Approx of L^2-inv. of locally sym. spaces. 24/09/2018.

Werner Müller.

1. L^2-inv. X cpt Riem, $\dim = n$, $\tilde{X} \to X$ universal cover; $\pi = \pi_1(X, x_0)$.
$\# \pi = \infty$

"classical inv" - Betti #, Ind D, signature, anal. torsion.

<u>L^2-Betti numbers</u> $\tilde{\Delta} : \Lambda^p(\tilde{X})$, $\mathcal{H}_2^p(\tilde{X}) = \{ \varphi \in \Lambda^p(\tilde{X}) : \tilde{\Delta}\varphi = 0,\ \varphi \in L^2 \}$
$(= \ker(\tilde{\Delta}, \Lambda^p(\tilde{X})))$.

$\{\varphi_j\}$ o.n. basis of $\mathcal{H}_2^p(\tilde{X})$, $f(\tilde{x}) = \sum_j |\varphi_j(\tilde{x})|^2$ - π inv.

$\Rightarrow \tilde{f}$ lift of a smooth function of $f \in C^\infty(X)$.

$$\dim_\pi \mathcal{H}_2^p(\tilde{X}) := \int_X f(x)\, dx. \quad \text{- von Neumann dim.}$$

L^2 $\quad b_p^{(2)}(X) := \dim_\pi \mathcal{H}_{(2)}^p(\tilde{X})$.

$D : C^\infty(X, E) \to C^\infty(X, F)$ elliptic,

$\tilde{D} : C^\infty(\tilde{X}, \tilde{E}) \to C^\infty(\tilde{X}, \tilde{F})$

$$\mathrm{Ind}_\pi(\tilde{D}) = \dim_\pi \mathcal{H}(\tilde{D}) - \dim_\pi \mathcal{H}(\tilde{D}^*).$$

<u>Th (Atiyah)</u> $\mathrm{Ind}\, D = \mathrm{Ind}_\pi(\tilde{D})$.

$$\chi(X) = \mathrm{Ind}_\pi(\tilde{d} + \tilde{d}^*) = \sum_{p=0}^n (-1)^p b_p^{(2)}(X).$$

2. Analytic Torsion

$\rho : \Gamma \to GL(V)$ rep; $\dim_{\mathbb{C}} V < \infty$

$\rightsquigarrow E_\rho \to X$ flat vector bundle.

$\Delta_p(\rho) : \Lambda^p(X, E_\rho) \circlearrowleft$, $0 \le \lambda_1 \le \lambda_2 \le \dots \to +\infty$ eigenvalues

$$\zeta_p(s,\rho) = \sum_{\lambda_j > 0} \lambda_j^{-s}, \quad \mathrm{Re}(s) > n/2 .$$

- $\zeta_p(s,\rho)$ admits merom. ext. to $\mathbb{C}$, hol @ $s=0$.

$$\det \Delta_p(\rho) := \exp\left(-\frac{d}{ds} \zeta_p(s,\rho)\Big|_{s=0}\right).$$

$$T_X(\rho) = \prod_{p=1}^{n} [\det \Delta_p(\rho)]^{(-1)^{p+1} (p/2)} .$$

(Ray-Singer-analytic torsion).

$$\zeta_p(s,\rho) = \frac{1}{\Gamma(s)} \int_0^\infty \left(\mathrm{Tr}\,(e^{-t\Delta_p(\rho)}) - b_p(\rho)\right) t^{s-1}\, dt .$$

<u>L^2-torsion</u>. $\Delta := \Delta_p(\rho)$, $\tilde{\Delta} : \Lambda^p(\tilde{X}; \tilde{E}_\rho) \circlearrowleft$

But $\tilde{\Delta} e^{-t\tilde{\Delta}}$ cannot be expected to be of trace-class because $\tilde{X}$ is noncpt.

$e^{-t\tilde{\Delta}}$ smoothing op., $\tilde{k}(x,y,t)$ kernel of $e^{-t\tilde{\Delta}}$ $\in \mathrm{Hom}(\Lambda^p T_y^* \tilde{X} \otimes E_{\rho,y}, \Lambda^p T_x^* \tilde{X} \otimes E_{\rho,x})$

Let $F \subset \tilde{X}$ fund. dom. for Γ

$$\mathrm{Tr}_\Gamma (e^{-t\tilde{\Delta}}) := \int_F \mathrm{tr}\, \tilde{k}(x,x,t)\, d\tilde{x} .$$

$$b_p^{(2)}(\tilde{X}) = \lim_{t\to\infty} Tr_\pi(e^{-t\tilde{\Delta}'})$$

Need to understand asymp. of $Tr_\pi(e^{-t\tilde{\Delta}'})$ as $t \to 0$ and ∞.

Studied by J. Lott & M. Varghese.

a) As $t \to 0$, $Tr_\pi(e^{-t\tilde{\Delta}}) \sim t^{-n/2} \sum_{k\geq 0} a_k t^k$.

b) $\alpha(\Delta_p(\rho)) =. \sup\{\beta : Tr_\pi(e^{-t\tilde{\Delta}}) =. b_p^{(2)}(x) + O(t^{-\beta/2})\}$.
$t \to \infty$.

Novikov–Shubin inv., homotopy inv. of X.

$\alpha(\Delta_p(\rho)) > 0$:

$$\log \det_\pi (\Delta_p(\rho)')$$

$$= \frac{d}{ds}\left(\frac{1}{\Gamma(s)} \int_0^\varepsilon Tr_\pi(e^{-t\tilde{\Delta}'}) t^{s-1}\right)\Big|_{s=0}$$

$$+ \int_\varepsilon^\infty Tr_\pi(e^{-t\tilde{\Delta}'}) t^{-1}\, dt.$$

$\Delta_p(\rho)' = \Delta_p(\rho)|_{\ker^\perp} = \Delta_p(\rho)|_{\overline{\mathrm{Dom}}}$, $\tilde{\Delta}'$ also.

$$T_X^{(2)}(\rho) = \prod_{p=1}^{n} [\det_\pi(\Delta_p(\rho)')]^{(-1)^{p+1}\cdot p/2}.$$

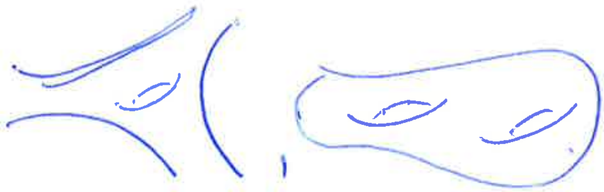
2) X locally symmetric.

- G semisimple LG. non-cpt.
- $K \subset G$, cpt?
- $\tilde X = G/K$ Riem. sym. space. $\Gamma \subset G$ discrete subgr. $\mathrm{vol}(\Gamma\backslash G) < \infty$, torsion free.

$X := \Gamma\backslash\tilde X$ locally sym. mfld.

Ex. $\mathbb{H}^2 = SL(2;\mathbb{R})/SO(2)$, $\Gamma(N) = \ker\big(SL(2;\mathbb{Z}) \to SL(2,\mathbb{Z}/N\mathbb{Z})\big)$.

$X(N) = \Gamma(N)\backslash\mathbb{H}^2$,



<u>Flat bundles</u>: $\tau : G \to GL(V)$. rep; $\dim V < \infty$,

$\rho = \tau|_\Gamma : \Gamma \to GL(V)$. $\rightsquigarrow E_\rho$, $\sigma := \tau|_K$,

$\tilde E = G \times_\sigma V = (G \times V)/K$.

