14/05/2018.

$$\mathcal{E}[z] = \int_\Omega \underbrace{\left(2|\nabla z|^2 - z^2 \log z^2\right)}_{e(z)} \gamma_{n+1}$$

For instance, if $z_0 \equiv$ const, (then it is in $L^2(\gamma_{n+1})$) and $\mathcal{E}(z_0) \in \mathbb{R}$

Take z_0 s.t. $\mathcal{E}[z_0] \in \mathbb{R}$ and solve

$$(\partial_t - L) z = 0 \quad \text{in } \Omega \text{ for } t > 0$$
$$z(0) = z_0$$
$$\nabla z \cdot \nu = 0 \quad \text{on } \partial\Omega, \forall t \geq 0.$$

In particular, $\nabla z^2 \cdot \nu = 0$ on $\partial\Omega$.

We showed: if $\partial\Omega$ is convex ($\partial\Omega \in C^2$, $A_{\partial\Omega} \geq 0$)

then $\partial_t \, \mathcal{E}(z(t)) \leq 0 \quad \forall t > 0$

$$\Rightarrow \mathcal{E}(z(t)) \leq \mathcal{E}(z_0) \quad \forall t \geq 0.$$

Non-optimal version of log-Sobolev (holds for any domain on which standard Sobolev embed. holds).
Gagliardo Nirenberg?

$\forall \varepsilon > 0$, $\forall \varphi \in C^1(\overline{\Omega})$ with $\int_\Omega \varphi \, dx = 1$.

$$\int_\Omega \left(\varepsilon |\nabla \varphi|^2 - \varphi^2 \log \varphi^2\right) dx \geq -c(n, C_S(\Omega)) - \frac{1}{\varepsilon}.$$

Very unsatisfactory (but sufficient for our purposes)

~~Gal~~ Gagliardo–Nirenberg: $\exists c(n,\Omega)$ $\forall w \in C^1(\bar\Omega)$

$$\int_\Omega |w|^{\frac{n+1}{n}} r_{n+1} \leq c(n,\Omega) \int_\Omega (|\nabla w| + |w|)\, r_{n+1}.$$

So, we get:

$\forall \varepsilon > 0$, $\forall \varphi \in C^1(\bar\Omega)$ with $\int_\Omega \varphi^2 r_{n+1} = 1$.

$$(*) \qquad \int_\Omega \left(\varepsilon |\nabla \varphi| - \varphi^2 \log \varphi^2\right) r_{n+1} \geq -c(n, c_S(\Omega)) - \frac{1}{\varepsilon}.$$

~~Use $(*)$ with~~

~~Show: $\int_\Omega |\nabla u(t)|^2 r_{n+1}$ is controlled~~

Show $$\int_\Omega |\nabla u(t)|^2 r_{n+1} \leq c\left(\mathcal{E}(u(t)) + \|u(t)\|^2\right)$$

By log-Sob:

$$\mathcal{E}(u) = \int_\Omega \left(2 |\nabla u|^2 - u^2 \log u^2\right) r_{n+1}$$

$$= \int_\Omega |\nabla u|^2 r_{n+1} + \int \left(|\nabla u|^2 - u^2 \log u^2\right) r_{n+1}$$

$$\geq \int_\Omega |\nabla u|^2 r_{n+1} - (c(n, c_S(\Omega)) + 1).$$

$\square$

But recall: $\int u^2 r_{n+1} = 1$.

We know already that $z_f^2(t) \xrightarrow[t\to\infty]{} c$ in $L^2(\Omega, r_{n+1})$ exp. fast.

$$c = \frac{\int_\Omega z_f^2 \, r_{n+1}}{\int_\Omega r_{n+1}} = \frac{1}{\int_\Omega r_{n+1}}.$$

Ex. Once we have shown that, also

$$\mathcal{E}(z_f(t)) \xrightarrow[t\to\infty]{} \mathcal{E}(\sqrt{c}) \quad \text{thus}$$

$$\mathcal{E}(z_f(t)) \to \log\left(\int_\Omega r_{n+1}\right).$$

<u>Claim</u> $\{z_j\} \subset H^1(\Omega)$ ~~s.t.~~ s.t. $\exists C > 0$

$$\int_\Omega z_j^2 + \int_\Omega |\nabla z_j|^2 \leq C \quad \forall j.$$

$\Rightarrow \exists$ subsequence (again called z_j) and $z_f \in H^1$

s.t. $z_j \rightharpoonup z_f$ in $H^1(\Omega)$.

H^1 cptly $\hookrightarrow$ in L^p $\forall p < \frac{2(n+1)}{n-1}$,

$z_j \to z_f$ strongly.

(For later purposes, note $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$ <u>cptly</u>.

$z_j \to z_f$ on $\partial\Omega$ in L^2 $\Rightarrow$ $\int_{\partial\Omega} \beta z_j^2 \to \int_{\partial\Omega} \beta z_f^2$.)

Rothaus "trick": mean value inequality

$$|z_j^2 \log z_j^2 - z^2 \log z^2| \le 2 \sup\big((\log \theta^2 + 1)\,\theta\big)\,|z_j - z|$$

$$\theta \le \max\{|z_j|, |z|\}.$$

Also, $\theta \log \theta \le \frac{1}{e\gamma}\,\theta^{1+\gamma} \quad \forall\, \gamma > 0.$

$$\Rightarrow \int_\Omega |z_j^2 \log z_j^2 - z^2 \log z^2|$$

$$\le c(\gamma)\left(\int_\Omega |z_j - z|^p\right)^{\frac{1}{p}} \times$$

$$\left(\int_\Omega \left(\max\{|z_j|,|z|\}^{(\gamma+1)q} + \max\{|z_j|,|z|\}^q\right)\right)^{\frac{1}{q}}$$

Choose $2 < p < \frac{2(n+1)}{n-1}$. $(\gamma+1)q = 2$.

Use Hölder in last integral

and RHS $\to 0$.

Conclusion: $\mathcal{E}(z_j) \to \mathcal{E}(z)$.

Analogous argument $\Rightarrow \mathcal{E}(z_j) + \alpha \int_{\partial\Omega} \beta z_j^2$

$$\to \mathcal{E}(z) + \alpha \int_{\partial\Omega} \beta z^2.$$