Lecture 9

We had

- $W_\beta(\Omega, f, \tau) = \int_\Omega (\tau|\nabla f|^2 + f - (n+1))u + 2\tau \int_{\partial\Omega} \beta u$,

$\tau > 0$, $\beta : \partial\Omega \to \mathbb{R}$, $f : \bar{\Omega} \to \mathbb{R}$, $u = \frac{e^{-f}}{(4\pi\tau)^{\frac{n+1}{2}}}$.

- $\mu_\beta(\Omega, \tau) := \inf \{W_\beta(\Omega, f, \tau), \int_\Omega u = 1\} \in \mathbb{R}$

if Ω satisfies Gagl.-Nirenberg,

$\partial\Omega \in C^{0,1}$, $\beta \in L^\infty(\partial\Omega)$

(uses log-Sob. (non-optimal suff.) + $H^1(\Omega) \underset{\text{cont}}{\hookrightarrow} L^2(\partial\Omega)$).

- $\exists$ min. f of $\mu_\beta(\Omega, \tau)$ (using $H^1(\Omega) \underset{\text{compact}}{\hookrightarrow} L^2(\partial\Omega)$)

<u>Proposition</u> f is a min. for $\mu_\beta(\Omega, \tau)$

$\iff$

(1) $W_\tau(f) := \tau(2\Delta f - |\nabla f|^2) + f - (n+1) = \mu_\beta(\Omega, \tau)$ in Ω.

(2) $\nabla f \cdot \nu = \beta$ on $\partial\Omega$

(3) $\int_\Omega u = 1$

Remark: If a function f satisfies $\nabla f \cdot \nu = \beta$ on $\partial\Omega$

then $\quad W_\beta(\Omega, f, \tau) = \int_\Omega W_\tau(f) u$.

$$\nu_\beta(\Omega) := \inf_{\tau > 0} \mu_\beta(\Omega, \tau)$$

Prop. $\partial\Omega \in C^2$, $\beta : \partial\Omega \to \mathbb{R}$ smooth and positive (or at least $\int_{\partial\Omega} \beta u > 0$), Ω bounded

$\Rightarrow \nu_\beta(\Omega) > -\infty$.

Prop. (no proof here; 2nd paper on website; for Ricci flow i.e. with $\beta \equiv 0$ see Sesum, Tian) If $\beta \geq 0$ on $\partial\Omega$ and smooth then $\displaystyle\lim_{\tau \searrow 0} \mu_\beta(\Omega, \tau) \geq 0$.

(in RF $\displaystyle\lim_{\tau \to 0} \mu(g, \tau) = 0$) (see also [P1]).

Sketch: $\int_\Omega |\nabla \varphi_\tau|^2 - \varphi_\tau^2 \log \varphi_\tau^2$

$\varphi_\tau^2 = (\varphi\, \tau^{\frac{n+1}{4}})^2$

$dV_\tau = \dots$ with τ in it

... proof by contradiction

$\Rightarrow$ **Cor.** ($\beta \geq 0$) If $\nu_\beta(\Omega) < 0$, then $\inf_{\tau > 0} \mu_\beta(\Omega, \tau)$ is attained for a $\tau > 0$.

One then has a minimising pair (f, τ) for $\nu_\beta(\Omega)$, i.e.

$$\nu_\beta(\Omega) = \mu_\beta(\Omega, \tau) = W_\beta(\Omega, f, \tau).$$

Prop. If the min. pair (f, τ) of $\nu_\beta(\Omega)$ satisfies $\tau > 0$, then we have the equivalent statements (see also Cao, Hamilton, Ilmanen for RF):

(4a) $\displaystyle\int_\Omega |\nabla f|^2 u = \frac{n+1}{2\tau} - 2\int_{\partial\Omega} \beta u$. $\quad$ (4b) $\displaystyle\int_\Omega \left(\frac{n+1}{2\tau} - \Delta f\right) u = \int_{\partial\Omega} \beta u$.

(4c) $\int_\Omega fu = \frac{n+1}{2} + \nu_\beta(\Omega)$

Proof: $0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \left(W_\beta(\Omega, f_\varepsilon, \tau) + \lambda \int_\Omega \frac{e^{-f_\varepsilon}}{(4\pi\tau)^{\frac{n+1}{2}}} \right)$

$f_0 = f$
$\tau_0 = \tau$ $\quad \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f_\varepsilon = \eta$
$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \tau_\varepsilon = \sigma$

$\rightsquigarrow$ (1) – (3) and $\lambda = 1 - \mu_\beta(\Omega, \tau)$.

