Klaus Ecker – Lec 6. 04/06/2018.

$\mu_\beta(\Omega,\tau) := \inf\{w_\beta(\Omega,f,\tau): \int_\Omega u = 1\}$.

$w_\beta(\Omega,f,\tau) = \int_\Omega \left(\tau|\nabla f|^2 + f - (n+1)\right) u + 2\tau \int_{\partial\Omega} \beta u$.

Have seen $\mu_\beta(\Omega,\tau) > -\infty$ under mild cond on $\Omega$ and $\partial\Omega$ and $\exists$ min $f$ of $\mu_\beta(\Omega,\tau)$. (uniqueness).

<u>Thm</u>, if $\beta > 0$, $\Omega$ bdd, $\partial\Omega \in C^2 \Rightarrow \nu_\beta(\Omega) = \inf_{\tau>0} \mu_\beta(\Omega,\tau) > -\infty$.

Can be weakened without change of pf $\int_\Omega \beta u > 0$.
(Roughly relevant to MCF).

<u>Prop</u>. $f$ min of $\mu_\beta(\Omega,\tau) \Leftrightarrow f$ sat:

(1) $W_\tau(f) = \tau(2\Delta f - |\nabla f|^2) + f - (n+1) = \mu_\beta(\Omega,\tau)$. in $\Omega$.

(2) $\nabla f \cdot \nu = \beta$

(3) $\int_\Omega u = 1$.

"$\Rightarrow$" used $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left(w_\beta(\Omega,f_\varepsilon,\tau) + \lambda\int_\Omega u_\varepsilon\right)$, $u_\varepsilon = \frac{e^{-f_\varepsilon}}{(4\pi\tau)^{\frac{n+1}{2}}}$.

$\int_\Omega u_\varepsilon = 1$ <u>NOT</u> required, $\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} f_\varepsilon(x) = \eta(x)$ arbitrary.

- $\nabla f \cdot \nu = \beta$ arises from considering $\eta \in C^1(\bar\Omega)$ (Lagrange mult).
- Restricting $\eta \in C^1_0(\Omega)$ and int. by parts. $\Rightarrow w_\beta(\Omega,f,\tau) = 1 - \lambda$.
- $\int u = 1$ (automatic) $\mu_\beta(\Omega,\tau) \overset{f_{min}}{=} w_\beta(\Omega,f,\tau) = \int_\Omega W_\tau(f) u$.

**Prop** Suppose $\nu_\beta(\Omega) > -\infty$, and $(f,\tau)$ ~~t~~ with $\tau > 0$ s.t.

$$\nu_\beta(\Omega) = \mu_\beta(\Omega,\tau)$$

(such a $\tau$ exists if $\beta > 0$ etc.).

[Later, $\nu_\beta(\Omega) = \lim_{\tau\to 0} \inf \mu_\beta(\Omega,\tau)$, then $\liminf_{\tau\to 0} \mu_\beta(\Omega,\tau) \geq 0$, $\beta \geq 0$ smu, $\partial\Omega \in C^2$, $\Omega$ bdd]

Then, TFAE:

a) $$\int_\Omega |\nabla f|^2 u = \frac{n+1}{2\tau} - 2\int_{\partial\Omega} \beta u$$

b) $$\int_\Omega \left(\frac{n+1}{2\tau} - \Delta f\right) u = \int_{\partial\Omega} \beta u$$

c) $$\int_\Omega f u = \frac{n+1}{2} + \nu_\beta(\Omega).$$

**Pf**. We could have started with ~~$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}$~~

(+) $$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \left(w_\beta(\Omega, f_\varepsilon, \tau_\varepsilon) + \lambda \int_\Omega \frac{e^{-f_\varepsilon}}{(4\pi\tau_\varepsilon)^{\frac{n+1}{2}}}\right).$$

Do not assume $\int_\Omega \frac{e^{-f_\varepsilon}}{(4\pi\tau_\varepsilon)^{\frac{n+1}{2}}} = 1$ $\forall\varepsilon$, $\partial_\varepsilon\big|_{\varepsilon=0} f_\varepsilon = \eta$, $\partial_\varepsilon\big|_{\varepsilon=0} \tau_\varepsilon = \sigma$

$f_0 = f$, $\tau_0 = \tau$, $(f,\tau)$ minimizing pair.