<u>Lem.</u> $\exists$ can. isom. $E_\rho \cong \Gamma\backslash\tilde E$.

$\langle\cdot,\cdot\rangle$ on V, K inv. $\rightsquigarrow$ G-inv. metric on $\tilde E$.

$\tilde\Delta_p(\rho) : \Lambda^p(\tilde X, \tilde E) \circlearrowleft$, $\tilde\Delta_p(\rho)$ commutes with action of G.

$\Rightarrow$ $e^{-t\tilde\Delta_p(\rho)}$ is a conv. op.

$\Rightarrow$ $\mathrm{tr}\,\tilde k(\tilde x,\tilde x,t)$ indep of $\tilde x \in \tilde X$.

$\Rightarrow$ $\mathrm{Tr}_\Gamma\left(e^{-t\tilde\Delta_p(\rho)}\right) = \int_F \mathrm{tr}\,\tilde k(\tilde x,\tilde x;t)\,d\tilde x$

$= c_p(t)\,\mathrm{vol}(X)$.

G-inv. crucial!

Computation of $c_p(t)$: $\Lambda^p(\tilde{X}, \tilde{E}) \simeq \left(C^\infty(G) \otimes \Lambda^p \mathfrak{g}^* \otimes V_\tau\right)^K$.

$\mathfrak{g} = \mathfrak{k} \underset{\perp}{\oplus} \mathfrak{s}$ $\mathfrak{k}$-Killing form.

$\exists: H_t^p : G \to \operatorname{End}(\Lambda^p \mathfrak{g}^* \otimes V_\tau) \cdot \left(e^{-t\tilde{\Delta}_p(\rho)}\varphi\right)(g)$.

$$= \int_G H_p^t(g^{-1}g')(\varphi(g'))\, dg.$$

$h_t^p(g) = \operatorname{tr} H_t^p(g) \quad g \in G$.

$c_p(t) = h_t^p(e)$, $h_t^p \in \mathcal{C}(G)$ Harish-Chandra Schwartz.

K-finite $h_t^p \in \mathcal{C}(G)_{K\times K} \rightsquigarrow$ H.-Ch. Plancherel Th^{m}.

$$h_t^p(g) = \sum_P \sum_{\xi \in \hat{M}_d} \int_{\mathfrak{a}} \operatorname{Tr}\left(\pi_{\xi,\nu}(h_t^p)\, \pi_{\xi,i\nu}(g)^{-1}\right) P_\xi(i\nu)\, d\nu.$$

sums over parabolic subgroups.

Th^m (Ollmann, $\tau = 1$, Bergeron-Venkatesh) $\alpha_p(\rho) > 0$. $\forall p = 0, \dots, n$.

$\Rightarrow$. $T_X^{(2)}(\rho)$ well defined.

3) . Approx. of L^2-inv.: $X = \Gamma \backslash \tilde{X}$, $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \dots \Gamma_j \supset \{e\}$

$\bigcap_j \Gamma_j = \{e\}$, $X_j = \Gamma_j \backslash \tilde{X} \to X$

normal coverings

~~Normal~~

"normal subgroup."

α classical inv. $\dfrac{\alpha(X_j)}{[\Pi : \Pi_j]}$ has limit as $j \to \infty$?

Thm. (Lück) $\lim_{j \to \infty} \dfrac{b_j(X)}{[\Pi : \Pi_j]} = b_p^{(2)}(X)$.

Thm. locally sym, $b_p^{(2)}(X) \neq 0 \Leftrightarrow$ rk G = rk K; $p = \frac{n}{2}$.

What about $\dfrac{\log |H^p(X_j; \mathbb{Z})_{\text{tors}}|}{[\Pi : \Pi_j]}$, $\dfrac{\log T_{X_j}(\rho)}{[\Pi : \Pi_j]} \to$?

$\tau : G \to GL(V)$, $\theta : G \to G$ Cartan inv. $\tau_\theta = \tau \circ \theta$.

Thm. (B-V) X cpt, $\tau_\theta \neq \tau$.

$$\lim_{j \to \infty} \frac{\log T_{X_j}(\rho)}{\mathrm{vol}(X_j)} = t_{\tilde{X}}^{(2)}(\tau).$$

Thm. (Pfaff-Mü) $X = \Pi \backslash \mathbb{H}^n$, fin. vol., $\tau_\theta \neq \tau$.

Same result as B-V.

Complication: non-cpt, need a certain regularism.

Assume: $\exists$ $M \subset V$ Π-inv. lattice $\rightsquigarrow \mathcal{M} \to X$ loc. sym.

$H^*(X, \mathcal{M})$, $H^*(X, \mathbb{Z})$, $T_X(\rho) = \prod_{p=0}^{n} |H^p(X, \mathcal{M})_{\text{tors}}|^{(-1)^{p+1}}$

$\Pi = SL(2, \mathbb{Z}[i])$. $\subset SL(2, \mathbb{C})$.

$\Pi(N)$.

Strong A_∞-weights and compactness of conformal metrics.

Clara Aldana.

(M, g_0) closed mfld, $\dim \geq 3$. g conformal. $\exists\, u > 0$.
$u \in C^\infty$ $\quad g = u^{\frac{4}{n-2}} g_0 = e^{2f} g_0, \quad f \in C^\infty$.

sequence of conformal metrics $(g_k)_{k \in \mathbb{N}}$, $g_k = e^{2f_k} g_0$

$(g_k)_k$ conv. ~~iff~~ $\Longleftrightarrow$ $(f_k)_k$ converges.

Compactness / precompactness: find subsequence that converges to a limit.

<u>Motivation</u>: • <u>Isospectrality</u>: same spectrum of Laplace.

~~Thm: (Osgood-Phillips-Sarnak, $n=2$, closed), Brooks-Perry~~.

<u>Thm</u> (lots of ppl)

Isospectral metrics are compact in the C^∞-top of the set of metrics.

• <u>Yamabe prob.</u> (M, g_0) as before, find $\tilde{g}$ conformal to g_0 s.t. $\mathrm{Scal}_{\tilde{g}} = \text{const}$.

$\tilde{g} = u^{\frac{4}{n-2}} g_0$. has $\mathrm{Scal}_{\tilde{g}}$ (const) iff u satisf.

$$\frac{4(n-1)}{n-2} \Delta_{g_0} u - \mathrm{Scal}_{g_0} u + (\mathrm{Const}) \cdot u^{\frac{n+2}{n-2}} = 0$$

$(*)$

Metrics of +ve scal. curv.

- (M,g) confml (S^n, g_{round}), set of sol^{n} not compact.
- Compactness conj: (R. Schoen) (M, g_0) conf. (S^n, g_{round}) even fixing (const) in eq^{n}, set of sol^{n}s is cpt in the C^2-top.

  True $n \leq 24$

  False $n \geq 25$.

- Pinching Prob: Given sequence $(g_n)_n$ on M s.t.

$$\exists a \in \mathbb{R} . \quad a - \varepsilon \leq \mathrm{Sect}_{g_n}(x, \pi) \leq a + \varepsilon .$$

$$\forall x, \forall \pi .$$

$$\Downarrow \quad \exists \, g_0 \text{ curv. unb. s.t. } g_n \longrightarrow g_0 .$$

$$\text{as } n \to \infty , \quad \varepsilon \to 0 .$$

Th^m. (Gursky 93) (M, g_0) $n \geq 3$, $p > n/2$, $v, \Lambda > 0$,

$$0 < \alpha < \frac{2p - n}{p} < 1 , \quad \text{let:}$$

$$\mathcal{M}(p, v, \Lambda) = \{ f \in C^\infty(M) : \mathrm{vol}(M, g_n) = v ,$$
$$\int |\mathrm{Riem}_{g_f}|^p \, dv_{g_f} < \Lambda \} .$$

precompact in $C^{1,\alpha}(M)$, $\quad g_f = e^{2f} g_0$.

