|x|^2}{2R^2} + \log\left(\int_{B_{\sqrt{n}}} \gamma_{n+1}\right)$$

(5) for $\tau>0$, $\mu_{\sqrt{\frac{n}{2\tau}}}(B_{\sqrt{2n\tau}}, \tau)$.

(think of τ as time left before collapse)

then $\exists!\, f^\tau$ given by

$$f^\tau(x) = \frac{|x|^2}{4\tau} + \log\left(\int_{B_{\sqrt{n}}} \gamma_{n+1}\right).$$

(mean curv of $B_{\sqrt{2n\tau}}$ is $\sqrt{\frac{n}{2\tau}}$).

<u>Euler-Lagrange</u> – do for one with the f.

<u>Prop</u>. Let $\tau>0$ be fixed, Ω, β <u>suitable</u> (ie $\mu_\beta(\Omega,\tau)$ well-def, div. thm holds on Ω).

Then, $f:\overline{\Omega}\to\mathbb{R}$ is a minimizer of $\mu_\beta(\Omega,\tau)$ <u>iff</u>

(I) $W_\tau(f) := \tau(2\Delta f - |\nabla f|^2) + f - (n+1) = \mu_\beta(\Omega,\tau)$

(II) $\nabla f\cdot\nu = \beta$ on $\partial\Omega$.

(III) $\int_\Omega u = 1$.

(I)–(III) $\Rightarrow$ min.

Exercises: $u = \frac{e^{-f}}{(4\pi\tau)^{\frac{n+1}{2}}}$

$\nabla_i u = -u\,\nabla_i u$.

$\Delta u = u(|\nabla f|^2 - \Delta f)$.

$$-\int_{\partial\Omega}\beta u \overset{(2)}{=} \int_{\partial\Omega} -u\nabla f\cdot\nu = \int_{\partial\Omega}\nabla u\cdot\nu = \int_\Omega \Delta u = \int_\Omega(|\nabla f|^2-\Delta f)\,u.$$

$$\Rightarrow \int_\Omega W_\tau(f)\, u \overset{①}{=} \int_\Omega 2\tau\left(\Delta f - |\nabla f|^2\right) u$$

$$+ \int_\Omega \tau |\nabla f|^2 u + \int_\Omega f u - \int_\Omega (n+1) u .$$

$$\mu_\beta(\Omega,\tau) \int_\Omega u = \mu_\beta(\Omega,\tau) . = \mathcal{W}_\beta(\Omega, f, \tau).$$