(4a): $0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \left(W_\beta(\Omega, f, \tau_\varepsilon) + (1 - \mu_\beta(\Omega)) \cdot \int_\Omega \frac{e^{-f}}{(4\pi\tau_\varepsilon)^{\frac{n+1}{2}}} \right)$

$\rightsquigarrow [\ldots]\sigma = 0 \quad \forall \sigma \in \mathbb{R}$

Second variation:

$$0 \le \frac{d^2}{d\varepsilon^2} \left(W_\beta(\Omega, f_\varepsilon, \tau) + (1 - \mu_\beta(\Omega)) \int_\Omega \frac{e^{-f_\varepsilon}}{(4\pi\tau)^{\frac{n+1}{2}}} \right)$$

$\forall \eta$ with $\int_\Omega \eta u = 0$, $\eta = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f_\varepsilon$,

$$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_\Omega \frac{e^{-f_\varepsilon}}{(4\pi\tau)^{\frac{n+1}{2}}} = -\int_\Omega \eta u$$

$\left(\eta \in T \left\{ \int_\Omega u = 1 \right\} \right)$

$\Rightarrow$ Prop. f min. for $\mu_\beta(\Omega, \tau) \Rightarrow \frac{1}{2\tau} \int_\Omega \eta^2 u \le \int_\Omega |\nabla \eta|^2 u$

$\forall \eta$ with $\int_\Omega \eta u = 0$.

Linearise (1) – (3): $\delta f = \eta$.

(1) $\Rightarrow 2\tau(\Delta\eta - \nabla f \cdot \nabla \eta) + \eta = 0$ in Ω

or $\underbrace{\Delta\eta - \nabla f \cdot \nabla \eta + \frac{1}{2\tau}\eta}_{L_f \eta} = 0$ in Ω

$$\boxed{\text{Ex: } \int_\Omega v \cdot L_f w \cdot u = \int_\Omega w \cdot L_f v \cdot u \text{ if } \nabla v \cdot \nu = \nabla w \cdot \nu = 0 \text{ on } \partial\Omega.}$$

$f = \frac{|x|^2}{2} \rightsquigarrow \Delta\eta - x \cdot \nabla\eta$

$$(2) \Rightarrow \nabla\eta \cdot \nu = 0.$$

$$(3) \Rightarrow \int_\Omega \eta u = 0.$$

?? Is $\eta \equiv 0$??

If (f, τ) min. pair, (4c) can be lin. to also give

$$0 = \int_\Omega \delta f \cdot u + f \cdot \delta u = \int_\Omega \eta u - f\eta u = \int_\Omega (1-f)\eta u$$

$$\int_\Omega \eta u = 0$$

$$\Rightarrow \int_\Omega f\eta u = 0$$

One can show ~~that~~ if $\Omega = B_{\sqrt{2n\tau}}$, $\beta = H_{\partial\Omega} = \sqrt{\frac{n}{2\tau}}$ then f^τ given by $f^\tau(x) = \frac{|x|^2}{4\tau} + c$ is the unique min. for suitable c.

c s.t. $\int_\Omega \frac{e^{-f_\tau}}{(4\pi\tau)^{\frac{n+1}{2}}} = 1$

$\int_{B_{\sqrt{n}}} \gamma_{n+1}$

(proof quite involved, very explicit)

<u>Check</u>: (1) $W_\tau(f_\tau) = \tau(2\Delta f_\tau - |\nabla f_\tau|^2) + f_\tau - (n+1)$

$\nabla f_\tau = \frac{x}{2\tau}$ $\quad = \tau\left(\frac{n+1}{\tau} - \frac{|x|^2}{4\tau^2}\right) + \frac{|x|^2}{4\tau} + c - (n+1),$

$\Delta f_\tau = \frac{n+1}{2\tau}$ $\quad \dots \quad c = \mu_{H_{\partial\Omega}}(\Omega, \tau)$

$\nu(x) = \frac{x}{|x|}$

$\nabla f^\tau(x) \cdot \nu = H_{\partial\Omega}$

For <u>no</u> $\tau > 0$ can (f_τ, τ) be a min. pair for $\gamma_{H_{\partial\Omega}}(\Omega)$ for this set Ω.