Rm $p = n/2$ critical, if $\| \mathrm{Riem}_{g_f} \|_{n/2} < \Lambda$, can have degeneracies.

Relation to analytic prop of the volume density:

$d\mu_f = e^{nf} d\mu_0$, critical elt $\mathcal{F}$ in scalar curv.

$\|R_{scal}\, g_f\|_{L^{n/2}(M, g_f)} \leq \Lambda$. $\Rightarrow$ analytical unders. on e^{nf}. (Strong A_∞ weight)

inspired by γ_f-Wang

$\Rightarrow$ cpctness in GH. (M, d_{g_f}).

Brendle conjecture for Conformal S^n using Q-curv.

<u>A_∞-weights:</u> $w \in L^1(M, d\mu_0)$ $w \geq 0$. an A_∞ weight w.r.t. g_0 if $\exists q > 1$ s.t. $\exists C$ $\forall B \subset M$ balls (w.r.t. g_0)

$$\left(⨍_B w^q \, d\mu_0\right)^{1/q} \leq C ⨍_B w \, d\mu_0.$$

(Reverse Hölder ineq.)

<u>Results</u> David-Semmes, Bonk-Heinonen-Saksman etc.

Implies (I) doubling of $d\mu_w = w\, d\mu_0$ w.r.t. g_0-balls,

(II) $\exists \tilde{B} = \tilde{B}(C, q, g_0)$ $\forall x, y \in M$,

constant

$$d_f(x,y) \leq \tilde{B} \int_{B(x, d_0(x,y))} e^{nf} d\mu_0.$$

$w = e^{nf}$.

$$= \tilde{B}\, M_w(B(x, d_0(x,y))).$$

Prop. (A - Carron - Topic), Let $c, v > 0$, ~~$R_0 \in (0, \mathrm{diam}$~~
$R_0 \in (0, \mathrm{diam}(M, g_0)]$, $q > 1$.

Define $\mathcal{M}_{v, R_0, q, c} \ni f \iff$ (I) for $g_f = e^{2f} g_0$, $\mathrm{vol}(M, g_f) \le V$.
(II) $w = e^{nf}$ an A_∞ weight w.r.t g_0 with consts (R_0, q, c).

Then: • $\{ d_f : f \in M_{v, R_0, q, c} \}$ precpt. C^α top; $\alpha \in (0, 1 - \frac{1}{q})$.

• $\{ (M, d_f) : f \in \mathcal{M}_{v, R_0, q, c} \}$ precpt G.H.

__Im Pt.__ • dist est.

• $\| \nabla^{g_0} d_f(x, \cdot) \|_{g_0}(y) = e^{f(y)}$.

• $d_f(x, \cdot) \in W^{1,p}$. $p = nq > n$.

__Strong A_∞ weights:__ $w = e^{nf}$ strong A_∞ w.r.t. g_0 if $\exists \eta, \theta > 0$,

(I) $\forall x \in M$. $\forall r < \eta$. $\mu_f(B_{g_0}(x, 2r)) \le \theta \mu_f(B_{g_0}(x, r))$.

(II) $\forall x, y \in M$: $d_0(x, y) \le \eta$ ~~$\frac{d_f(x,y)^n}{\theta^n}$~~

$\Rightarrow \frac{d_f(x, y)^n}{\theta^n} \le \mu_f(B_{g_0}(x, d_0(x, y))) \le \theta^n d_f(x, y)^n$ ~~$\mu_f(B_{g_0}(x, d_0(x,y))) \le \theta^n d_f(x,y)^n$~~

(10)

Rmks. · Strong $A_\infty \Rightarrow A_\infty$ but not conversely.
· Geodesic balls of g_f and g_0 are comparable.
· g_f doubling w.r.t. B_{g_f} balls.

Partial extensions: Poincaré ineq., Euclidean isoperimetric, Sobolev.

Main Th^m (A-Cann-Tapie) $\exists \Lambda = \Lambda(g_0)$.

s.t. $\forall R_0$, $0 < R_0 \leq \operatorname{diam}(M, g_0)$,

$$\forall x \in M \int_{B(x,R_0)} |\operatorname{Scal} g_f|^{n/2} \, d\mu_f \leq \Lambda_0$$

$\Rightarrow e^{nf}$ strong A_∞ wrt. (M, θ) depending only on (R_0, g_0, Λ_0).

Corollary $(g_u)_u$ s.t. e^{nf_u} uniformly S-A_∞-weights and if $\exists v, V > 0$ $v \leq \operatorname{vol}(M, g_u) \leq V$.

$\Rightarrow \exists (g_{u_\ell})$ $(M, d_{u_\ell}) \to (M, d_\infty)$

and d_∞ is bi-Hölder to d_0.

Th^m (A-Cann-Tapie) $g = e^{2f} \delta_{\mathbb{R}^n}$ s.t.
· $\operatorname{vol}(\mathbb{R}^n, g) = +\infty$,
· $\int_{\mathbb{R}^n} |\operatorname{Scal}_g|^{n/2} \, d\mu_g < \infty$ &

$\Rightarrow e^{nf}$ str. A_∞ in $\mathbb{R}^n$ w.r.t. Eucl. dist.

5

<u>Q.</u> to <u>Clara:</u> Bi-Hölder lemma for.

Ahlfors regular condition. in Simons Ax.

→ A fund. sol^ns. for wave eqs. – Drago.

Bdry triples:

$A : \mathcal{D}(A) \subset H \to H$. symmetric.

$(\mathcal{H}, \gamma_0, \gamma_1)$ bdry triple of A^*.

① $(\gamma_0, \gamma_1) : \mathcal{D}(A^*) \to h \times h$ lin, bdd, surj.

② $\langle A^*\varphi, \psi\rangle - \langle \varphi, A^*\psi\rangle = \langle \gamma_1\varphi, \gamma_0\psi\rangle - \langle \gamma_0\varphi, \gamma_1\psi\rangle$.

⇝ Indices, characterise all s.a. ext of A.

Grubb 68: $A = \operatorname{div} A \nabla$.

Hyper trace $t_0 : H^s(\Omega) \to H^{s-\frac{1}{2}}$,

Bdry triple $(L^2, \cdot\, t_0, i_+ t_0)$.

Some maps.

→ Gives control of spectrum also.

C.f. Derkach & Malamud 1991.

Mflds with bdd geom + uniform ℓ-L cond

Nistor.

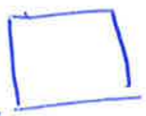
Motivation: Analysis on singular spaces. (BVPs).
- Noncpct mflds & QFT.

0) Ω = bdd C^∞ domain, $H^m(\Omega)$ Sobolev in L^2.
$H^1_0(\Omega)$ ← only need this

Classical result: $\Delta : H^{m+1}(\Omega) \cap H^1_0(\Omega) \to H^{m-1}(\Omega)$ is an isomorphism.

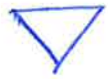
$\Rightarrow$ Δ = s.a. ~~for~~ Δ with $\mathrm{dom}(\Delta) = H^2(\Omega) \cap H^1_0(\Omega)$.

