Summary

- $(\Omega_t)_{t\in[0,T[}$, $M_t = \partial\Omega_t$ smoothly
- $\overline{\Omega_t} = \varphi_t(\overline{\Omega})$, $\varphi_t : \overline{\Omega} \to \mathbb{R}^{n+1}$ is smooth and smooth in $t \in]0,T[$

- $\overline{\Omega_t} \ni x = \varphi(p,t)$, $p \in \overline{\Omega}$.
- Normal speed of M_t: (1) $\beta = -\frac{\partial x}{\partial t}\cdot \nu$ outward normal to M_t

(Example: $\beta = H_{M_t}$: $\left(\frac{\partial x}{\partial t}\right)^\perp = \vec{H}_{M_t} = -H_{M_t}\nu$ MCF up to tang. diffeos)

More specifically assume

(2) $\frac{\partial x}{\partial t} = -\nabla f(x,t)$, $x \in \Omega_t$
f smooth

(2) compatible with (1) if $\nabla f\cdot \nu = \beta$ on $M_t = \partial\Omega_t$ (3)
which leads to $\frac{\partial x}{\partial t} = -\beta\nu - \nabla^{M_t} f$ on M_t (3')

Suppose f satisfies

(4) $(\partial_t + \Delta) f = |\nabla f|^2 + \frac{n+1}{2\tau(t)}$ in Ω_t, $t\in]0,T[$ dep. on x,t

Total derivative: $\frac{dh}{dt} = \frac{\partial h}{\partial t} - \nabla f\cdot\nabla h$, $h(x,t)$

$$\rightsquigarrow \frac{df}{dt} = \frac{\partial f}{\partial t} - |\nabla f|^2$$

$$\rightsquigarrow (4) \Leftrightarrow \left(\frac{d}{dt} + \Delta\right) f = \frac{n+1}{2\tau(t)}.$$

Assume $\tau(t) > 0$ $\forall t\in]0,T[$ and $\frac{d\tau}{dt}(t) = -1$ (i.e. $\tau(t) = a - t$ for some $a\in\mathbb{R}$, "backward time" or "time left").

Then (4) $\Leftrightarrow$ $\left(\frac{\partial}{\partial t} + \Delta\right) u = 0$ where $u = \frac{e^{-f}}{(4\pi\tau)^{\frac{n+1}{2}}}$.

Note: $x = \varphi(p,t)$ $\quad \tilde{f}(p,t) = f(\varphi(p,t),t)$
Then $\frac{\partial\tilde{f}}{\partial t}(p,t) = \frac{df}{dt}(x,t)$.

$$\frac{d}{dt} dx \overset{\text{Ex.}}{=} \operatorname{div}\left(\frac{\partial x}{\partial t}\right) dx = -\Delta f \, dx.$$

$\sqrt{\det g_{ij}(p,t)}$

$g_{ij}(p,t) = \frac{\partial \varphi}{\partial p_i}(p,t) \cdot \frac{\partial \varphi}{\partial p_j}(p,t)$

$$\hookrightarrow \frac{d}{dt}(u \, dx) = \left[\left(\frac{\partial}{\partial t} + \Delta\right) u\right] dx = 0$$

$$\Rightarrow \frac{d}{dt} \int_{\Omega_t} u \, dx = 0 \quad \text{so}$$

$$\boxed{\int_{\Omega_t} u = \int_{\Omega_{t_0}} u = 1 \quad \forall t, t_0 \in [\ldots]}$$

Prop. (Perelman, paper 1, ch. 9) on fixed mfd (X,g) rather than $(\Omega_t) \subset \mathbb{R}^{n+1}$

Let $(\Omega_t)_{t \in [0,T[}$ evolve by (2) and f evolve by (4).

Suppose $\tau(t) > 0$ satisfies $\frac{d\tau}{dt} = -1$. Then

$W = W_\tau(f) := \tau(2\Delta f - |\nabla f|^2) + f - (n+1)$ satisfies

$$(*) \quad \left(\frac{d}{dt} + \Delta\right) W = 2\tau \left|\nabla^2 f - \frac{I}{2\tau}\right|^2 + \nabla W \cdot \nabla f$$

$(*)$, $\frac{d}{dt}(u \, dx) = 0$ and $\nabla f \cdot \nu = 0$ on $\partial\Omega_t$

$$\left(\Rightarrow W_\beta(\Omega_t, f(t), \tau(t)) = \int_{\Omega_t} W u \, dx\right)$$

$$\frac{d}{dt} W_\beta(\Omega_t, f(t), \tau(t)) = \frac{d}{dt}\int_{\Omega_t} W u \, dx = \int_{\Omega_t} \frac{dW}{dt} u \, dx = \int_{\Omega_t} \left[\left(\frac{d}{dt} + \Delta\right) W\right] u \, dx$$

$$= 2\tau \int_{\Omega_t} |\ldots|^2 u \, dx + \int_{\Omega_t} \nabla W \cdot \nabla f \, u - \int_{\Omega_t} \Delta W u$$

$$= 2\tau \int_{\Omega_t} |\ldots|^2 u - \int_{\Omega_t} (\nabla W \cdot \nabla u + \Delta W u)$$

$$= 2\tau \int_{\Omega_t} |\ldots|^2 u - \int_{\Omega_t} \operatorname{div}(\nabla W u)$$

$- \int_{M_t} \nabla W \cdot \nabla u$

For later:

Let f^{t_0} be min. of $\mu_\beta(\Omega_{t_0}, \tau(t_0))$, $t_0 > 0$ and $f(t)$, $t \in]t_0 - \varepsilon, t_0[$ sol. of (4) and $f(t_0) = f^{t_0}$.

