k. Ecker.

09/07/2018.

$$W_\beta(\Omega, f, \tau) = \int_\Omega (\tau|\nabla f|^2 + f - (n+1))\, u \, dx + 2\tau \int_{\partial\Omega} \beta u \, ds.$$

Had assumed: $\overline{\Omega_T} = \varphi_T(\overline{\Omega})$, $x = \varphi(p, t)$, $p \in \overline{\Omega}$

① $\dfrac{\partial x}{\partial t} = -\nabla f(x,t) \qquad x \in \overline{\Omega_T}.$

② $\dfrac{\partial x}{\partial t} \cdot \nu = -\beta$ on $M_T = \partial_r \Omega_T$ (later take $\beta = H_{M_T}$).

① & ② compat. if $\nabla f \cdot \nu = \beta$ on M_t (3).

Assume now that f satisfies:

④ $(\partial_t + \Delta) f = |\nabla f|^2 + \dfrac{n+1}{2\tau}$, $\dfrac{\partial \tau}{\partial t} = 1.$

Set $\dfrac{dh}{dt} = \dfrac{\partial h}{\partial \tau} + \nabla h \cdot \dfrac{\partial x}{\partial t} = \dfrac{\partial h}{\partial t} - \nabla f \cdot \nabla h.$ So

so $\dfrac{df}{dt} = \dfrac{\partial f}{\partial t} - |\nabla f|^2.$

$\Rightarrow$ ⑤ $\left(\dfrac{d}{dt} + \Delta\right) f = \dfrac{n+1}{2\tau}$ (5).

Then $\dfrac{d\tau}{dt} = -1$, we conclude $\left(\dfrac{\partial}{\partial t} + \Delta\right) u = 0$

Also, $\dfrac{d}{dt} dx = x \operatorname{div}\left(\dfrac{\partial x}{\partial t}\right) dx.$

$\rightsquigarrow \;\; \dots \rightsquigarrow \dots \;\; \dfrac{d}{dt}(u \, dx) = \left[\left(\dfrac{\partial}{\partial t} + \Delta\right) u\right] dx = 0$ (1).

$\Rightarrow$ $\boxed{\int_{\Omega_T} u \quad \text{indep of } t\,!}$

$\nabla f \cdot \nu = \beta$ on $\partial\Omega$ $\Rightarrow$ $\mathcal{W}_\beta(\Omega, f, \tau) = \int_\Omega W u$ ~~with~~.

with $W = \tau(2\Delta f - |\nabla f|^2) + f - (n+1)$.

Perelman calculated in Ch 9. (paper I)

$\left(\frac{d}{d\tau} + \Delta\right) w$ and concluded (for our situation)

$$\frac{d}{dt} \mathcal{W}_\beta(\Omega_T, f(t), \tau(t)) =$$

$$= 2\tau(t) \int_{\Omega_T} \left|\nabla^2 f(t) - \frac{I}{2\tau(t)}\right|^2 u(t)\, d\mu$$

$$- \int_{M_\tau} \nabla w \cdot \nu\, u.$$

Lemma On M_τ, ~~$\nabla w \cdot \nu$~~

$$-\nabla w \cdot \nu = 2\tau\left(\frac{\partial \beta}{\partial t} - 2\nabla^M \beta \cdot \nabla^M f + A_M(\nabla^M f, \nabla^M f) - \frac{\beta}{2\tau}\right)$$

Recall $Z(v) = \frac{\partial H}{\partial t} + 2\nabla^{M_\tau} H \cdot v + A_{M_\tau}(v, v)$, v tangent to M_τ.

Hamilton: Harnack ineq. for MCF (convex, closed soln's).

$Z(v) + \frac{H}{2\tau} \geq 0$. $\forall T > 0$ on closed convex MCF soln.

" $=$ " on homoth. exp. soln's.

Ex. $Z(v) - \frac{H}{2\tau(t)} = 0$ on horiz. shrinking soln.

Pf. $W = \tau(2\Delta f - |\nabla f|^2) + f - (n+1)$.

$$\frac{df}{dt} = -\Delta f + \frac{n+1}{2\tau} \iff -2\tau\Delta f + (n+1) = -2\tau\frac{df}{dt}.$$

$$-\frac{df}{dt} = -\Delta f - \frac{(n+1)}{2\tau}, \quad \Delta f = -\frac{df}{dt} + \left(\frac{n+1}{2\tau}\right).$$

$$2\Delta f = -2\frac{df}{dt} + \frac{n+1}{\tau}.$$

$$W = \tau\left(-2\frac{df}{dt} - |\nabla f|^2\right) + f.$$

$$\frac{\partial x}{\partial t}\cdot \nu = -\beta, \quad \frac{\partial x}{\partial t} = -\nabla f(x,t), \quad \nabla f\cdot\nu = \beta \text{ on } M_T.$$

$$\boxed{\frac{\partial x}{\partial t} = -\beta\nu - \nabla^M f.}$$

$$\boxed{\frac{\partial \nu}{\partial t} = \nabla^M \beta - A_M(\nabla^M f, \cdot).}$$