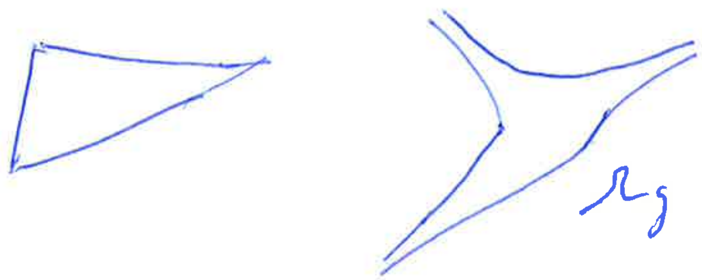
$\Rightarrow$ $\Delta u = f \in C^\infty \Rightarrow u \in C^\infty$ (reg).

Ω =  $f \in H^2(\Omega)$, if $u \in H^4$.

(0,0)

$\partial_x^2 u(0,0) = 0 = \partial_y^2 u(0,0) \Rightarrow \Delta u = 0$ (~~expl~~ contradiction of reg.)

Kondratiev '67: Any Ω = , $g = \frac{g_E}{r^2}$?
r = "dist to vertices".



Ω_g.

Ω_g satisfies Poincaré inequality, and

Pf of isom of $\Delta : H^{m+1} \cap H^1_0 \to H^{m-1}$.
requires Poincaré + Regularity.

Mflds with ~~bdry~~ bdd geom:

(I) $\|\nabla^k \mathrm{Riem}\| \le R$
(II) inj Rad > 0.
$\}$ for mfld w/o bdry.

Mfld w/ bddy ~~geom~~ + bdd geom:

$M \subset \hat{M}$ ~~ambient~~ mfld with bdd geom;
∂M sub mfld with bdd geom.

<u>Tech.</u> $\exists\, \mathcal{U}_\alpha$ = covering, $\mathcal{U}_\alpha = B_r(x_\alpha)$, $r \ll$ inj rad(M)

$\exists\, \varphi_\alpha$. p.o.u. $<(\mathcal{U}_\alpha)$.
$\|\nabla^k \varphi_\alpha\|_{C^0} \le C_k$.

<u>Def^n</u> Let $A \subset \partial M$, we say that (M, A) has finite width if $d(x, A)$ is bdd on M.

<u>Th^m</u> (Ammann - Grosse - V. N).
$A \subset \partial M$ open and closed, (M, A) finite width,

$$\int_M |u|^2 \, d\mathrm{vol} \le c \int |\nabla u|^2 \, d\mathrm{vol} + \int_A |u|^2 \, d\mathrm{vol}'$$

Corollary (M,A) finite width,

$$\Delta : H^1_D(M;E) \xrightarrow{\cong} H^1_D(M;E)^*$$

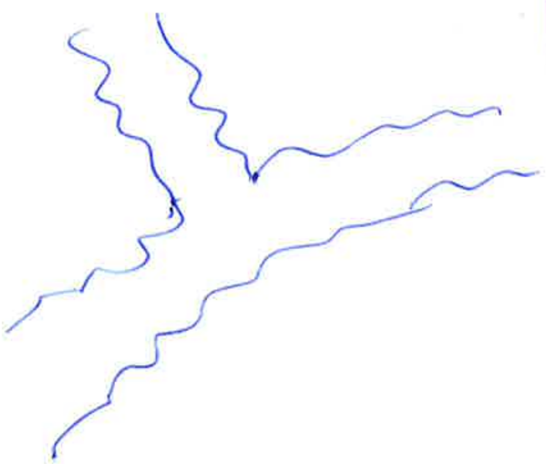
$$H^1_D = \{u : u|_A = 0\}$$

To get to Neumann bdy conds — full Neumann, need more regularity because need to define $\partial_\nu u$.

Thm $\Delta : \underbrace{H^{m+1}(M;E) \cap \{u|_A = 0,\ \partial_\nu u|_{A^c} = 0\}}_{} \longrightarrow H^{m-1}(M;E).$

As before, Δ s.a. ~~on~~ $\underline{\underline{H^2 \cap}} \{ \text{———} \}$

Examples.



← curve at ∞

mfld with bdry with bdd geom.

Δ not an isom $H^1_0 \to H^{-1}$.

$0 \in \sigma(\Delta)$.

~~But~~ But, regularity ok.

← curvature blows up

We have Poincaré, so $\Delta : H^1_0 \xrightarrow{\sim} H^{-1}$.

Regularity: Def$^{\underline{n}}$ (D,C) be a BVP,

D - 2nd order, C bdry conds of order j.

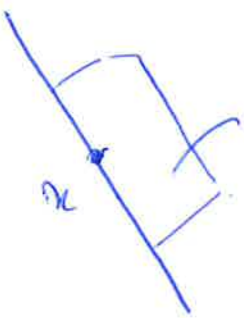
(D,C) satis. **Reg** if $\exists\, \tilde C > 0$.

$$\|u\|_{H^{k+1}} \le \tilde C\left(\|Du\|_{H^{k-1}} + \|Cu\|_{H^{k-j+1/2}} + \|u\|_{H^k}\right).$$

$u \in H^1$, Spt u cpct.

If $S = \{(D,C)\}$ and $\tilde C$ can be chosen indep. of $(D,C) \in S$, then S ~~[illegible]~~ satisfies a uniformly reg. est.

(D,c) on M.



U_x coord chart,

(D_x, C_x) on $B_r(0) \times [0,r) \subset T_x^+M \cong \mathbb{R}^n_+$. $x \in \partial M$.

Th$^{\underline{m}}$ (Grosse-V.N.) (D,C) sat. reg. iff

$S = \{ (D_x, C_x) ;\ x \in \partial M,\ \mathrm{dist}(x, \partial M) \ge \frac{r}{2} \}$

satisfies regularity condition.

Prop Assume $S \subset \overline{S}$ = cpt in $W^{\infty,\infty}$ and each $(D,C) \in \overline{S}$ sat. a reg. cond. Then S satisfies a <u>uniform reg cond.</u>

Lemma Assume D = uniformly strongly elliptic, ie $\operatorname{Re} \operatorname{sym}_D(x,\xi) > 0$.

C = Dirichlet or Neumann. (Can take diff on diff parts of bdry)

S satisfies uniformly ell. condition ($S = \{(D_x, C_x)\}$) of prop. Ie, $(D_n, C_n) \longrightarrow (D, C)$ in $W^{\infty,\infty}$; then (D,C) sat. reg.

<u>Shapiro - Lopatinsky</u>: $(D_x, C_x) \mapsto (D^{(0)}_x, C^{(0)}_x)$. minimal ops via freezing coefficients + keeping highest order terms.

Def (D,C) sat. uniform reg. cond. if $\exists\, c > 0, k$,

$$\|u\|_{H^{k+1}(\mathbb{R}^n_+)} \leq \tilde{C} \Big(\|D^{(0)}_x u\|_{H^{k-1}(\mathbb{R}^n_+)} + \| C^{(0)}_x u\|_{H^{k-j+\frac{1}{2}}} + \|u\|_{H^k(\mathbb{R}^n_+)} \Big).$$

Thm Uniform SL $\Leftrightarrow$ Reg.

$\sigma(\Delta)$ on forms - Nelia Charalambous.

General s.a. H.

$\sigma(H) = \sigma_{ess}(H) \cup \sigma_{pt}(H)$.

cluster points, infinite multiplicity.

Weyl Criterion: H d.d., s.a. on $\mathcal{H}$.

