~~I) Homology~~ AE I Lecture day C. Bär 14/03/2018

I) Homology (simplicial)

Remarks:

$$\cdots \longrightarrow C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \longrightarrow \cdots$$

v. spaces.

It is a <u>complex</u> if $\partial_k \circ \partial_{k+1} = 0$.

$\iff \operatorname{Im}(\partial_{k+1}) \subset \ker(\partial_k)$.

<u>Exact</u> sequence if $\operatorname{Im} \partial_{k+1} = \ker \partial_k$.

For a complex, we can define $H_k(C_\cdot, \partial_\cdot) = \ker(\partial_k) / \operatorname{Im}(\partial_{k+1})$

<u>Ex</u>. $0 \xrightarrow[(A)]{} C_1 \xrightarrow{\partial} C_0 \xrightarrow[(B)]{} 0$. <u>Exact</u>.

$\underbrace{0 = \operatorname{Im} A = \ker \partial}_{\partial \text{ injective}}$. $\underbrace{\operatorname{Im} \partial = \ker B = C_0}_{\partial \text{ surjective}}$.

Similarly,

$$\cdots \longrightarrow C^{k-1} \xrightarrow{d_{k-1}} C^{k} \xrightarrow{d_k} C^{k+1} \longrightarrow \cdots$$

$$d_k \circ d_{k+1} = 0 \quad \forall k .$$

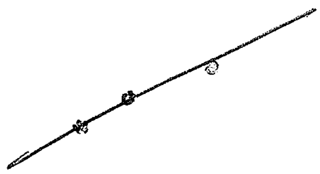
$$H^k(C^{\cdot}, d_{\cdot}) := \ker(d_k)/\mathrm{Im}(d_{k-1}).$$

"Cohomology"

<u>Def</u>

Def$^{\underline{n}}$. $v_0, \dots, v_k \in \mathbb{R}^N$ are in <u>general position</u> iff they are <u>not</u> contained in affine subspace of $\dim = k-1$.

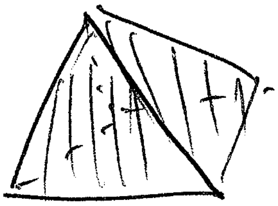
in genl pos.



not in genl pos.

If $v_0, \dots, v_k$ are in genl pos, then

$$|v_0 \cdots v_k| := \left\{ \sum_{j=0}^{k} a_j v_j : a_j \geq 0, \ \sum_{j=0}^{k} a_j = 1 \right\}$$



is called a <u>k-simplex</u>.

A finite (Euclidean) <u>simplicial complex</u> is a finite set K of k-simplices s.t.

① $\forall \sigma \in K$, all faces of σ also in K.

$\{w_0, \dots, w_l\} \subseteq \{v_0, \dots, v_n\}$. then $|w_0 \dots w_l|$ is called a <u>face</u> of $|v_0 \dots v_n|$

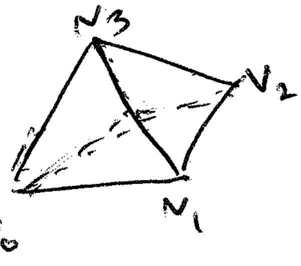
② $\forall \sigma, \tau \in K$: $\sigma \cap \tau$ is a common face.

allowed   not allowed.

$$|K| = \bigcup_{\sigma \in K} \sigma \subset \mathbb{R}^n$$ <u>geometric realisation</u>

<u>Rmk</u> K contains the combinatorial data, it is $\subset \mathcal{P}(\mathbb{R}^n)$. But $|K|$ is geometric, it is $\subset \mathbb{R}^n$.

<u>Ex.</u> tetrahedron.



14 faces, pts, edges, facets

$$C_k(K,\mathbb{R}) := \Big\{ \underbrace{\sum_j a_j \sigma_j}_{\text{finite linear combination}} : \sigma_j \ k\text{-simplex} \Big\}, \ a_j \in \mathbb{R} \Big\}.$$

$$\dim C_k(K,\mathbb{R}) := \#\{k\text{-simplexes in } K\}. \ (<\infty).$$

To define linear map ∂_k, need to have ordering.

$(v_0,\dots,v_k) \sim (v_{s(0)},\dots,v_{s(k)}) \iff s$ even permutation.

orientation of $|v_0,\dots,v_k|$ is equiv. class of ordering of vertices.

