K. Ecker.

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$\frac{d}{dt} W_{H_{M_t}}(\Omega_t, f(t), \tau(t))$.

$$= 2\tau \int_{\Omega_t} \left|\nabla^2 f - \frac{I}{2\tau}\right|^2 u +$$

$$2\tau \int_{M_t} \Big( \underbrace{\frac{\partial}{\partial t} - 2\nabla^M H \cdot \nabla^M f + A(\nabla^M f, \nabla^M f)}_{Z(-\nabla^M f)} - \frac{H}{2\tau} \Big) u$$

(does not use the E.L. for $\mathcal{W}$)

Hamilton Harnack ineq. for MCF.

$\frac{d}{dt} \mu_{H_{M_t}} \geq \ldots$

(*) If $(f, \tau)$, $\tau > 0$ is min pair for $\nu_{H_{M_t}}(\Omega_t)$ (ie, $\nu_{H_{M_t}}(\Omega_t) < 0$)

$$\nu_\beta(\Omega) = \inf_{\tau > 0} \mu_\beta(\Omega, \tau)$$

$$= \inf_{\tau > 0} \inf_f \{ W_\beta(\Omega, f, \tau) : \int_\Omega u = 1 \}.$$

One can show then that $t \to \nu_{H_{M_t}}(\Omega_t)$ is Lipschitz if $(\Omega_t)$ is smooth in $t$. ($\Omega_t = \varphi_t(\bar\Omega)$, $\varphi_t$ smooth).

Then $\frac{d}{dt} W_{H_{M_t}}(\Omega_t, f(t), \tau(t))$

$$= \underbrace{\int_{\Omega_t} \left(\frac{n+1}{2\tau} - \Delta f\right) u}_{\text{(Li-Yau Harnack)}} + 2\tau \int_{M_t} H\left(|A|^2 + \nabla^2 f(\nu, \nu) - \frac{1}{\tau}\right) u$$

(using E.L. Eq$^{n}$).

$\frac{d}{dt} \mu_{H_{M_t}}(\Omega_t, \tau(t)) \geq \ldots$

$$\frac{d}{dt}\gamma_{H_{M_t}}(\Omega_t) \geq 2\tau \int_{M_t} H\left(|A|^2 + \nabla^2 f(\nu,\nu) - \frac{1}{2\tau}\right) u .$$

peculiarity $2\tau$ not $\tau$.

$$\int_{\Omega_t}\left(\frac{n+1}{2\tau} - \Delta f\right)u = \int_{M_t} H_{M_t} u = 2\tau \int_{M_t} \frac{H}{2\tau} u .$$

Note: this is well defined if $H_{M_0} \geq 0$.

Another formula for RHS: $\frac{d}{dt}$ of something else

Claim. $\frac{d}{dt}\left(2\tau\int_{M_t} Hu\right) = 2\tau \int_{M_t} H\left(|A|^2 + \nabla^2 f(\nu,\nu) - \frac{1}{\tau}\right) u .$

Pf: $(f,\tau)$ $\tau>0$ minimizing pair for $\gamma_{H_{M_t}}(\Omega_t)$

and let $\frac{d\tau}{ds}(s) = -1$,

$$\left(\frac{\partial}{\partial s} + \Delta\right) f = |\nabla f|^2 + \frac{n+1}{2\tau} , \quad \nabla f \cdot \nu = H_{M_s} \text{ on } M_s$$

$s<t$.
in $\Omega_s$.

Then,

$$\lim_{s\to t} \frac{d}{ds}\left(2\tau(s,t)\int_{M_s} H_{M_s} u(s,t)\right)$$

$$= 2\tau \int_{M_t} H\left(|A|^2 + \nabla^2 f(\nu,\nu) - \frac{1}{\tau}\right) . \qquad (\ast\ast)$$

$\tau(t,t) = \tau_t$.
$f(t,t) = f_t$.
$(f_t,\tau_t)$ min. pair for $\gamma_{H_{M_t}}(\Omega_t)$.

$\Rightarrow (f_t,\tau_t)$ diff a.e. in $t$ So,

$$\overset{?}{=} \frac{d}{dt}\left(2\tau_t \int_{M_t} \frac{e^{-f_t}}{(4\pi\tau_t)^{\frac{n+1}{2}}} H_{M_t}\right).$$

Combining these things:

$$\frac{d}{dt}\gamma_{H_{M_t}}(\Omega_t) - \lim_{s\nearrow t}\frac{d}{ds}\left(2\tau(s,t)\int_{M_s} H_{M_s}\, n(s,t)\right)$$

$$= \int_{M_t} H_{M_t}\, n(t) \geq 0.$$

Since we know $\frac{d}{dt}\left(\gamma_{H_{M_t}}(\Omega_t) - 2\tau_t\int_{M_t} H_{M_t}\, n_t\right) \geq 0.$

Then, $\gamma_{H_{M_t}}(\Omega_t) - 2\tau_t\int_{M_t} H_{M_t} n_t$

$$\geq \gamma_{H_{M_0}}(\Omega_0) - 2\tau_0\int_{M_0} H_{M_0} n_0 + \underbrace{2\tau_t\int_{M_t} H_{M_t} n_t}_{\geq 0}.$$