$\lambda \in \sigma(H)$ iff $\exists \{\varphi_j\}_j \subset \mathrm{dom}(H)$

(I) $\|\varphi_j\| = 1$. $\forall j$

(II) $\|(H-\lambda)\varphi_j\|_{L^2} \to \dots$

- $\sigma(k, \Delta, \mathbb{R}^n) = [0,\infty)$ on k-forms.

$k=0$: $\Delta \propto -\frac{\partial^2}{\partial r^2}$, $e^{i\sqrt{\lambda}r}\varphi(r)$, $\varphi(r)$ cutoff

- $\sigma(k, \Delta, \mathbb{H}^{n+1}) = [(n/2-k)^2, \infty)$ $k \le n/2$.

- $M = \mathbb{R} \times B^3$ B-flat 3-mfld. has vanishing 1st & 2nd Betti numbers (no Harmonic 1,2 forms).

$\sigma(1, \Delta, M) = \sigma_{ess}(1, M) = [0,\infty)$.

? $\sigma(2, \Delta, M) = [a,\infty)$ $a > 0$.

1st eigenvalue of B.

Noncpct flat mfld. $\mathbb{R}^n/\Gamma$, Γ subset of $\mathcal{A}(\mathbb{R}^n)$.

$$\mathbb{R}^n/\Gamma \neq \mathbb{R}^s \times B^{n-s}.$$

in general.

$$\sigma(k, \Delta, \mathbb{R}^n/\Gamma) = \sigma_{ess}(k, \Delta, \mathbb{R}^s \times B^{n-s}).$$

$B = \mathbb{R}^{n-s}/\Gamma_0$ cpct $\Gamma_0 \subset \Gamma$ subgroup.

B - not smooth but reflects structure @ ∞.

Th^m (C. - Lu 2007) $M = \mathbb{R}^n/\Gamma$.

$$\sigma(k, \Delta, \mathbb{R}^n/\Gamma) = \sigma_{ess}(k, \Delta, \mathbb{R}^n/\Gamma) = [a_n, \infty).$$

a_n either 0 or depends on $\left(\begin{matrix} \ell \leq k & k \text{ form} \\ & \text{harp of } B^{n-s} \end{matrix} \right)$.

Result of K. T. Sturm $\rightarrow$ asymptotically flat mfld, spectrum independent of p.

C - 2005. L^p independence for forms.

M: mfld with pole., strong decay cond.

Always. $\boxed{L^1 \text{ spec} \subset L^2 \text{ spec}.}$

So dimishing + go for L^p indep.

$\Rightarrow$ <u>Generalize Weyl Criteria</u>: (simplified).

- H d.d., s.a., non-neg. over $\mathcal{H}$.

(I) $\forall j \in \mathbb{N}$, $\|\varphi_j\| = 1$.

(II). $m = 1, 2$ $\forall j$, $|\langle (H+1)^{-m}\varphi_j, (H-\lambda)\varphi_j\rangle| \le \delta$.

Then $\exists$ const. $c(\lambda) > 0$ s.t.

$$\mathrm{dist}(\lambda, \sigma(H)) < c(\lambda)\, \delta^{\frac{1}{3}}.$$

(III) $\varphi_j \to 0$ weakly as $j \to \infty$ in $\mathcal{H}$ then.

the upper bd holds for σ_{ess}.

on the arxiv.

<u>Point</u>: $(\Delta + 1)^{-1}$: ——— $\to L^\infty$ so enough to

find <u>bdd</u> test functions.

~~Thm (C. Lu, 2018).~~

~~M complete, noncpct. Sps $\lambda > 0$ $\lambda \in \sigma_{ess}$~~

<u>Cts perturbation of ops.</u> $\langle \cdot, \cdot \rangle_0$, $\langle \cdot, \cdot \rangle_1$,

on $\mathcal{H}$, H_0, H_1 s.a., common core.

$\varepsilon > 0$, ε-close, $Q_0(u,u) \sim_\varepsilon Q_1(u,u)$

$(1-\varepsilon) \ldots (1+\varepsilon)$.

$\Rightarrow$ $(1+H_1)^{-1}$ and $(1+H_0)^{-1}$ ε-close.

Prop. $A \geq 0$ fixed, $\lambda \in \sigma(H_1) \cap [0, A]$

$d(\sigma(H_0), \lambda) \leq \underline{(A+1)^{-} \varepsilon^{\frac{1}{3}}}$.

g $*$

Applica. g_0, g_1, $D_i = d + d_i^*$, $\Delta_i = D_i^2$.

but Δ_i **not** ~~ste~~ close. but $d_0 d_0^*$ and $d_1 d_1^*$ **are**.

Use hodge $*$ to relate $\sigma(k, \Delta_i)$ to

$\sigma(k, d_i d_i^*) \cup \sigma(k, d_i^* d)$.

Rmk when M has Ric assump. non, neg, bottom of ess spec. captured by sequence of lim geo balls.

Local flexibility I. — C. Bär. 25/09/2018.

$X \to V$ fibre, $J^k X \to V$ k^{m} Jet bundle.

coordinate independent way to talk about taylor expansion.

Feynman prop on curved spacetime - Gérard.

(I) Minkowski $(\mathbb{R}^{1,d}, \eta)$ $P = \partial_t^2 - \Delta_x + m^2$, $m > 0$.

$x = (t, x)$, $\xi = (\tau, k)$, 4 distinguished inverses:

$I \in \{ret, adv, F, \bar{F}\}$.

Fourier multipliers:

ret/adv: $\hat{G}_{ret/adv}(\tau, k) = \left((\tau \mp i0)^2 - (k^2 + m^2) \right)^{-1}$.

Fey/anti-Fey: $\hat{G}_{F/\bar{F}}(\tau, k) = \left(\tau^2 - (k^2 + m^2) \pm i0 \right)^{-1}$.

$i0 = \lim_{\varepsilon \to 0^+} i\varepsilon$, $C_\pm$ future/past light cone.

Use of $G_{ret/adv}$: $u = G_{ret/adv}\, v$, $v \in C_0^\infty(M)$.

Solves: $Pu = v$, $\mathrm{supp}\, u \subset \mathrm{supp}\, v \subset C_\pm$.

$G = G_{ret} - G_{adv}$. Solves Cauchy prob.

use of G_F: related to quantisation of KG eq^{n}

$\varphi(t,x)$, $\varphi^*(t,x)$ state (vacuum state)

$w(\varphi(t,x), \varphi^*(t',x')) = w(t-t', x-x')$.

$G_F = G_{ret} + i^{-1} w_2$.

(II) Curved Spacetime

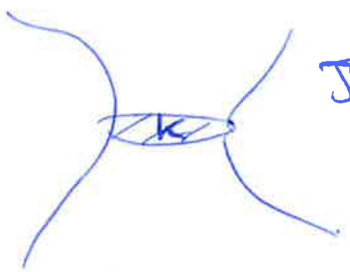
(M,g) Lorentzian mfld, globally hyp ($\exists\, \Sigma$ Cauchy surf.).

$P = -\Box_g + m(x)$, $m \in C^\infty(M;\mathbb{R})$.

~~Is~~ $P = P^*$? approach $\langle u, Pv\rangle_M = \int_M \bar{u} v \, dvol_g$.

Leray (1952) $\exists!$ $G_{ret/adv} : C_0^\infty(M) \to C^\infty(M)$.

s.t. $u = G_{ret/adv} v$ soln $Pu = v$, supp $u \subset J^\pm(\text{supp } v)$.



$J^\pm(K)$.

$G_F / G_{\bar F}$? (1) when does it deserve "Feynman" name.