Since (+) has to hold for the "f-variation part" (which gave us $\lambda = 1 - \mu_\beta(\Omega,\tau)$), we

$$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \left(w_\beta(\Omega, f, \tau_\varepsilon) + (1-\mu_\beta(\tau)) \int_\Omega \frac{e^{-f}}{(4\pi\tau_\varepsilon)^{\frac{n+1}{2}}}\right).$$

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{e^{-f}}{(4\pi\tau_\varepsilon)^{\frac{n+1}{2}}} = -u \frac{n+1}{2\tau}\sigma. \Rightarrow \ldots \Rightarrow$$

$$\int_\Omega |\nabla f|^2 u - \frac{n+1}{2\tau}\int_\Omega \left(\tau|\nabla f|^2 + f - (n+1)\right) u = \cancel{\frac{n+1}{2\tau}}$$

$$= \lambda \frac{n+1}{2\tau} - \left(2-(n+1)\right)\int_{\partial\Omega} \beta u.$$

But $\lambda = 1 - \mu_\beta(\Omega, \tau)$, ~~~ $\mu_\beta(\Omega,\tau) = \nu_\beta(\Omega)$

cancellation.

$$\Rightarrow \int_\Omega |\nabla f|^2 u = \frac{n+1}{2\tau} - 2\int_{\partial\Omega} \beta u.$$

$\nabla u = u \cdot \nabla f$, $\int_\Omega \Delta u = \int_{\partial\Omega} \nabla u \cdot \nu = -\int_{\partial\Omega} u \nabla f \cdot \nu = -\int_{\partial\Omega} \beta u.$

$\Rightarrow$ (b).

$$\nu_\beta(\Omega) = \mathcal{W}_\beta(\Omega, f, \tau) = \tau\int_\Omega |\nabla f|^2 u - (n+1) + 2\tau\int_{\partial\Omega} \beta u + \int_\Omega f u$$

$$= \frac{n+1}{2} - 2\tau\int_{\partial\Omega} \beta u + 2\tau\int_{\partial\Omega} \beta u - (n+1) + \int_\Omega f u.$$

$\Rightarrow$ (c).

Remark. Sps $\delta f = \eta$ $\left(\delta f = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f_\varepsilon, \quad f_0 = f\right)$.

Assume $u_\varepsilon = \dfrac{e^{-f_\varepsilon}}{(4\pi\tau)^{\frac{n+1}{2}}}$ sat. $\int_\Omega u_\varepsilon = 1$, for $\varepsilon$

in small nbhd and $\varepsilon = 0$.

Then $\int_\Omega u\eta = 0$, because $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} u_\varepsilon = -u\eta$.

~~$g \ni$~~ $f : (x \in) \mathbb{R}^{n+1} \to \mathbb{R}$ with $g(x) \le g(y) \; \forall y \in S$.

$S = \{ z \in \mathbb{R}^{n+1} : g(z) = 0 \}$.

$g, f \in C^1 \Rightarrow \exists \lambda \in \mathbb{R}$, s.t. $\nabla (f + \lambda g)(x) = 0$.

$\alpha : I \to S$, $\alpha(\varepsilon) \in S$, $\forall \varepsilon \in I$, $0 \in I$.

$g \in C^1 \Rightarrow \alpha \in C^1$ or $g \in C^2 \Rightarrow \alpha \in C^2$.

$v \in T_pS$ (tangent vector), $\exists \alpha : I \to S$ with $\alpha(0) = p$, $\dot\alpha(0) = v$.

One can show ~~...~~ $v \cdot \nabla g(x) = 0 \Rightarrow v \in T_pS$.

(A) $f(x) \le f(y)$ $\forall y \in S$.

Let $\alpha : I \to S$ with $\alpha(0) = x$, $\dot\alpha(0) = v \in \mathbb{R}^{n+1}$ arbitrary.

Then, (A) $\Rightarrow f(\alpha(0)) \le f(\alpha(\varepsilon))$ $\forall \varepsilon \in I$.

$\Rightarrow \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} f(\alpha(\varepsilon)) = 0$.

In general, $\frac{d}{d\varepsilon} f(\alpha(\varepsilon)) = \nabla f(\alpha(\varepsilon)) \cdot \dot\alpha(\varepsilon)$

$\Rightarrow 0 = \nabla f(\alpha(0)) \cdot \dot\alpha(0) = \nabla f(x) \cdot v$.

$\Rightarrow \nabla f(x) \perp T_x S$, and $\nabla g(x) \perp T_x S$.

