K. Ecker. 12/07/2018

Recall corollary from last time:

If $\frac{d}{dt} \mathcal{V}_{H_{M_T}}(\Omega_T, f(t), \tau(t)) \geq 0$ $\quad t \in [0,T)$, $\quad T < \infty$.

Then $\exists\, k = k(c_0, c_1, T) > 0$ s.t.

$$\forall \Omega_t \quad \frac{|\Omega_t \cap B_r(x_0)|}{r^{n+1}} \geq k \qquad \forall r \leq \sqrt{T}.$$

as long as

$$(\forall) \quad \frac{|\Omega_t \cap B_r(x_0)| + r^2 \int_{M_t \cap \partial B_r(x_0)} |H|}{|\Omega_t \cap B_{r/2}(x_0)|} \leq c_1.$$

Let $\lambda_j \searrow 0$, $t_j \nearrow$, $x_j \in \mathbb{R}^{n+1}$ (Not at x_0, but some x_j).

$$\Omega^j_s := \frac{1}{\lambda_j}\left(\Omega_{\lambda_j^2 s + t_j} - x_j\right), \quad t = \lambda_j^2 s + t_j \in [0,T).$$

$$\left(\Rightarrow s \in \left(-\lambda_j^{-2} t_j,\ \lambda_j^{-2}(T - t_j)\right)\right) =: (a_j, b_j).$$

$\exists$ smooth limit $(M'_s)_{s \in (-\infty, b)}$ bounding set $(\Omega'_s)_{s \in (-\infty, b)}$ under suitable assumptions on (λ_j).

$(\forall)$ scale invariant.

$\Rightarrow$ $(\circledast_j)$ $\qquad \dfrac{|\Omega^j_s \cap B_r(x_0)|}{r^{n+1}} \geq k > 0$ same k as in $\circledast$.

$\forall s \in (a_j, b_j)$, $r \in \left(0, \dfrac{\sqrt{T}}{\lambda_j}\right)$, ~~$\forall B_r(x_0)$ s.t. $O(\Omega^j_s, B_r(x_0))$~~
$\forall B_r(x_0)$ s.t. $\quad O(\Omega^j_s, B_r(x_0)) \leq C_1$.

$j \to \infty \quad \Rightarrow \dfrac{\sqrt{T}}{\lambda_j} \to \infty$ and so.

$\Rightarrow \quad \dfrac{|\Omega'_s \cap B_r(x_0)|}{r^{n+1}} \geq k > 0 \quad \forall s \in (-\infty, b)$ $\boxed{\forall r > 0}$

as long as $O(\Omega'_s; B_r(x_0)) \leq C_1$.

$\Rightarrow$ Examples (1) – (3) cannot arise as rescaling limits
(slab, catenoid soln, grim reaper $\times \mathbb{R}^n$).

<u>Prop</u>. $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^{n+1}$ open, smooth, $\tau > 0$,
$u = \dfrac{e^{-f}}{(4\pi\tau)^{\frac{n+1}{2}}}$ satisfies

$$\tau(2\Delta f - |\nabla f|^2) + f \equiv \text{const} \quad \text{in } \Omega.$$

Then, $T = (T_{ij})$ be defined by $T_{ij} = \nabla_i \nabla_j f - \dfrac{\delta_{ij}}{2\tau}$.
satisfies $\quad \dfrac{1}{u}\operatorname{div}(u \nabla T_{ij}) = \displaystyle\sum_{k=1}^{n+1} T_{ik} T_{kj} + \frac{1}{2\tau} T_{ij}$

<u>Note</u>: $\quad \dfrac{1}{u}\operatorname{div}(u \nabla h) = \Delta h - \nabla f \cdot \nabla h$, $\quad f = \dfrac{|x|^2}{2} \Rightarrow$ O.U. op.

<u>Also</u>. $\operatorname{tr} T = \Delta f - \dfrac{n+1}{2\tau}$ (see Li-Yau) [illegible]

$$\frac{1}{n}\operatorname{div}(n\nabla \operatorname{tr} T) = \left|\nabla_i\nabla_j f - \frac{\delta_{ij}}{2\tau}\right|^2 + \frac{1}{2\tau}\operatorname{tr} T.$$

On $\partial\Omega$, we have.

$$\nabla(\operatorname{tr} T)\cdot\nu = \nabla^2 f(\nabla f,\nu) - \frac{1}{2\tau}\nabla f\cdot\nu = T(\nabla f,\nu).$$

Furthermore,

$$(**)\quad 2\tau\int_\Omega \left|\nabla_i\nabla_j f - \frac{\delta_{ij}}{2\tau}\right|^2 n = \int_\Omega \left(\frac{n+1}{2\tau} - \Delta f\right) n + 2\tau\int_{\partial\Omega} T(\nabla f, n)$$

pf Calculation, similar to ~~Hamilt~~ Hamilton formula.