(2) $\exists$, $!$, "covariance under causal embedding"

Thm: D-Hörmander $\tilde{G} : \mathcal{E}'(M) \to \mathcal{D}'(M)$ Parametrix for P.

$PG - I$, $GP - I$ smoothing ops.

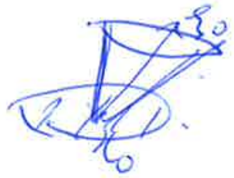
$u \in \mathcal{D}'(\mathbb{R}^n)$ WF set. $\quad WF\,u \subset \mathbb{R}^n \times \mathbb{R}^{n*} \setminus \{0\}$.

$(x_0, \xi_0) \notin WF\,u \quad \exists \quad \chi \in C_0^\infty(\mathbb{R}^n) \quad \chi(x_0) = 1$.

$\exists\, \Gamma$ cone taking ξ_0 s.t.

$|\widehat{\chi u}(\xi)| \le C_N (1 + |\xi|)^{-N} \quad \forall N, \xi \in \Gamma$.

$(x_0, \xi_0) = $ ct. (picture $\overset{\xi}{\underset{x}{\nearrow}}$ wrong)



$K : C_0^\infty(M) \to D'(M) \quad k(x, \tilde{x})$ its kernel.

$WF(K) = \{ (x,\xi), (x',\xi'), \dots \}$ conic subset of $T^*M \times T^*M \setminus 0$.

$WF(K)' = \{ (x,\xi), (x',-\xi') : (x,\xi), (x',\xi') \in WF(K) \}$.

conic relation th? ..

$K = I \;;\; WF(I) = \Delta$ diagonal.

$p(x,\xi) = \xi\, g^{-1}(x)\, \xi. \quad N = \{ (x,\xi) : p(x,\xi) = 0 \} = N^+ \cup N^-$.
$(\tau = \pm |k|)$.

$\xi \rhd 0$ iff $\xi \cdot v > 0 \quad \forall v \in C_+(x)$

$(x,\xi) = X \qquad X \sim X'$ iff $X, X' \in N$ and X, X' lie on same Hamiltonian curve.

$C_{ret} = \{ (X, X') : X \sim X'$ and $x > x' \}$.

$C_{adv} = \{ (X, X') : X \sim X'$ and $x < x' \}$.

$C_F = \{ (X, X') : X \sim X' \quad x > x'$ if $\xi \rhd 0$; $x < x'$ if $\xi \lhd 0 . \}$.

Thm (DH) $I =$ ret/adv, F, $\exists \tilde{G}_I$ unique mod smooth. s.t. $WF(\tilde{G}_I)' = \Delta \cup C_I$.

diagonal.

Role of Feynman inv: $\varphi(x)$, $\varphi^*(x)$, $P\varphi(x) = P\varphi^*(x) = 0$.

$$G = G_{ret} - G_{adv}$$

$$[\varphi(x), \varphi^*(x')] = i\, G(x, x')\, I.$$

'w' state $\quad w(\varphi(x), \varphi^*(x')) = \Lambda^+(x, x')$ $\quad \Lambda^+ - \Lambda^- = iG$

$w(\varphi^*(x), \varphi(x')) = \Lambda^-(x, x')$ $\quad \Lambda^\pm \geq 0$.

Hadamard Cond: $WF(\Lambda^\pm) \subset N^\pm \times N^\pm$, then

$G_F = G_{ret} + i\Lambda^+$ Feynman inv.

Vasy's idea: use Fredholm theory!

For wave.

Asymp flat $\quad M = \mathbb{R}^{1+d}$, g glob hyp ($\Leftrightarrow$ no trapped geo)

$g - \eta \in O(\langle x\rangle^{-\delta})$

$\exists \tilde{t}$ time funct ($\nabla \tilde{t}$ time like). $\tilde{t} - t \in O(\langle x\rangle^{1-\varepsilon})$.

$\Rightarrow \exists \tilde{t}$ Cauchy time funct. $t = \tilde{t}$ outside compact set.

Can split $\quad g = -c(t,x)\,dt^2 + h(t,x)\,dx^2$.

Hilbert Spaces for Fredholmness

$$Y^m = \langle t \rangle^{-\gamma} L^2(\mathbb{R}, H^m(\mathbb{R}^d)), \quad \gamma > \frac{1}{2}.$$

$$\subset L^1(\mathbb{R}; H^m(\mathbb{R}^d)).$$

$$X^m = \{ u \in C^0(\mathbb{R}, H^{m+1}) \cap C^1(\mathbb{R}; H^m) : Pu \in Y^m \}$$

$$\|u\|^2_{X^m} := \|Pu\|^2_{Y^m} + \|P_0 u\|^2_{\mathcal{E}^m}$$

$$\begin{cases} P_0 u = \begin{pmatrix} u(0,x) \\ \frac{1}{i} \partial_t u(0,x) \end{pmatrix}. \\ \mathcal{E}^m = H^{m+1} \oplus H^m. \end{cases}$$

$(X^m, \|\cdot\|^2_{X^m})$ Hilbert.

Boundary Conditions.

$$P_t u = \begin{pmatrix} u(t,\cdot) \\ \frac{1}{i} \partial_t u(t,\cdot) \end{pmatrix}$$

$$\pi^{\pm} = \frac{1}{2} \begin{pmatrix} 1 & \pm(-\Delta_x + m^2)^{\frac{1}{2}} \\ \pm(-\Delta_x + m^2)^{\frac{1}{2}} & 1 \end{pmatrix}.$$

$$X^m_F = \{ u \in X^m : \lim_{t \to \mp\infty} \pi^{\pm} P_t u = 0 \text{ in } \mathcal{E}^m \} \subset X^m \text{ closed}.$$

$$X^m_{ret} = \{ u \in X^m : \lim_{t \to -\infty} P_t u = 0 \}.$$

<u>Thm</u> [GW, 2008] $P: X^m_F \to Y^m$ is invertible; its inverse G_F is called the Feynman prop. of P. $WF(G_F)' = \Delta \cup C_F$.

<u>Pf</u> Uses ~~[illegible]~~ "positive commutator method".

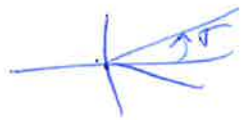
Perturbative Functional Calculus – Phillipp Hanns.

<u>Question</u> $A \xmapsto{C^\infty/C^\omega} f(A)$.. ?

<u>Motivation</u> $A = \Delta_g$ on (M,g) perturbation of g, $f = \zeta^\alpha$, i.e., fractional power of Δ_g.

<u>Context</u> $t \mapsto A(t) \rightsquigarrow \lambda_i(t)$ Lipschitz.
So diff not smooth, $e_i(t)$ discts.
$z \mapsto A(z) \rightsquigarrow \lambda_i(z), e_i(z)$ C^ω.

<u>Setting</u> $A: \mathcal{D}(A) \subset X \to X$ X: $\mathbb{C}$-Bspace.

- A d.d., invertible.
- A sectorial: $\sigma \in (0,\pi)$ angle.  $S_\sigma \subset \mathbb{C}$ open.

$\mathrm{Spec}(A) \subset S_\sigma$, $\omega > \sigma \Rightarrow$ Res. bd.

6

Usual f.c. via Cauchy integral formula
for f holomorphic on S_φ $\sup_{\lambda \in S_\varphi} |\lambda^r f(\lambda)| < \infty$.

$$\boxed{\dot{X}_r := \overline{\mathrm{dom}(A^r)}^{\|A_r \cdot\|_X}.}$$

<u>Lemma</u>. $\mathcal{H}(\dot{X}_1, \dot{X}_0) \ni B \longrightarrow f(B) \in \mathcal{L}(\dot{X}_0, \dot{X}_r)$.
is well defined and holomorphic map.