Why? $\Delta f_\tau - \frac{n+1}{2\tau} = 0$

$$(4b) \quad \int_{\Omega} \underbrace{\left(\Delta f - \frac{n+1}{2\tau}\right)}_{=0} u = \underbrace{\int_{\partial\Omega} \beta u}_{>0} \quad !$$

$f \equiv$ const. also does not work with any β:

$$\int_{\Omega} f u \overset{(4c)}{=} \gamma_\beta(\Omega) + \frac{n+1}{2} = \mu_\beta(\Omega, \tau) + \frac{n+1}{2}.$$

$\equiv C$ since $\int_{\Omega} u = 1$

$$\mu_\beta(\Omega, \tau) \overset{(1)}{=} W_\tau(f) = \underbrace{\tau\left(2\Delta f - |\nabla f|^2\right)}_{=0 \text{ since } f \equiv C} + f - (n+1) = C - (n+1)$$

$$\leadsto \boxed{C = \mu_\beta(\Omega, \tau) + (n+1)} \quad ⚡$$

Upper bound on $\mu_\beta(\Omega, \tau)$

Prop. $\Omega \subset \mathbb{R}^{n+1}$, "reas. bdry", $\beta \in L^1(\partial\Omega)$

$$\Rightarrow \mu_\beta(\Omega, \tau) \leq \log\left(\frac{|\Omega \cap B_{\sqrt{\tau}}(x_0)|}{\tau^{\frac{n+1}{2}}}\right) + \underbrace{c(n) \cdot \left(\frac{|\Omega \cap B_{\sqrt{\tau}}(x_0)| + 2\tau \int_{\partial\Omega \cap B_{\sqrt{\tau}}(x_0)} |\beta|}{|\Omega \cap B_{\frac{\sqrt{\tau}}{2}}(x_0)|}\right)}_{=: c(\Omega, \tau, x_0)}$$

$\forall \tau > 0 \ \forall B_{\sqrt{\tau}}(x_0)$ s.t. $|\Omega \cap B_{\frac{\sqrt{\tau}}{2}}(x_0)| > 0$.

Proof: Next week!

Corollaries Ω open and bounded

$\Rightarrow$ (a) $\sup_{\tau>0} \mu_0(\Omega,\tau) \leq c(n,\Omega) < \infty$

(b) $\nu_0(\Omega) = \inf_{\tau>0} \mu_0(\Omega,\tau) = -\infty$

Pf Ω open $\Rightarrow \exists\, B_{\sqrt{\tau_0}}(x_0) \subset \Omega$, $\tau_0 > 0$ dep. on Ω

(a) $\Rightarrow$ for $0<\tau\leq\tau_0$ $\mu_0(\Omega,\tau) \leq \log \omega_{n+1} + c(n)\, 2^{n+1}$.

& for $\tau \geq \tau_0$ $\mu_0(\Omega,\tau) \leq \log\left(\frac{|\Omega|}{\tau_0^{\frac{n+1}{2}}}\right) + c(n)\cdot 2^{n+1}\cdot \frac{|\Omega|}{\omega_{n+1}\cdot\tau_0^{\frac{n+1}{2}}}$

(b) Ω bdd $\Rightarrow \Omega \subset B_{\frac{\sqrt{\tau_1}}{2}}(0)$, τ_1 dep. on Ω

$$\Rightarrow \mu_0(\Omega,\tau) \leq \underbrace{\log\left(\frac{|\Omega|}{\tau^{\frac{n+1}{2}}}\right)}_{\xrightarrow{\tau\to\infty} -\infty} + c(n) \longrightarrow -\infty. \quad \square$$

$\forall \tau \geq \tau_1$

Rmk More generally, we have instead of (b)

$\nu_\beta(\Omega) = -\infty$ if $b_\beta(\Omega) = \inf\left\{\int_\Omega 4|\nabla\varphi|^2 + 2\int_{\partial\Omega} \beta\varphi^2,\ \int_\Omega \varphi^2 = 1\right\} \leq 0$

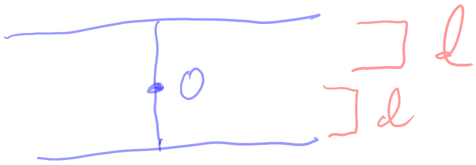
(see 2^{nd} paper for proof).