$$2\tau\Delta f - \tau|\nabla f|^2 + f \equiv \text{const} \Leftrightarrow \Delta f = \frac{1}{2}|\nabla f|^2 - \frac{f}{2\tau} \quad (***)$$

$$\Delta\nabla_i\nabla_j f = \nabla_i\nabla_j\Delta f \overset{*}{=} \frac{1}{2}\nabla_i\nabla_j|\nabla f|^2 - \frac{1}{2\tau}\nabla_i\nabla_j f.$$

$$\rightsquigarrow \dots \rightsquigarrow$$

□

Prop. $\nabla f\cdot\nu = \beta$ on $\partial\Omega$.

$$\Rightarrow \nabla(\operatorname{tr} T)\cdot\nu = \nabla^M f\cdot\nabla^M\beta + \beta\left(\nabla^2 f(\nu,\nu) - \frac{1}{2\tau}\right) - A_M(\nabla^M f,\nabla^M f).$$

Pf. $\nabla(\operatorname{tr} T)\cdot\nu = \nabla^2 f(\nabla f,\nu) - \frac{\beta}{2\tau} \leftarrow \nabla f\cdot\nu$.

$$\nabla^M f\cdot\nabla^M\beta = (\nabla^M f)(\beta) = \nabla^M f(\nabla f\cdot\nu) = \nabla^2 f(\nabla^M f,\nu) + \underbrace{\nabla^M f\cdot A_M(\nabla^M f,\cdot)}_{= A_M(\nabla^M f,\nabla^M f)}.$$

$$= \nabla^2 f(\nabla f, v) - \beta \nabla^2 f(v, v) + A_n(\nabla^M f, \nabla^M f).$$

$$= \nabla(\partial_\tau T)\cdot v + \frac{\beta}{2\tau} - \beta \nabla^2 f(v,v) + A_M(\nabla^M f, \nabla^M f).$$

$\nabla^M f = \nabla f - \beta v.$

Corollary. $2\tau \int_\Omega |\ldots|^2 u = \int_\Omega \left(\frac{n+1}{2\tau} - \Delta f\right) u$

$$+ 2\tau \int_M \left(\nabla^M f \cdot \nabla^M \beta + \beta(\nabla^2 f(v,v) - \tfrac{1}{2\tau}) - A(\nabla^M f, \nabla^M f)\right) u.$$

Corollary. If (f,τ) min pair for $\mathcal{V}_\beta(\Omega)$ with $\tau > 0$, then

$$\int_\Omega \left(\frac{n+1}{2\tau} - \Delta f\right) u = \int_M \beta u \left(= 2\tau \int_M \frac{\beta}{2\tau} u \right).$$

Last week: $\frac{d}{dt} W_\beta(\Omega_t, f(t), \tau(t))$

$$= 2\tau \int_{\Omega_t} |\ldots|^2 u + 2\tau \int_{M_t} \left(\frac{\partial \beta}{\partial t} - 2\nabla^M f \cdot \nabla^M \beta + A(\nabla^M f, \nabla^M f) - \frac{\beta}{2\tau}\right) u.$$

$$= \int_{\Omega_t} \left(\frac{n+1}{2\tau} - \Delta f\right) u + 2\tau \int_M \frac{\partial \beta}{\partial t} - \nabla^M f \cdot \nabla^M \beta + \beta\left(\nabla^2 f(v,v) - \frac{1}{2\tau}\right) u$$

$$= \int_{M_t} \left[\left(\frac{\partial}{\partial t} - \Delta_{M_t}\right)\beta + \beta\left(\nabla^2 f(v,v) - \frac{1}{2\tau}\right)\right] u.$$

anything to be $\frac{1}{t}$.

$-\int \nabla^M f \, \nabla^M \beta u = -\int_M \Delta_M \beta \, u$

integration by parts, M closed mfld.

~~RSE: $\left(\frac{\partial}{\partial t} - \Delta_{M_t}\right) A = H|A|^2$ on MCF.~~

Here $\left(\frac{\partial}{\partial t} - \Delta_{M_t}\right) H = H|A|^2$ on MCF ($\beta = H$)

Then, maybe min pair (f_t, τ_t) at time t for $\gamma_{H_{M_t}}(\Omega_t)$ where $\tau_t > 0$.

$$\frac{d}{dt}\gamma_{H_{M_t}}(\Omega_t) \geq 2\tau_t \int_{M_t} H_{M_t}\left(|A_{M_t}|^2 + \nabla^2 f(\nu,\nu) - \frac{1}{2\tau_t}\right)u .$$

Forecast.

maybe only $\frac{1}{\tau_t}$?

$$\frac{d}{dt}\left(2\tau\int_{M_t} Hu.\right).$$