<u>Thm</u> <u>If</u> $\sup_{f \in H^\infty(S_\varphi)\setminus\{0\}} \frac{\|f(A)\|}{\|f\|_\infty} < \infty$. $(H^\infty - \text{f.c.})$.

$\Rightarrow$. $\mathcal{L}(\dot{X}_1, \dot{X}_0) \cap \mathcal{H}(\dot{X}_{1+\delta}, \dot{X}_\delta) \ni B \to f(A) \in \mathcal{H}(\dot{X}_0, \dot{X}_r)$.

(Sectoriality open condition allows the sector to get slightly larger. Similarly, Bdd of H^∞ f.c. is open condition.)

<u>Pf</u>. Kalton, Kunstmann, Weis;
commutative calculus; $A(z)$ hol. curve,
$B(z) := f(A(z))$.
Move to function close to ~~worst~~ Elements of $B(z)$.

$$H(z) = \frac{1}{2\pi i} \oint f(\lambda) \cdot \frac{R_\lambda(A(r)) - R_\lambda(A(z))}{z - r} \, d\lambda. \quad (*)$$

But lower reg. $H(z) = B'(t) \in \mathcal{L}(\dot{X}_0, \dot{X}_{<r})$.

~~Replace by multiplying replace~~

Replace $f(\lambda)$ with $f(\lambda)\lambda^{r-z} e^{(z-r)^2}$, $A(z)$ by $A(\varphi(z))$.

Fix $x \in \dot{X}_0$:

$$H_x : \begin{array}{l} \{0 \le \operatorname{Re} z \le 1\} \xrightarrow{C_b} \dot{X}_0 \\ \{0 < \operatorname{Re} z < 1\} \xrightarrow{H^\infty} \dot{X}_0 \\ \{\operatorname{Re} z = 1\} \xrightarrow{C_b} \dot{X}_1 \end{array} \quad B'(r) \in [\dot{X}_0, \dot{X}_1]_r = \dot{X}_r.$$

B_x $\mathbb{C}$-diff w/values in $\dot{X}_{<r}$.
$B'x$ is locally bdd w/values in $\dot{X}_r$ $\Rightarrow$ B_x $\mathbb{C}$-diff ∎

Parallel Spinors on Riem mflds – B. Ammann.

M closed, $\tilde{M}$ spin, $n = \dim \geq 3$.

$$\mu(M,[g]) = \inf_{\tilde g \in [g]} \mathcal{E}(g), \qquad \mathcal{E}(g) = \frac{\int_M \mathrm{Scal}_{\tilde g}\, d\mathrm{vol}_{\tilde g}}{\mathrm{vol}(M,\tilde g)^{\frac{n-2}{n}}}.$$

Yamabe const.

$\mathcal{M}(M) = \{g \text{ Riem metrics}\}$.

$\cup$

$\mathcal{M}_0(M) = \{g \in \mathcal{M}(M) : \mathrm{Ric}(g) = 0\}$.

$\cup$

$\mathcal{M}_{\mathrm{inf\text{-}stab}}(M) = \{g \in \mathcal{M}_0(M) : \mathrm{spec}\, \Delta_E \in [0,\infty)\}$.

$\Updownarrow$

$\mathrm{Hess}\, \mu \leq 0$.

$$\Delta_E h = \nabla^*\nabla h - 2\mathring{R}h, \quad g(R(\cdot,\cdot)\cdot,\cdot)\,h(\cdot,\cdot).$$

$h \in \Gamma(T^*M \odot T^*M)$.

$\cup$

$\nexists\, g_t : g_0 = g,\ \frac{d}{dt}\Big|_{t=0} \mathrm{Scal}_{g_t} > 0$.

$\mathcal{M}_{\mathrm{stab}}(M) = \mathcal{M}_0(M) \setminus \overline{\{\mathrm{scal} > 0\}}$. μ has a local max.

$\cup$

$\mathcal{M}_{\parallel}(M) = \{g : \exists \varphi \in \Gamma(\Sigma^g \tilde{M}),\ \nabla\varphi = 0,\ \varphi \neq 0\}$.

"structured Ricci-flat mets."

Open problem: $\mathcal{M}_{\parallel}(M) \overset{?}{=} \mathcal{M}_0(M)$.

Thm A. 1) $\mathcal{M}_{\parallel}(M)$ open & closed in $\mathcal{M}_0(M)$.

2) $\mathcal{M}_0(M) \longrightarrow \mathbb{N}_0$

$$s(g) = \dim \Gamma_{\parallel}(\Sigma^g M) = \{\varphi \in \Gamma(\Sigma M) : \nabla\varphi = 0\}.$$
$$\tilde{s}(g) \quad \text{———} \quad \text{—}(\Sigma^g \tilde{M}) \quad \text{———}$$

are locally const.

3). $\mathcal{M}_{\parallel}(M) \xrightarrow[\mathrm{Hol}_0]{\mathrm{Hol}} O(T_xM)/\mathrm{conj}$. locally const.

$$\mathrm{Hol}(M,g)_x = \{P^\gamma \in O(T_xM) : \gamma = \text{loop at } x \in M\} \subset O(T_xM).$$
$$\mathrm{Hol}_0(M,g)_x = \{ \text{———} \quad \gamma \text{ contractible}\} = \mathrm{Hol}(\tilde{M}, g).$$

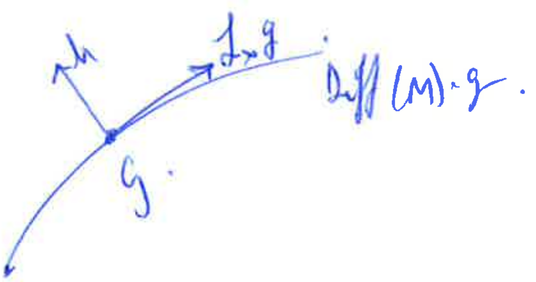
4) $\mathcal{M}od := \mathcal{M}_0(M)/\mathrm{Diff}_0(M)$.

$\mathrm{Diff}_0(M)$ identity component of $\mathrm{Diff}(M)$.

C^∞ mfd.

$h \in \Gamma(T^*M \otimes T^*M)$.

$$h \in TT_g \iff \begin{cases} \operatorname{tr} h = 0 & \text{traceless.} \\ \operatorname{div}^g h = 0. & (\iff h \perp \mathrm{Diff}(M)\cdot g). \end{cases}$$



$$TT \cap \ker(d\,\mathrm{Ric}).$$
$$\parallel$$
$$TT \cap \ker \Delta_E.$$

$$\mathrm{Hess}\left(g \mapsto \int_M \mathrm{scal}_g \, dv_g\right) = -\tfrac{1}{2}\Delta_E \quad \text{on } TT_g.$$

Q. Do Ricci flat metrics preserve product structure?

Need a parallel spinor $\varphi \in \Gamma_{\parallel}(\Sigma\tilde{M})$.

$\Rightarrow \exists$ par spin on N_i

$$(\hat{M}, g) \cong \underbrace{(N_1, h_1) \times \cdots \times (N_k, h_n) \times \mathbb{R}^{\ell}/\Gamma}.$$

because (M,g) RF, closed conn. $\longrightarrow$ finite cover $\searrow$ $\Gamma = \pi_1(N_i)$ cpt, Ricci-flat, irred. hol., dim ≥ 2.

$$\mathrm{Hol}(N_i) \subset \begin{cases} SU(n) \\ Sp(n) \\ G_2,\ Spin(7). \end{cases}$$

h_i inf-stable.