Examples

(1) Slab: $\Omega = \{x\in\mathbb{R}^{n+1},\ -d<x_{n+1}<d\}$ $d>0$

$\beta = H_{\partial\Omega} = 0$

$B_R \hat{=} B_R(0)$

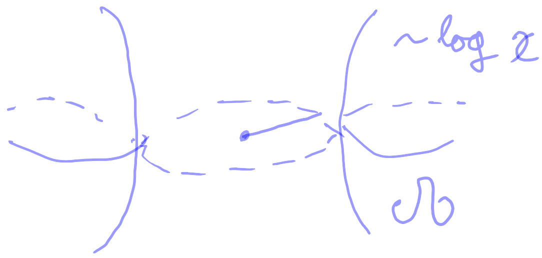


$\forall\, B_r$ $|\Omega\cap B_{\frac{r}{2}}| > 0$

and $\frac{|\Omega\cap B_r|}{|\Omega\cap B_{\frac{r}{2}}|} \leq c(n)\ \forall r$.

$C(\Omega,r^2,0) \leq \tilde{c}(n)$ so $\mu_0(\Omega,r^2) \leq \log\left(\frac{|\Omega\cap B_r|}{r^{n+1}}\right) + \text{const}(n)$ $\xrightarrow{r\to\infty} -\infty$ since $|\Omega\cap B_r| \leq C(n,d)\, r^n$

(2) $\Omega = \{ x = (\hat{x}, x_3) \in \mathbb{R}^3, |\hat{x}| \geq 1, |x_3| < \cosh^{-1} |\hat{x}| \}$.



$\beta = H_{\partial\Omega} = 0$ (~~catenoid~~ min. surface)

<u>Ex.:</u> $\exists c_1$ $\forall r > 2$ (dep. on n)

$$\frac{|\Omega \cap B_r|}{|\Omega \cap B_{\frac{r}{2}}|} \leq c_1$$

and $\exists c_2$ $\forall r \geq 2$

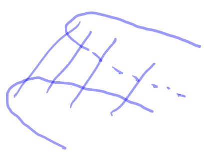
$$|\Omega \cap B_r| \leq c_2 r^2 \log(1+r)$$

$$\Rightarrow \lim_{r \to \infty} \frac{|\Omega \cap B_r|}{r^3} = 0$$

$$\Rightarrow \lim_{r \to \infty} \mu_0(\Omega, r^2) = -\infty.$$

(3) $\Omega = \mathbb{R}^{n-1} \times G$, G grim reaper

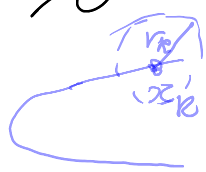
$\partial G \cong$ ~~transl.~~ soliton of CSF



$$G = \left\{ (x_n, x_{n+1}) \in \mathbb{R}^2 : -\frac{\pi}{2} < x_n < \frac{\pi}{2}, x_{n+1} > -\log \cos x_n \right\}$$

$$\Rightarrow \boxed{H_{\partial\Omega}(x) = e^{-x_{n+1}} \quad \forall x \in \partial\Omega}$$

$\Rightarrow \exists (B_{r_k}(x_k))$, $r_k \to \infty$ and $|\Omega \cap B_{\frac{r_k}{2}}(x_k)| > 0$



and $\dfrac{|\Omega \cap B_{r_k}(x_k)|}{|\Omega \cap B_{\frac{r_k}{2}}(x_k)|} \leq \underset{\text{indep. of } k}{c(n)}$ and $\dfrac{r_k^2 \int_{\partial\Omega \cap B_{r_k}(x_k)} H}{|\Omega \cap B_{\frac{r_k}{2}}(x_k)|} \leq 1$

so $c(\mathcal{M}, r_k, x_k) \leq c(n)$ indep. of k, but $|\mathcal{M} \cap B_{r_k}(x_k)| \leq c r_k^n$

$$\Rightarrow \log\left(\frac{\mathcal{M} \cap B_{r_k}(x_k)}{r_k^{n+1}}\right) \xrightarrow{k \to \infty} 0 .$$