$\Rightarrow \exists \lambda \in \mathbb{R}$ s.t. $\nabla f(x) = \lambda \nabla g(x)$.

$$\frac{d^2}{d\varepsilon^2}\Big|_{\varepsilon=0}\Big(f(\alpha(\varepsilon)) + \lambda g(\alpha(\varepsilon)).\Big)$$

$$= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Big( \nabla f(\alpha(\varepsilon)) \cdot \dot\alpha(\varepsilon) + \lambda \nabla g(\alpha(\varepsilon))\cdot\dot\alpha(\varepsilon) \Big).$$

$$= \nabla^2 f(\dot\alpha(0))\,(\dot\alpha(0),\dot\alpha(0)) + \nabla f(\alpha(0))\cdot\ddot\alpha(0)$$

$$+ \lambda \nabla^2 g(\dot\alpha(0))(\dot\alpha(0),\dot\alpha(0)).$$

$$+ \lambda \nabla g(\alpha(0))\cdot\ddot\alpha(0)$$

$$= \nabla^2(f+\lambda g)(x)\,(v,v) + \underbrace{\nabla(f+\lambda g)(x)}_{\overset{\shortparallel}{0}}\cdot\ddot\alpha(0)$$

$$\nabla^2(f+\lambda g)(x)(v,v) \geq 0 \quad \forall v \in T_x S.$$

For us, $S = \{f \;,\; g(f) = \int_\Omega n - 1 = 0\}$.

$n$.

Sem: $\delta\left(\mathcal{W}_\beta(\Omega, f, \tau) + \lambda \int_\Omega n\right) = 1$.

$$\delta f = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f_\varepsilon, \quad f_0 = f \leftrightarrow n.$$

$$\overset{\shortparallel}{\eta} \in T_f S. \qquad f_\varepsilon \leftrightarrow \eta(\varepsilon).$$

Lag. mult. sec. variation gives

$$0 \leq \frac{d^2}{d\varepsilon^2}\Big|_{\varepsilon=0}\left(\mathcal{W}_\beta(\Omega, f_\varepsilon, \tau) + (1-\mu_\beta(\Omega))\int_\Omega n_\varepsilon\right)$$

$$\forall \eta \text{ with } \int_\Omega \eta n = 0.$$

$\Rightarrow$ Corollary $\forall \eta$ with $\int_\Omega \eta u = 0$, we have

$$\frac{1}{2\tau}\int_\Omega \eta^2 u \leq \int_\Omega |\nabla \eta|^2 u.$$

Remark: w.r.t. $u$.

$$\mathrm{Var}_f = \int_\Omega \{\eta^2 u = \int_\Omega |\eta - \int_\Omega \eta u|^2 u$$

$$\leq \int_\Omega |\nabla \eta|^2 u.$$

Linearize $W_\tau(f)$ around $f$ (dep on $f$ and $\tau$).

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} W_\tau(f_\varepsilon) = \tau\,(2\Delta f_\varepsilon - |\nabla f_\varepsilon|^2) + f_\varepsilon - (n+1).$$

$\nabla f_\varepsilon \cdot \nu = \beta$ on $\partial\Omega$.

$\int_\Omega u_\varepsilon = 1 \quad \forall \varepsilon$.

$$\Rightarrow \quad 2\tau\,(\Delta \eta - \nabla f \cdot \nabla \eta) + \eta = 0 \text{ in } \Omega.$$

$\nabla \eta \cdot \nu = 0$ on $\partial\Omega$, $\int_\Omega \eta u = 1$.

Set $L_f = \Delta - \nabla f \cdot \nabla$,

$$L_f \eta + \frac{\eta}{2\tau} = 0 \text{ in } \Omega.$$

$\nabla \eta \cdot \nu = 0$ on $\partial\Omega$.

$\int_\Omega \eta u = 0$.

$L_f$ s.a. $(\Omega, u\,dx)$, $\quad \frac{1}{2\tau}\int_\Omega |\eta|^2 u \leq \int_\Omega |\nabla \eta|^2 u = -\int_\Omega \eta L_f \eta u$

For minimizing pairs $(f, \tau)$:

$$\begin{cases} (I) & \int_\Omega f \eta \; n = 0. \\ (II) & \int_\Omega |\nabla f|^2 \eta \; n = 0. \end{cases}$$

$$(III) \quad \int_\Omega \Delta f \cdot \eta \; n = 0.$$

(I). $\{f\}$ not a subspace, $f = 0$ is not a minimum.