Kröncke: (Q_i, h_i) closed inf-stable Ricci flat.

$$\ker \Delta_E^{Q_1 \times Q_2} = \ker \Delta_E^{Q_1} \oplus \ker \Delta_E^{Q_2} \oplus \underbrace{\Gamma_{\parallel}(T^*Q_1) \odot \Gamma_{\parallel}(T^*Q_2)}_{\text{trivial to trace symm. prod.}}.$$

Ricci curv. + flow via Brownian motion

26/09/2018.

Thalmaier

$$d\mathrm{vol}_g = \left(1 - \tfrac{1}{6} \mathrm{Ric}_{ij}(p) x^i x^j + o(|x|^3)\ d\mathrm{vol}_{\mathrm{Eucl}}\right).$$

$$\mathrm{vol}(B(p,r)) = \left(1 - \mathrm{Scal}(p)\, c r^2 + o(r^4)\right) \mathrm{vol}_{\mathbb{R}^n}(B^{\mathbb{R}^n}(p,r))$$

Heat flow $L = \Delta + Z$ $\quad Z$ v. field.

$$\partial_t u = L u \qquad M \times \mathbb{R}_+.$$

Gradient est: $|\nabla u|$ or $\frac{|\nabla u|}{u}$.

Harnack est. $u(x,s)$ and $u(y,t)$.

Explicit formula $(\nabla u)(\cdot,t)_x$ in terms of L-diffusion.

$X_t = X^x_t$. $t < \zeta(x)$.

L-diff. X_t have a comm prop. with d.

$Ric^Z = Ric - \nabla Z$.

$Ric^Z_{//_t} := //_t^{-1} \circ Ric^Z_{X_t} \circ //_t$. $\in End\, T_xM$.

back to x. $T_xM \to T_{X_t}M$ move to X_t.

$//_t$: done via stochastic analysis.

Probabilistic formulas. for minimal sol^n.

Semigroup, Derivative and Bismut.

Also. explicit Log-Sob., etc.

Idea: characterize $k_1 \le Ric^Z \le k_2$, in terms of P_t.

- Natural ext: $k_1(x) \le Ric^Z_x \le k_2(x)$.
- mflds with bdry.
- Evolve under geometric flow.

Trace formulae for spect. spect. spectrum. –

A. Spectrum.

$\Box = \partial_t^2 - \Delta_h$ (Σ, h) Riem manifold.

$\sigma(\Delta_h) = \{\lambda_j^2\}$. $e^{\pm i\lambda_j t}\phi_j$ solves wave eq.

$\mathrm{tr}\, V(t) = \sum_j e^{i\lambda_j t}$. wave trace.

sing spt $(\mathrm{tr} V) \subset \mathrm{Lsp}(\Sigma, h)$.

$\sum \langle e^{i\lambda t}, f(t)\rangle = \sum \hat{f}(\lambda_j)$.

Singularity at zero:

$$a_{0,-d}(t+i0)^{-d} + a_{0,-d+1}(t+i0)^{-d+1} + \cdots$$

Isolated closed geo's contribute to

$$a_{L,-1}(t-L+i0)^{-1} + a_{L,0} \log(t-(L+i0)) + \cdots$$

Principal wave inv. $t = L$.

$$a_{L,-1} = \sum_{\gamma : L_\gamma = L} \frac{i^{m_\gamma} L_\gamma^{\#}}{|\det(1-P_\gamma)|^{\frac{1}{2}}}.$$

Contains a lot more info than heat trace.

$\mathbb{R}\times\Sigma$. $-dt^2+h$. special case of stationary spacetime.

Stand. stat. sp-ti (M^n, g). w/ cpct Cauchy surface Σ has

$$g = -N dt^2 + h_{jk}(dx^j+\beta^j dt)\otimes(dx^k+\beta^k dt).$$

$N : \Sigma \to \mathbb{R}$ lapse func.

(M,g) stat. $\Leftrightarrow$ $\exists$ complete timelike Killing v.f. Z;

locally $-N^2dt^2 + w\otimes dt + dt\otimes w + h$, $Z = \frac{\partial}{\partial t}$.

cross term $w\otimes dt$ can be made to vanish $(Z)^\perp$ integrable.

(M,g) spatially cpt, stat., globally hyperbolic sp-ti.

$\Rightarrow$ $(M,g) \cong (\mathbb{R}\times\Sigma, -Ndt^2 + h_{jk}(dx^j+\beta^j dt)(dx^k+\beta^k dt))$.

Wave eq^n in form $\partial_t^2 - \partial_n\partial_t + \Delta_h \cdots$

PP-wave spacetime $-H(y)\,dt^2 + 2\,dx\,dy + dy^2$.

on $\mathbb{R}_t\times\mathbb{R}_x\times\mathbb{R}_y^{n-2}$.

$\to$ Lense-Thirring metric : like Schwarzschild but mixed terms : Gravity Probe B.

$\Rightarrow$ Frame dragging $\to$ presence of mixed terms.

Sep. variables: $\lambda^2 + 2i\lambda\beta + \Delta_{\Sigma} + W$.

"Quadratic operator pencil" ← Russian School.

(M,g) - stationary (but not ..?) timelike killing Z.

$\square = \square_g + V$. invariant under killing flow $ZV = 0$.

ker $\square$. space of sol^{n}. w/ Cauchy data $H^1(\Sigma) \oplus L^2(\Sigma)$.

→ Gives Hilbert Space topology, but in gen. not possible to find invariant inner prod.

⇝ Eigenvalues.

Spectrum of iZ on ker $\square$: discrete in $\mathbb{R}$.

$$\operatorname{tr}\left(e^{Zt}\big|_{\ker\square}\right) = \sum_j m_j e^{i\lambda_j t}.$$

↑ algebraic multi.

⇝ Weyl law.

3 Gradient estimates for H.K.

(M,g) cplte nonexp.

$$e^{-t\Delta} f(x) = \int_M p_t(x,y) f(y)\, dy$$

$p_t > 0$, sym., smooth.

(15)

$$p_t(x,y) \le \frac{C}{V(x,\sqrt{t})} e^{-\frac{d^2(x,y)}{ct}} = \mathrm{Gauss}(x,y).$$

$\forall t>0, \forall x,y \in M.$

Li–Yau 80's. → characterized in terms of functional inequ. (Sobolev, isoperi).

• $|\nabla p_t(x,y)| \le \frac{1}{\sqrt{t}} \mathrm{Gauss}(x,y)$. (GUE).

if $Ric \ge 0$.

Q. prove (GUE) more generally.

Q. Is it true (GUE) depends only on geometry at ∞.

Another approach: H.K. on 1-forms.

$\vec{\Delta} = dd^* + d^*d$ (1-forms). has ~~commutation~~ relation with d.

$de^{-t\Delta} = e^{-t\vec{\Delta}} d$.

$\vec{\Delta} = \nabla^*\nabla + Ric$. v. valued Schrödinger.

$\vec{P}_t(x,y)$. = h.k. of $\vec{\Delta}$

$\|\vec{p}_t(x,y)\| \le \mathrm{Gauss}(x,y)$

$(\vec{ME}) + (ME) \Rightarrow (GUE)$.

Devyver recent: Green function, Ric₋ small at ∞ etc 2016.