<u>Oriented simplex</u> is simplex equipped with orientation; $\langle v_0,\dots,v_k\rangle$.

$$\partial_k \langle v_0,\dots,v_k\rangle = \sum_{j=0}^{k} (-1)^j \langle v_0,\dots,\hat{v}_j,\dots,v_k\rangle.$$

Lemma. $\partial_n \circ \partial_{n+1} = 0$.

$\rightsquigarrow$ Simplicial Homology $H_n(K,\mathbb{R})$.

How do we see this? ∂_n gives us the border, faces of dim less, and sum up with the sign $(-1)^i$. When we identify two we [illegible], we are identifying with fixed index.

Ex. Tetrahedron. $\partial_2\langle v_0,v_1,v_2\rangle = \langle v_1,v_2\rangle$.

$$\partial_2\langle v_0,v_1,v_2\rangle = \overset{\sigma_1}{\langle v_1,v_2\rangle} - \overset{\sigma_2}{\langle v_0,v_2\rangle} + \overset{\sigma_3}{\langle v_0,v_1\rangle}.$$

$$\partial_2\langle v_0,v_2,v_3\rangle = \overset{\sigma_4}{\langle v_2,v_3\rangle} - \overset{\sigma_5}{\langle v_0,v_3\rangle} + \overset{\sigma_2}{\langle v_0,v_2\rangle}.$$

$$\partial_2\langle v_1,v_2,v_3\rangle = \overset{\sigma_4}{\langle v_2,v_3\rangle} - \overset{\sigma_6}{\langle v_1,v_3\rangle} + \overset{\sigma_1}{\langle v_1,v_2\rangle}.$$

$$\partial_2\langle v_0,v_1,v_3\rangle = \underset{\sigma_6}{\langle v_1,v_3\rangle} - \underset{\sigma_5}{\langle v_0,v_3\rangle} + \underset{\sigma_3}{\langle v_0,v_1\rangle}.$$

$$\partial_2 - \begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \Rightarrow \operatorname{rk}(\partial_2) = 3.$$

$$\dim H_2(K,\mathbb{R}) = \underbrace{\dim \ker(\partial_2)}_{1} - \underbrace{\dim \operatorname{im}(\partial_3)}_{0}. = 1.$$

$$\boxed{b_n = \dim H_n(K,\mathbb{R}) \text{ — Betti numbers}}$$

2) (de Rham) - Cohomology.

M - mfld, $C^k = \mathcal{A}^k(M) = \{$smooth diff k-forms$\}$.

locally: $\omega = \sum\limits_{1 \le i_1 < \dots < i_k \le n} \omega_{i_1 \dots i_k} \, dx^{i_1} \wedge \dots \wedge dx^{i_k}$.

$d: C^k \to C^{k+1}$, $d\omega = \sum\limits_{1 \le i_1 < \dots < i_k \le n} d\omega_{i_1 \dots i_k} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$

<u>Lemma</u>. $d \circ d = 0$.

$\rightsquigarrow$ de Rham cohomology $H^k(M) = \dfrac{\text{closed } k\text{-forms}}{\{d\eta : \eta \in \mathcal{A}^{k-1}(M)\}}$

<u>Poincaré Lemma</u>: $M \overset{\text{diffeo}}{\cong} B_1(0)$; (open ball), then $\forall \omega \in \mathcal{A}^k$

$d\omega = 0 \Leftrightarrow \exists \eta \in \mathcal{A}^{k-1}$; $\omega = d\eta$.

In other words, $H^k(M) = 0$.

$f: M \to N$, smooth maps,

$\rightsquigarrow$ $f^*: \mathcal{A}^k(N) \to \mathcal{A}^k(M)$.

$f^*(\omega_{i_1 \dots i_k} \, dx^{i_1} \wedge \dots \wedge dx^{i_k}) = \omega_{i_1 \dots i_k} \circ f \; df^{i_1} \wedge \dots \wedge df^{i_k}$

$f^i = x^i \circ f$.

hence

$$\begin{tikzcd} \Omega^k(N) \arrow[r, "f^*"] \arrow[d, "d"'] & \Omega^k(M) \arrow[d, "d"] \\ \Omega^{k+1}(N) \arrow[r, "f^*"] & \Omega^{k+1}(M) \end{tikzcd}$$

commutes.

