Upper bound on $\mu_\beta(\Omega,\tau)$:

$$\mu_\beta(\Omega,\tau) \le \log\left(\frac{|\Omega\cap B_{\sqrt{\tau}}(x_0)|}{\tau^{\frac{n+1}{2}}}\right) + c(n)\left(\frac{|\Omega\cap B_{\sqrt{\tau}}(x_0)| + 2\tau\int_{\partial\Omega\cap B_{\sqrt{\tau}}(x_0)}|\beta|}{|\Omega\cap B_{\frac{\sqrt{\tau}}{2}}(x_0)|}\right)$$

$\forall B_r(x_0)\subset\mathbb{R}^{n+1}$ with $|\Omega\cap B_{\frac{\sqrt{\tau}}{2}}(x_0)|>0$.

Cor. Suppose $\mu_\beta(\Omega,r^2)\ge -c_0$ and we have for $|\Omega\cap B_{\frac{r}{2}}(x_0)|>0$ and

$$\frac{|\Omega\cap B_r(x_0)| + r^2\int_{\partial\Omega\cap B_r(x_0)}|\beta|}{|\Omega\cap B_{\frac{r}{2}}(x_0)|}\le c_1.$$

Then $\exists K=K(c_0,c_1)>0$ s.t. $\frac{|\Omega\cap B_r(x_0)|}{r^{n+1}}\ge K$.

(K-non-collapsed in $B_r(x_0)$)

Collapse!

$\nexists K>0$ s.t.

$|\Omega_\varepsilon\cap B_r(x_0)|\ge Kr^2$ $\forall\varepsilon>0$


$K_\varepsilon\equiv 0$.

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Examples for which $\gamma_\beta(\Omega)=\inf_{\tau>0}\mu_\beta(\Omega,\tau)=-\infty$:

$\Omega$ = slab, $\Omega$ = int. of catenoid, $\Omega=GR\times\mathbb{R}^n$ (these collapse)

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Proof. $e^{-f}=a\xi$, $\xi\in C^\infty$ to be chosen, $a\in\mathbb{R}$

w.l.o.g. $a\ne 0$, $\xi\not\equiv 0$. We have $\xi\ge 0$.

$$1=\int_\Omega u=\int_\Omega\frac{e^{-f}}{(4\pi\tau)^{\frac{n+1}{2}}}=\int_\Omega\frac{a\xi}{(4\pi\tau)^{\frac{n+1}{2}}}\Rightarrow a=\frac{(4\pi\tau)^{\frac{n+1}{2}}}{\int_\Omega\xi}$$

$$\left((4\pi\tau)^{\frac{n+1}{2}}=a\int_\Omega\xi \text{ or } \frac{1}{(4\pi\tau)^{\frac{n+1}{2}}}=\frac{1}{a\int_\Omega\xi}\right)$$

$-f=\log(a\xi)$, $f=-\log(a\xi)$

$$\nabla f=-\frac{1}{a\xi}\cdot a\nabla\xi=-\frac{\nabla\xi}{\xi}\rightsquigarrow|\nabla f|^2=\frac{|\nabla\xi|^2}{\xi^2}$$

$$\Rightarrow |\nabla f|^2u=|\nabla f|^2\cdot\frac{e^{-f}}{(4\pi\tau)^{\frac{n+1}{2}}}=\frac{|\nabla\xi|^2}{\xi^2}\xi a\cdot\frac{1}{(4\pi\tau)^{\frac{n+1}{2}}}=\frac{|\nabla\xi|^2}{\xi}a\cdot\frac{1}{(4\pi\tau)^{\frac{n+1}{2}}}$$

$$\left( \boxed{\begin{array}{l} \int_\Omega \tau |\nabla f|^2 u = \frac{a}{(4\pi\tau)^{\frac{n+1}{2}}} \int_\Omega \tau \frac{|\nabla g|^2}{g} \\ \hline \int_\Omega f u = \frac{-a}{(4\pi\tau)^{\frac{n+1}{2}}} \int_\Omega g \log(a g) \end{array}} \right.$$

$$\Rightarrow \quad W_\beta(\Omega, f, \tau) = \frac{a}{(4\pi\tau)^{\frac{n+1}{2}}} \int_\Omega \left(\tau \frac{|\nabla g|^2}{g} - g \log(a g)\right)$$

$$- (n+1) + 2\tau \frac{\int_{\partial\Omega} \beta g}{\int_\Omega g} .$$

Approximate $g$ by $g$'s in $C_0^2(\mathbb{R}^{n+1})$ with $g \geq 0$ so w.l.o.g.