$\tilde{f}(p,t) = f(\varphi(p,t), t)$

$g_{ij}(p,t) = \frac{\partial \varphi}{\partial p_i}(p,t) \cdot \frac{\partial \varphi}{\partial p_j}(p,t)$, inverse metric $(g^{ij}(p,t))$

$$\frac{\partial x}{\partial t} = -\nabla f(x,t) \Leftrightarrow \frac{\partial \varphi}{\partial t}(p,t) = -\tilde{\nabla}\tilde{f}(p,t) \overset{Ex}{=} -g^{ij}(p,t)\cdot\frac{\partial \tilde{f}}{\partial p_i}(p,t)\cdot\frac{\partial \varphi}{\partial p_j}(p,t)$$

$$\text{"} |\nabla f|^2 \text{"} = g^{ij}\frac{\partial \tilde{f}}{\partial p_i}\frac{\partial \tilde{f}}{\partial p_j}.$$

$$\underline{Ex:}\quad \frac{\partial}{\partial t} g_{ij} = -2\tilde{\nabla}_i\tilde{\nabla}_j f \left(= -2\frac{\partial^2}{\partial p_i \partial p_j}\tilde{f}\right)$$

$$\frac{\partial}{\partial t} g^{ij} = 2\tilde{\nabla}^i\tilde{\nabla}^j\tilde{f}$$

$$\left[\frac{\partial}{\partial t} g^{il} = -g^{ik} g^{jl}\frac{\partial}{\partial t} g_{kj}\right]$$

Using normal coord. at a fixed point (p,t),

$$g_{ij}(p,t) = \delta_{ij},\ \partial_i g_{kj}(p,t) = 0$$

$1 \le i,j \le n$

$$\Updownarrow$$

$$\frac{\partial^2 \varphi}{\partial p_i \partial p_j}(p,t)\cdot\frac{\partial \varphi}{\partial p_j}(p,t) = 0$$

$$\underline{Ex:}\quad \boxed{g^{ij}\frac{\partial}{\partial t}\Gamma^k_{ij} = \tilde{\nabla}^k\tilde{\Delta}\tilde{f}}$$

$$\Delta f(x,t) = \tilde{\Delta}\tilde{f}(p,t),\quad \tilde{\Delta}\tilde{f} = g^{ij}\left(\frac{\partial^2}{\partial p_i \partial p_j}\tilde{f} - \Gamma^k_{ij}\frac{\partial \tilde{f}}{\partial p_k}\right)$$

$$\rightsquigarrow \boxed{\frac{d}{dt}|\nabla f|^2 = 2\nabla^2 f(\nabla f, \nabla f) + 2\nabla f\cdot\nabla\frac{df}{dt}}$$

$$\boxed{\frac{d}{dt}\nabla f = \nabla^2 f(\nabla f,\cdot) + \nabla\frac{df}{dt}}$$

$\underline{Ex}$

$$\boxed{\underline{Ex:}\quad \frac{d}{dt}\Delta f = \Delta\frac{df}{dt} + 2|\nabla^2 f|^2 + \nabla f\cdot\nabla\Delta f}$$

$$\Rightarrow \left(\frac{d}{dt}+\Delta\right)\frac{df}{dt} \underset{(4)}{=} \frac{n+1}{2\tau^2} - 2|\nabla^2 f|^2 - \nabla f\cdot\nabla\Delta f$$

$$\left(\left(\frac{d}{dt}+\Delta\right)f = \frac{n+1}{2\tau}\right)$$

Bochner: $\Delta|\nabla f|^2 = 2|\nabla^2 f|^2 + 2\nabla f\cdot\nabla\Delta f$

$$\Rightarrow \left(\frac{d}{dt}+\Delta\right)|\nabla f|^2 = 2|\nabla^2 f|^2 + 2\nabla^2 f(\nabla f, \nabla f).$$

$$W = \tau\left(2\Delta f - |\nabla f|^2\right) + f - (n+1)$$

$$\Rightarrow W = \tau w + f, \quad w = -2\frac{\partial f}{\partial t} - |\nabla f|^2 \quad \left(\overset{(4)}{=} 2\Delta f - |\nabla f|^2 - \frac{n+1}{2\tau}\right)$$

$$\left(\frac{d}{dt} + \Delta\right) w = 2|\nabla^2 f|^2 - \frac{n+1}{2\tau^2} \underbrace{- 2\nabla^2 f(\nabla f, \nabla f) + 2\nabla f \cdot \nabla \Delta f}_{\overset{\text{Ex.}}{=} \nabla f \cdot \nabla w}$$

$$\Rightarrow \left(\frac{d}{dt} + \Delta\right) w = 2|\nabla^2 f|^2 - \frac{n+1}{2\tau^2} + \nabla f \cdot \nabla w, \quad \frac{d\tau}{dt} = -1$$

$$\Rightarrow \left(\frac{d}{dt} + \Delta\right) W = \left(\frac{d}{dt} + \Delta\right)(\tau w + f) = -w + 2\tau|\nabla^2 f|^2 - \frac{n+1}{\tau} + \nabla f \cdot \nabla(\tau w) + \frac{n+1}{2\tau}$$

$$= \nabla f \cdot \nabla W - w - |\nabla f|^2 + 2\tau|\nabla^2 f|^2 - \frac{n+1}{2\tau}.$$

Note: $2\tau|\nabla^2 f|^2 = 2\tau\left|\nabla^2 f - \frac{I}{2\tau}\right|^2 + 2\Delta f - \frac{n+1}{2\tau}$

and $w = 2\Delta f - |\nabla f|^2 - \frac{n+1}{\tau}$. //