$\rightsquigarrow$ $f^*\{\text{closed } k \text{ forms on } N\} \subset \{\text{closed } k \text{ forms on } M\}$.

$f^*(\text{exact}) \subset \{\text{exact on } M\}$.

$f^* : H^k(N) \to H^k(M)$.

$[\omega] \mapsto [f^*\omega]$. well defined.

Functorial properties: $M \xrightarrow{f} N \xrightarrow{g} P$.

$(g \circ f)^* = f^* \circ g^*$, $\mathrm{id}_M^* = \mathrm{id}_{H^k(M)}$.

$f, g : M \to N$, homotopic. $(f \simeq g)$.

eg $\exists F : M \times [0,1] \to N$ smooth s.t.

$F(0,x) = f(x)$, $F(1,x) = g(x)$.

Fact: $f \simeq g \Rightarrow f^* = g^*$.

<u>Def^{n}</u> M, N are homotopy equiv.

if $\exists\, f: M \to N$, $g: N \to M$ s.t.

$f\circ g \simeq id_N$, $g\circ f \simeq id_M$.

$\Rightarrow (g^* \circ f^*) = (f\circ g)^* = id_N^* = id_{H^k(N)}$.

$\Rightarrow f^*: H^k(N) \to H^k(M)$. <u>isom</u>.

Weaker than even diffeo!

$M = \mathbb{R}^n$, $N = \{pt\}$. ~~$f: \mathbb{R}^n \to \{pt\}$~~.

$f: \mathbb{R}^n \to \{pt\}$, $g: pt \to 0$.

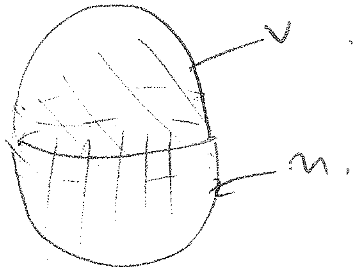
$\Rightarrow H^k(\mathbb{R}^n) \cong H^k(pt) = 0$.

<u>Mayer-Vietoris sequence.</u> $M = U \cup V$; $U, V \subset M$ open.

$$0 \to H^0(M) \to H^0(U) \oplus H^0(V) \to H^0(U\cap V) \to$$
$$\to H^1(M) \to H^1(U) \oplus H^1(V) \to H^1(U \cap V) \to \cdots$$

is <u>exact</u>.

Ex. $M = S^n$.



$U \simeq V \simeq$ open ball.

$$H^k(U) = H^k(V) = 0, \quad \forall k \geq 1.$$

$$\Rightarrow \quad 0 \to H^k(U \cap V) \to H^{k+1}(S^n) \to 0 \quad \underline{\text{exact}}.$$

$$\Rightarrow \quad H^{k+1}(S^n) \cong H^k(U \cap V).$$

But $U \cap V = S^{n-1} \times (-\varepsilon, \varepsilon) \simeq S^{n-1}$.

homotopy equiv.

$$\Rightarrow \quad H^{k+1}(S^n) \cong H^k(S^{n-1}).$$

Induction $\Rightarrow$ $H^k(S^n) = b^k(S^n) = \begin{cases} 1, & k=0,\ k=n. \\ 0, & \text{otherwise}. \end{cases}$

3 Homology vs. Cohomology.

M cpct mfld, M can be triangulated

$\exists$ simplicial cplx K and homeom.

$$h: |K| \to M$$

s.t. $h|_\sigma$ is smooth, $\forall \sigma \in K$.

$\Omega^k(M) \times C_k(K, \mathbb{R}) \longrightarrow \mathbb{R}$.

$$(\omega, \sum_j a_j \sigma_j) \longmapsto \sum_j a_j \int_{\sigma_j} h^* \omega .$$

If $\omega = d\eta$, on,

$$\int_\sigma h^* d\eta = \int_\sigma d h^* \eta \overset{\text{Stokes's Thm.}}{=} \int_{\partial\sigma} h^* \eta \Longrightarrow 0 \quad \text{if } \sum_j a_j \sigma_j \text{ has no boundary.}$$

If $\sigma = \partial\tau$

$$(\omega, \partial\tau) \mapsto \int_{\partial\tau} h^* \omega = \int_\tau d h^* \omega = \int_\tau h^* d\omega$$

$$= 0 \quad \text{if } d\omega = 0.$$

$\Rightarrow$ Bilinear map $H^k(M) \times H_k(K, \mathbb{R}) \to \mathbb{R}$ is well defined.