$g \geq 0$ satisfies $g \in C_0^2(\mathbb{R}^{n+1}) \rightsquigarrow \tau \cdot \frac{|\nabla g|^2}{g} \leq 2\tau |\nabla^2 g|^2 \leq c(n)\tau$

Assume $\chi_{B_{\frac{\sqrt{\tau}}{2}}(x_0)} \leq g \leq \chi_{B_{\sqrt{\tau}}(x_0)}$.

Set now $\tau = r^2$

$r^2 \frac{|\nabla g|^2}{g} \leq 2r^2 |\nabla^2 g|^2$

with

$\chi_{B_{\frac{r}{2}}(x_0)} \leq g \leq \chi_{B_r(x_0)}$

and $g \in C_0^2(\mathbb{R}^{n+1})$

$\Rightarrow |\nabla^2 g| \leq \frac{c(n)}{r}$

$\rightsquigarrow \leq c(n)$

$$\rightsquigarrow \quad \tau \frac{|\nabla g|^2}{g} \leq c(n) \quad \text{and} \quad \int_\Omega g \geq |\Omega \cap B_{\frac{\sqrt{\tau}}{2}}(x_0)| > 0$$

$$\frac{a}{(4\pi\tau)^{\frac{n+1}{2}}} \int_\Omega \tau \frac{|\nabla g|^2}{g} \leq c(n) \cdot \underbrace{\frac{a}{(4\pi\tau)^{\frac{n+1}{2}}}}_{\leq \frac{1}{\int_\Omega g}} |\Omega \cap B_{\sqrt{\tau}}(x_0)|$$

$$\leq c(n) \cdot \frac{|\Omega \cap B_{\sqrt{\tau}}(x_0)|}{|\Omega \cap B_{\frac{\sqrt{\tau}}{2}}(x_0)|}$$

Jensen's inequality: $\varphi : \mathbb{R} \to \mathbb{R}$ convex $\rightsquigarrow \varphi\left( ⨍_\Omega w \right) \leq ⨍_\Omega \varphi(w)$.

Apply with $w = a g$ and $\varphi(x) = x \log x$.

$$\rightsquigarrow \quad \frac{1}{|\Omega \cap \operatorname{supp} g|} \int_\Omega a g \log \left( \frac{1}{|\Omega \cap \operatorname{supp} g|} \int_\Omega a g \right) \leq \frac{1}{|\Omega \cap \operatorname{supp} g|} \int_\Omega (a g) \log(a g).$$

Multiply by $-\frac{1}{(4\pi\tau)^{\frac{n+1}{2}}}\cdot|\Omega\cap\operatorname{supp} g|$

$\frac{(4\pi\tau)^{\frac{n+1}{2}}}{|\Omega\cap B_\tau(x_0)|}$

$$\rightsquigarrow\ -\frac{1}{(4\pi\tau)^{\frac{n+1}{2}}}\int_\Omega (a g)\log(a g) \le -\frac{1}{(4\pi\tau)^{\frac{n+1}{2}}}\int_\Omega a g \log\left(\frac{1}{|\Omega\cap\operatorname{supp} g|}\int_\Omega a g\right),$$

$$\operatorname{supp} g = \overline{B_\tau(x_0)}$$

$$\int_\Omega a g = (4\pi\tau)^{\frac{n+1}{2}}.$$

$$= \log\left(\frac{|\Omega\cap B_{\sqrt{\tau}}(x_0)|}{(4\pi\tau)^{\frac{n+1}{2}}}\right) = \log\left(\frac{|\Omega\cap B_{\sqrt{\tau}}(x_0)|}{\tau^{\frac{n+1}{2}}}\right) - \tilde{c}(n).$$

∎

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$(\Omega_t)_{t\in[0,T[}$, $M_t = \partial\Omega_t$ smooth

$\overline{\Omega}_t = \varphi_t(\overline{\Omega})$, $M_t = \partial\Omega_t = \varphi_t(\partial\Omega)$

$\varphi_t = \varphi(\cdot,t)$ smooth

$\overline{\Omega}_t \ni x = \varphi(p,t)$, $p\in\overline{\Omega}$

Normal speed of $M_t$ w.r.t. inward unit normal

(1) $\beta = \beta_{M_t} = -\frac{\partial x}{\partial t}\cdot\nu$, $x\in M_t$ (outward pointing)

Ex: $\beta = H_{M_t}$, $\vec{H}_{M_t} = -H_{M_t}\nu_{M_t}$;
MCF of $M_t$ up to tangential diffeos
$\left(\frac{\partial x}{\partial t}\right)^\perp = \vec{H}$ on $M_t$.