Pf of ⊛ : Eval. equations:

$$\frac{\partial n}{\partial t} = -H\nu - \nabla^M f, \quad \frac{d}{dt}\, dS = -(H^2 + \Delta_M f)\, dS.$$

$$\left(\frac{\partial}{\partial t} + A\right)n = 0 \text{ on } \Omega_t, \quad \nabla f\cdot\nu = H.$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} - \nabla^M H\cdot\nabla^M f.$$

$$= \Delta_M H + H|A|^2 - \nabla^M H\cdot\nabla^M f.$$

$$= \frac{\partial n}{\partial t} + \nabla n\cdot\frac{dn}{dt}.$$

$$= \frac{\partial n}{\partial t} - H \nabla n \cdot \nu - \nabla^M n \cdot \nabla^M f.$$

$$= \frac{\partial n}{\partial t} + H n \nabla f \cdot \nu - \nabla^M n \cdot \nabla^M f.$$

$$= \frac{\partial n}{\partial t} + H^2 - \nabla^M n \cdot \nabla^M f.$$

$$\frac{d}{dt} \int_{M_t} H n \, ds = \int_{M_t} \frac{dH}{dt} n \, ds + \int_{M} H \frac{dn}{dt} \, ds.$$

$$+ \int_{M_t} H n \frac{ds}{dt}.$$

$$= \int_{M_t} \frac{dH}{dt} n \, ds + \int_{M_t} H \frac{dn}{dt} \, ds.$$

$$- \int_{M_t} H n \left( H^2 n + \Delta_M f \right).$$

$$= \cdots ?$$

$$= \int_{M_t} \left( \frac{\partial H}{\partial t} n + H \frac{\partial n}{\partial t} \right) ds.$$

$$= \int_{M_t} \left[ \left( \frac{\partial}{\partial t} - \Delta_{M_t} \right) H \right] n \, ds.$$

$$+ \int_{M_t} H \left( \frac{\partial}{\partial t} + \Delta_{M_t} \right) n \, ds.$$

$\Delta_M h = \mathrm{div}_M \nabla^M h. = \dots = \Delta h - \nabla^2 h(\nu, \nu) - H \nabla h \cdot \nu.$

For functions with an extension to a neighbourhood of surface.

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$\left(\frac{\partial}{\partial t} + \Delta\right) u = 0$ in $\Omega_t$, $\nabla f \cdot \nu = H_{M_t}$ on $M_t = \partial \Omega_t$.

$$\frac{\partial x}{\partial t} = -\nabla f(x,t) \rightsquigarrow \frac{d}{dt}(u\, dx) = 0.$$

$$N = -\int_\Omega u \log(u)\, dx, \quad u = \frac{e^{-f}}{(4\pi\tau)^{\frac{n+1}{2}}}.$$

$$\log u = -f - \frac{n+1}{2}\log(4\pi\tau), \quad \frac{d\tau}{dt} = -1.$$

$$\Rightarrow N = \int_\Omega \left\{ f u + \frac{n+1}{2}\log(4\pi\tau) \right\} \quad \left( \int u = 1 \right).$$

Prop. Evolution of $u$ and $(f,\tau)$ as above, MCF on $M_t$;
$\frac{\partial x}{\partial t} = -\nabla f$ on $\Omega_t$.

$$\frac{dN}{dt} = -\int_{\Omega_t} |\nabla f|^2 u - \int_{M_t} H_{M_t} u.$$

$$\Rightarrow \frac{d}{dt}\int_{\Omega_t} f u = \frac{n+1}{2\tau(t)} - \int_{\Omega_t} |\nabla f|^2 u - \int_{M_t} H_{M_t} u.$$

Let $(f_{t_0}, \tau_{t_0})$ main given and $f(t,t_0), \tau(t,t_0)$ solve
nonl PDE/ODE for $t < t_0$.

$$\lim_{t \to t_0}\left(\frac{d}{dt}\int_{\Omega_t} f u\right) = -\int_{\Omega_{t_0}} |\nabla f_{t_0}|^2 u_{t_0} - \int_{M_{t_0}} H_{M_{t_0}} u_{t_0} + \frac{n+1}{2\tau_{t_0}}.$$

Q: ? $\Uparrow$ ?

$$\frac{d}{dt}\Big|_{t=t_0} \int_{\Omega_t} f_{t_0} u_t = \int_{M_{t_0}} H_{M_{t_0}} u_{t_0} > 0.$$

$$= \int_{M_t} H|A|^2 u + \int_{M_t} H\left(\frac{\partial}{\partial t} + \Delta\right) u \, d\mu + \int_{M_t} H^3 u - H\nabla^2 u(\nu,\nu).$$

$$\nabla_i \nabla_j u = \nabla_i(-u\nabla_j f) = u(\nabla_i f \nabla_j f - \nabla_i \nabla_j f).$$

$$-\nabla^2 u(\nu,\nu) = u\nabla^2 f(\nu,\nu) - \underbrace{u(\underbrace{\nabla f \cdot \nu}_{H})^2}_{-H^2 u}.$$

$$\Rightarrow \frac{d}{dt}\int_{M_t} Hu = \int_{M_t} H\left(|A|^2 + \nabla^2 f(\nu,\nu)\right) u.$$

<u>Nash Entropy</u>: (Boltzmann - Shannon) entropy:

One can show $t \mapsto \gamma_{H_{M_t}}(\Omega_t)$ is Lipschitz if $\Omega_t$ evolves smoothly in $t$. (So diff a.e.)

Consider a $t_0 > 0$ s.t. a min pair $(f_{t_0}, \tau_{t_0})$ for $\gamma_{H_{M_{t_0}}}(\Omega_{t_0})$ sat. $\tau_{t_0} > 0$ and $\frac{d}{dt}\Big|_{t=0} \gamma_{H_{M_t}}(\Omega_t)$ exists.

Then,

$$\int_{\Omega_{t_0}} f_{t_0} u_{t_0} = \gamma_{H_{M_{t_0}}}(\Omega_{t_0}) + \frac{n+1}{2}.$$

<u>Note</u>: We also have $\int_{\Omega_{t_0}} |\nabla f_{t_0}|^2 u_{t_0} = \frac{n+1}{2\tau_{t_0}} - 2\int_{M_{t_0}} H_{M_{t_0}} u_{t_0}.$