<u>de Rham Thm</u>: Bilinear map is non-degenerate.

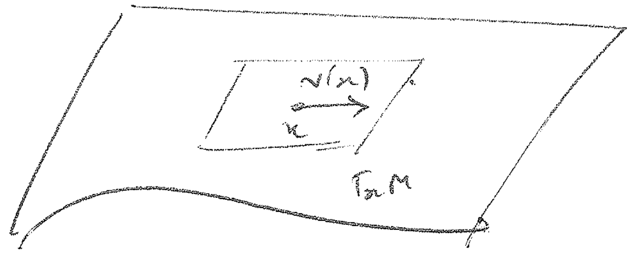
Equ. $H^k(M) \to H_k(K, \mathbb{R})^*$.

$$[\omega] \mapsto \left([\textstyle\sum_j a_j \sigma_j] \longmapsto \sum a_j \int_{\sigma_j} h^* \omega \right).$$

is an isomorphism.

<u>Corollary</u> $\underbrace{b^k(M)}_{\text{not clear at all that this, a priori, is finite dim!}} = b_k(K)$.

(II) V. Bundles.



$V(x) \in T_xM$. depends on x!

Def^n. Let $\pi : E \to M$ be smooth surjection s.t. each $E_x = \pi^{-1}(x)$ comes equipped with $\mathbb{K}$-v. space structure, ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).

(π, E, M) is called $\mathbb{K}$-v.b. if $\forall x \in M$, $\exists x \in U \subset M$ open, smooth maps $s_1, \dots, s_r : U \to E$ s.t. $(s_1(y), \dots, s_r(y))$ is a basis of E_y $\forall y \in U$.

"bundle of vector spaces over mfld".

M: base space, E: total space, r: rank,
$s = (s_1, \dots, s_r)$ local frame.

Def^n. A map $s : M \to E$ is called a **section** if $s(x) \in E_x$ $\forall x \in M \Rightarrow \pi \circ s = id_M$.

Ex. $E = TM = \bigcup_{x \in M} T_xM$, sections = v. fields.

E	TM	T^*M	$\Lambda^k T^*M$	$M \times \mathbb{K}^r$	Spinor bundles
$\mathbb{K}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{K}$	$\mathbb{C}$.
Sections	v-fields	1-forms	diff k-forms	$\mathbb{K}^r$ valued fns	spin fields.

Algebraic constructions: (I) $E_1, E_2 \rightsquigarrow E_1 \oplus E_2$.

$$(E_1 \oplus E_2)_x = E_{1,x} \oplus E_{2x}.$$

(II) $E \rightsquigarrow E^*$

(III) $E \rightsquigarrow \Lambda^k E$.

(IV). $E_1 \rightsquigarrow E_1 \otimes E_2$., $(E_1 \otimes E_2)_x = E_{1,x} \otimes E_{2,x}$.

$S = (s_1, \dots, s_r)$. local frame, on $U \subset M$. s (local)-sectn on M,

$$s(x) = \sum_{j=1}^{r} a^j(x)\, s_j(x). \quad , \quad s \leftrightarrow (a^1, \dots, a^r) \text{ locally}.$$

Second frm $\tilde{S} = (\tilde{s}_1, \dots, \tilde{s}_r)$. on $\tilde{U}$.

on $\tilde{U} \cap U$, $\quad s_j(x) = \sum_{i=1}^{r} \tau^i_j(x)\, \tilde{s}_i(x)$.

$$\rightsquigarrow (\tau^i_j) : U \cap \tilde{U} \to GL(r, \mathbb{K}).$$

transition functions.

$$s = \sum_j a^j s_j = \sum_{i,j} a^j \tau^i_j \tilde{s}_i.$$

$$(a^1, \dots, a^r) \longmapsto \underbrace{(\tau^i_j)}_{\text{matrix}} (a^1, \dots, a^r).$$

If each E_x carries a scalar product, the vector bundle is called Riemann ($\mathbb{K} = \mathbb{R}$) or Hermitian ($\mathbb{K} = \mathbb{C}$). Then, can choose frames orthogonal.

Then $T \in \begin{cases} O(r), & \mathbb{K} = \mathbb{R}. \\ U(r), & \mathbb{K} = \mathbb{C}. \end{cases}$

<u>Q.</u> How to differentiate sections? Need a connection.