We saw last week:

$$\begin{cases} \dfrac{d}{dt}\nabla f = \nabla^2 f(\nabla f, \cdot) + \nabla \dfrac{df}{dt} \\ \beta = \nabla f\cdot\nu. \end{cases}$$

$$\Rightarrow \frac{d\beta}{dt} = \nabla^2 f(\nabla f, \nu) + \nabla\frac{df}{dt}\cdot\nu + \nabla^M\beta\cdot\nabla^M f$$
$$- A(\nabla^M f, \nabla^M f).$$

$$\nabla w \cdot \nu = -\tau\left(2\nabla \frac{df}{dt}\cdot\nu + \nabla|\nabla f|^2\cdot\nu\right) + \nabla f\cdot\nu .$$

and $\nabla|\nabla f|^2\cdot\nu = 2\nabla^2 f(\nabla f,\nu)$.

Also, $\nabla \frac{df}{dt}\cdot\nu = \frac{d}{dt}\nabla f\cdot\nu - \nabla^2 f(\nabla f,\nu)$.

$$\rightsquigarrow \nabla w\cdot\nu = -\tau\left(2\nabla\frac{df}{dt}\cdot\nu + 2\nabla^2 f(\nabla f,\nu)\right) + \beta .$$

{[Bry] ?} (Thm of RF due to Fezza):

Sps that we have $\left(\frac{\partial}{\partial s}+\Delta\right)f = |\nabla f|^2 + \frac{n+1}{2\tau(s)}$, $\frac{d\tau}{ds} = 1$. $s \leq t$.

$(s\in(t-\varepsilon,t]$, ε small$)$.

Then $\int_{\Omega_s} u(s)\, d\mu = 1 \quad \forall s\leq t$. $s\in[t-\xi, t]$.

and $\frac{d}{ds} W_{\mathbb{1}_{M_s}}(\Omega_s, f(s), \tau(s)) \geq 0. \quad s\leq t.$

(see Fezza).

Then, $\mu_{\mathbb{1}_{M_{t_2}}}(\Omega_{t_2}, \tau(t_2)) \geq \mu_{\mathbb{1}_{M_{t_1}}}(\Omega_{t_1}, \tau(t_1))$.

$$T \geq t_2 \geq t_1 \geq 0 .$$

Pf. Fix $t_0 < T$, let f^{t_0} be a min. for $\mu_{\mathbb{1}_{M_{t_0}}}(\Omega_{t_0}, \tau(t_0))$ and $f(t)$ sol^{n} of $(*)$ for $t < t_0$, with $f(t_0) = f^{t_0}$.

$$\Rightarrow \mathcal{W}_{H_{M_t}}(\Omega_t, f(t), \tau(t)) \overset{(+)}{\leq} \mathcal{W}_{H_{M_{t_0}}}(\Omega_{t_0}, \underbrace{f(t_0)}_{f^{t_0}\ \text{minimiser!}}, \tau(t_0)).$$

$$= \mu_{H_{M_{t_0}}}(\Omega_{t_0}, \tau(t_0)).$$

$$\geq \mu_{H_{M_t}}(\Omega_t, \tau(t))$$. (Since this is in the admissible class. $t < t_0$).

Set $t_0 = t_2$, $t = t_1$,

$$\boxed{\Rightarrow \mu_{H_{M_{t_1}}}(\Omega_{t_1}, \tau(t_1)) \leq \mu_{H_{M_{t_2}}}(\Omega_{t_2}, \tau(t_2))}$$

Monotonicity formula.

Cor. (See Perelman: lower volume ratio bound).

Apply above with $a = r^2 + t$, $t < T$, $\tau(t) = a - t$.

$t_1 = 0$, $t_2 = t$.

If $r \leq \sqrt{T}$, $t < \sqrt{T}$, then,

$\tau(0) = a = r^2 + t$.

$$\mu_{H_{M_0}}(\Omega_0, r^2 + t) \leq \mu_{H_{M_t}}(\Omega_t, r^2).$$

If $r^2 \leq T$ and $t < T$, then $r^2 + t < 2T$.

So if $T < \infty \Rightarrow$

$$\mu_{H_{M_0}}(\Omega_0, r^2+t) \geq - c_0(n, \Omega_0, M_0, T) > -\infty.$$

$\quad$ from that depends on log-Sobolev.

$\Rightarrow$ ~~A~~.

$$-c_0(n, \Omega_0, M_0, T) \leq \mu_{H_{M_t}}(\Omega_t, r^2).$$

$$\leq \log\left(\frac{|\Omega_t \cap \mathring{B}_r(x_0)|}{r^{n+1}}\right) +$$

$$c(n)\left(\frac{|\Omega_t \cap B_r(x_0)| + 2r^2 \int_{M_t \cap B_r(x_0)} |H|}{|\Omega_t \cap B_{r/2}(x_0)|}\right).$$

see before.

$\Longrightarrow \exists\, k = k(c_0, c_1) > 0 \quad \forall t < T, \ \forall r \leq \sqrt{T}.$

$$\forall \Omega_t. \ \frac{|\Omega_t \cap B_r(x_0)|}{r^{n+1}} \geq k \ \text{ if } \ O(r, t, x_0) \leq c_1.$$