(2) $\frac{\partial x}{\partial t} = -\nabla f(x,t)$, $x\in\Omega_t$

(2) is compatible with (1) if $\boxed{\nabla f\cdot\nu = \beta \text{ on } M_t \quad (3)}$

$f: U\times[0,T[\to\mathbb{R}$
with $\bigcup_{t\in[0,T[}\overline{\Omega}_t \subset U$

$\left(\text{so } \left(\frac{\partial x}{\partial t}\right)^\perp = -\beta\nu \quad \text{so } \frac{\partial x}{\partial t} = -\beta\nu - \nabla^{M_t} f\right)$

Suppose $f(t)$ satisfies

(4) $\left(\frac{\partial}{\partial t}+\Delta\right)f = |\nabla f|^2 + \frac{n+1}{2\tau}$ in $\Omega_t$, $t\in[0,T[$.

Total time derivative $\frac{df}{dt} = \frac{\partial f}{\partial t} + \nabla f\cdot\frac{\partial x}{\partial t} = \frac{\partial f}{\partial t} - |\nabla f|^2 \Rightarrow \left(\frac{d}{dt}+\Delta\right)f = \frac{n+1}{2\tau}$. (5)

If $\tau(t)>0$ satisfies $\frac{d\tau}{dt}=-1$, i.e. $\tau(t)=a-t$ for some $a\in\mathbb{R}$, then

(5) resp. (4) $\Longleftrightarrow$ $\left(\frac{\partial}{\partial t}+\Delta\right)u=0$ in $\Omega_t$ with $u=\frac{e^{-f}}{(4\pi\tau)^{\frac{n+1}{2}}}$.

Note: For $x=\varphi(p,t)$ and $\tilde{f}(p,t)=f(\varphi(p,t),t)$,

$$\frac{\partial\tilde{f}}{\partial t}(p,t)=\frac{df}{dt}(x,t).$$

Prop. (Due to Perelman [P1] for Riemannian mfd evolving by RF)
Let $(\Omega_t)_{t\in[0,T[}$ evolve by (2) and let $f$ evolve by (4).
Suppose $\frac{d\tau}{dt}=-1$. Then the function $W=W_\tau(f)=\tau(2\Delta f-|\nabla f|^2)+f-(n+1)$

satisfies $$\left(\frac{d}{dt}+\Delta\right)W=2\tau\left|\nabla_i\nabla_j f-\frac{\delta_{ij}}{2\tau}\right|^2+\nabla W\cdot\nabla f.$$

Remark: Perelman has

$$\left(\frac{\partial}{\partial t}+\Delta\right)W=2\tau|\dots|^2+2\nabla W\cdot\nabla f \text{ but } \frac{dW}{dt}=\frac{\partial W}{\partial t}-\nabla W\cdot\nabla f.$$

In fact, Perelman shows in Ch. 9,

$$\left(\frac{\partial}{\partial t}+\Delta\right)(Wu)=2\tau|\dots|^2u \quad (*)$$

and $\left(\frac{\partial}{\partial t}+\Delta\right)u=0$ give (+) (exercise).

Volume evolution: $$g_{ij}(p,t)=\frac{\partial\varphi}{\partial p_i}(p,t)\cdot\frac{\partial\varphi}{\partial p_j}(p,t)$$

$$\frac{\partial\varphi}{\partial t}(p,t)=X(p,t)$$

$$dx=\sqrt{\det g_{ij}(p,t)}\,dp$$

$$\rightsquigarrow \frac{d}{dt}dx=\frac{1}{2\sqrt{\det g_{ij}(p,t)}}\det(g_{ij}(p,t))\,g^{ij}(p,t)\cdot\frac{\partial}{\partial t}g_{ij}(p,t)$$

$$=\operatorname{div}X\ \left(=g^{ij}(p,t)\frac{\partial X}{\partial p_i}(p,t)\cdot\frac{\partial\varphi}{\partial p_j}(p,t)\right.$$

$$\left.=g^{ij}\left(\frac{\partial X_i}{\partial p^j}-\Gamma_{ij}^k(p,t)X_k\right)\right).$$

Here $X=-\nabla f$ $\rightsquigarrow$ $\boxed{\frac{d}{dt}dx=-\Delta f}$,

$$\frac{\partial x}{\partial t} = -\nabla f = \frac{1}{u}\nabla u \quad \text{since} \quad \nabla u = -u\nabla f.$$

$$\rightsquigarrow \frac{du}{dt} = \frac{\partial u}{\partial t} - \nabla u \cdot \nabla f = \frac{\partial u}{\partial t} + \frac{|\nabla u|^2}{u}$$

$$\rightsquigarrow \Delta u = (|\nabla f|^2 - \Delta f)u$$

$$\Rightarrow \dots \Rightarrow \frac{d}{dt}(u\,dx) = \left(\left(\frac{\partial}{\partial t} + \Delta\right)u\right)dx = 0$$

$\Rightarrow$ If $\int_{\Omega_0} u = 1$, then $\int_{\Omega_t} u = 1$.