<u>Def^n.</u> A connection is a linear map

$$\nabla : C^\infty(M, E) \longrightarrow C^\infty(M, T^*M \otimes E).$$

Satisfying Leibniz rule.

$$\nabla(fs) = df \otimes s + f \nabla s.$$

$$E_p \ni (\nabla s)_p(X) =: (\nabla_X s)(p).$$
$$X \in T_pM.$$

(LR)
$$(\nabla_X (fs))(p) = (\partial_X f\, s)(p) + f(p)\, (\nabla_X s)(p)$$

local description: $S = (s_1, \dots, s_r)$ local frame

$$\nabla_X s_i = \sum_{j=1}^{r} \underbrace{w(X)^j_i}_{\text{exists uniquely.}} s_j(p).$$

$\Rightarrow w^j_i \in \Omega^1(U)$, $(w^j_i) \in \mathrm{Mat}(r, \Omega^1(U))$.

In $\tilde{\beta} = (s_1, \ldots, s_r)$

$$\nabla_x \tilde{s}_j =: \sum_i \tilde{\omega}(x)^i_j \tilde{s}_i = \sum \tilde{\omega}^i_j \tau^k_i s_k .$$

$$\nabla_x (\tau^i_j s_i) =: \partial_x \tau^i_j s_i + \tau^i_j \nabla_x s_i = (\partial_x \tau^k_j + \tau^i_j \omega(x)^k_i) s_k .$$

$$\tilde{\omega}(x) =: (\partial_x \tau)\tau^{-1} + \tau \omega(x) \tau^{-1} .$$

<u>Def$^{\underline{n}}$</u>. E Riem or Hermitian. ∇ is called <u>metric</u>

if $\partial_x \langle s, t \rangle = \langle \nabla_x s, t \rangle + \langle s, \nabla_x t \rangle$.

$$= \cancel{\langle \omega(x)^k_i}$$

$$0 = \partial_x \underbrace{\langle s_i, s_j \rangle}_{\delta_{ij}} = \langle \nabla_x s_i, s_j \rangle + \langle s_i, \nabla_x s_j \rangle .$$

$$= \langle \omega(x)^k_i s_k, s_j \rangle + \langle s_i, \omega(x)^k_j s_k \rangle .$$

$$= \omega(x)^i_j + \overline{\omega(x)^j_i} .$$

$$\Rightarrow \omega(x) \in \begin{cases} \mathfrak{o}(r), & \mathbb{K} = \mathbb{R}. \quad (\text{skew adj}) \\ \mathfrak{u}(r), & \mathbb{K} = \mathbb{C}. \quad (\end{cases}$$

E Hermitial. Line bundle (rank 1),

$\tau \in U(1)$, $\omega(x) \in \mathfrak{u}(1) = i\mathbb{R}$.

$$\tilde{\omega} = d\tau \cdot \tau^{-1} + \omega .$$

~~R(X,Y) s =~~

<u>Def^n</u> Curvature tensor.

$$R(X,Y)s =. \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s.$$

Locally, $R(X,Y)s_i = \sum_j \Omega(X,Y)^j_i \; S_j$. $\Omega^j_i \in \mathcal{A}^2(M)$.

In metric case: $\Omega \in \begin{cases} \mathfrak{o}(r), & \mathbb{K} \in \mathbb{R} \\ \mathfrak{u}(r), & \mathbb{K} \in \mathbb{C} \end{cases}$.

<u>Lemma.</u> (a) $\Omega = d\omega + \omega \wedge \omega$,

(b) Bianchi Id. $d\Omega = \Omega \wedge \omega - \omega \wedge \Omega$.

<u>Pf</u> (a) $\Rightarrow$ (b). $d\Omega = dd\omega + d\omega \wedge \omega - \omega \wedge d\omega$.

$$= (\Omega - \omega\wedge\omega)\wedge\omega - \omega \wedge (\Omega - \omega\wedge\omega).$$

$$= \Omega\wedge\omega - \omega\wedge\Omega.$$

<u>Ex.</u> $\omega = iA$, $A \in \mathcal{A}^1$; $\Omega = iF$, $F \in \mathcal{A}^2$; $F = dA$.

$\tilde{A} = d\varphi + A$ (gauge transform).

$M =$ 4d Lorentz; local coords t, x^1, x^2, x^3:

$$F = E \wedge dt + B, \quad E = E_1 dx^1 + E_2 dx^2 + E_3 dx^3.$$

$$B =. B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2$$

(E_1, E_2, E_3) – electric field

(B_1, B_2, B_3) – magnetic field.

$$0 = dF =. \frac{\partial E_1}{\partial x_2} dx^2 \wedge dx^1 \wedge dt + \frac{\partial E_1}{\partial x_3} dx^3 \wedge dx^1 \wedge dt + \ldots E_2 + \ldots E_3$$

$$+ \frac{\partial B_1}{\partial t} dt \wedge dx^2 \wedge dx^3 + \frac{\partial B_1}{\partial x_1} dx^1 \wedge dx^2 \wedge dx^3 + \ldots B_2 + \ldots B_3$$

(*)

$$\Leftrightarrow \begin{cases} \dfrac{\partial B}{\partial x_1} + \dfrac{\partial B}{\partial x_2} + \dfrac{\partial B}{\partial x_3} = 0 \\ -\dfrac{\partial E_1}{\partial x^2} + \dfrac{\partial E_2}{\partial x^1} + \dfrac{\partial B_3}{\partial t} = 0. \end{cases}$$

$\Leftrightarrow$ $\operatorname{div} B = 0$, (Gauss' law) $\qquad \dfrac{\partial B}{\partial t} + \operatorname{curl} E = 0$. (Faraday's law).

So, encoded in the geometric setup.
line bundle E plays no role, $\rightarrow$ only curvature of ∇ which is in $\mathcal{E}^2$.

Change of frame for curvature:

$$\tilde{\Omega} = \tau \wedge \Omega \wedge \tau^{-1}.$$

(III) Characteristic Classes.

1. Chern classes.

<u>Def^{n}</u> A polynomial function $P: \mathrm{Mat}(r, \mathbb{C}) \to \mathbb{C}$.

is called <u>invariant</u> if

$$P(T \cdot x \cdot T^{-1}) = P(x).$$

$\forall x \in \mathrm{Mat}(r, \mathbb{C})$, $\forall T \in GL(n, \mathbb{C})$.

Ex. $P =$ det, $P =$ tr.

Given $\mathbb{C}$-v.b. with connection ∇, Ω curvature matrix.

w.r.t. frame over U, $P(\Omega) \in A^{\text{even}}(U)$.

$\tilde{\Omega}$ w.r.t. frame,

$$P(\tilde{\Omega}) = P(\tau \wedge \Omega \wedge \tau^{-1}) = P(\Omega).$$

$P(\Omega)$ yields globally well-def $\in A^{\text{even}}(M)$.

<u>Ex.</u> $\mathrm{tr}_\nabla = \Omega^1_1 + \dots + \Omega^r_r \in A^2(M)$.

$$\det_\nabla = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \cdot \Omega_1^{\sigma(1)} \wedge \dots \wedge \Omega_r^{\sigma(r)} \in A^{2r}(M).$$

<u>Lemma</u> $dP_\nabla = 0. \rightsquigarrow [P_\nabla] \in H^{\text{even}}(M)$.

<u>Th^{m}</u>. $[P_\nabla]$ is independent of choice of ∇

Pf. WLOG, P homogenous of degree m, polarize:

$$P : \underbrace{\mathrm{Mat}(n,\mathbb{C}) \times \dots \times \mathrm{Mat}(n,\mathbb{C})}_{m} \to \mathbb{C}.$$

multilin & symmetric.

$\nabla, \tilde{\nabla}$ connections, ${}^t\nabla = t\tilde{\nabla} + (1-t)\nabla$. $t \in [0,1]$.

$$\frac{d}{dt} P(\Omega_t, \dots, \Omega_t) = m \cdot dP(\tilde{\omega} - \omega, \Omega_t, \dots, \Omega_t).$$

Integrate:

$$P(\tilde{\Omega}) - P(\Omega) = m \cdot d \int_0^1 P(\tilde{\omega} - \omega, \Omega_t, \dots, \Omega_t) = dQ(\nabla, \tilde{\nabla})$$

$\rightsquigarrow \quad P(E) := [P_\nabla] \in H^{\mathrm{even}}(M).$

Def^n. $P(X) = \det(\mathbb{1}_r + \frac{1}{2\pi i} X)$.

$\rightsquigarrow \quad P(E) =: c(E)$ total Chern-class.

$P(X) = \det(1 + \frac{1}{2\pi i} X)$, homogenous part:

$$P_0(X) = 1, \quad P_1(X) = \frac{\mathrm{tr}(X)}{2\pi i}, \dots, \quad P_r(X) = \frac{\det(X)}{(2\pi i)^r}.$$

$$c(E) = 1 + \underbrace{\frac{1}{2\pi i}[\mathrm{tr}_\nabla]}_{= c_1(E) \in H^2} + \dots + \underbrace{\frac{1}{(2\pi i)^r}[\det{}_\nabla]}_{= c_r(E) \in H^{2r}}.$$

Thm (1). $c(E_1 \oplus E_2) = c(E_1) \cdot c(E_2)$.

2. $c_k(E^*) = (-1)^k c_k(E)$.

3. $E_1 \cong E_2 \Rightarrow c(E_1) = c(E_2)$.

Pf 1. Choose connections ∇^i on E_i, $\nabla = \nabla^1 \oplus \nabla^2$,

$s^{(1)}, s^{(2)}$ local frames on E_1, E_2.

$\rightsquigarrow s = (s^{(1)}, s^{(2)})$ local frame for $E_1 \oplus E_2$.

$$\Rightarrow \Omega = \left(\begin{array}{c|c} \Omega^{(1)} & \\ \hline & \Omega^{(2)} \end{array}\right) \Rightarrow P(\Omega) = \det\left(1 + \frac{1}{2\pi i}\Omega\right)$$

$$= \det\left(\begin{array}{c|c} 1+\frac{1}{2\pi i}\Omega_1 & \\ \hline & 1+\frac{1}{2\pi i}\Omega_2 \end{array}\right)$$

$$= \det\left(1+\frac{1}{2\pi i}\Omega_1\right) \cdot \det\left(1+\frac{1}{2\pi i}\Omega_2\right)$$

$$= P(\Omega_1) \cdot P(\Omega_2).$$

Lemma $c(E) \in H^{even}(M, \mathbb{R}) \subset H^{even}(M, \mathbb{C})$.

Pf. choose Herm. prod. and metric connection,

$\Omega(X, Y) \in \mathfrak{u}(r)$.

$$\det\left(1 + \frac{1}{2\pi i}\Omega\right) = \det\left(1 - \frac{1}{2\pi i}\overline{\Omega}\right) = \det\left(1 - \frac{1}{2\pi i}\Omega^*\right)$$

$$= \det\left(1 + \frac{1}{2\pi i}\Omega\right).$$

$\Rightarrow \det\left(1 + \frac{1}{2\pi i}\Omega\right) \in \mathbb{R}$. form.

(This is why we have $\frac{1}{i}$ in 2nd part, always get $\mathbb{R}$).

Rmk. $c(E)$ is integral: If σ is a smooth ~~k-simplex in M~~ simplicial chain with $\partial\sigma = 0$, with $\mathbb{Z}$-coefficients, then

$$\int_\sigma c(E) \in \mathbb{Z}.$$

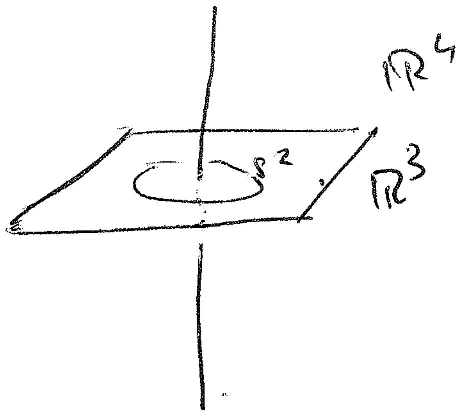
This is the reason for the $\frac{1}{2\pi}$ in the normalisation.

Cor. If E is trivial, then $c(E) = \underline{1} \in H^0(M)$.

Pf WLOG, $E = M \times \mathbb{C}^r$, choose $\nabla = \partial \rightsquigarrow R = 0$.
$\Rightarrow \Omega = 0$ w.r.t. all frames.
$\Rightarrow \det\left(1 + \frac{1}{2\pi i}\Omega\right) = \underline{1}$.

Back to EM example.

Choose $M = \mathbb{R}^4 \setminus (\mathbb{R} \times \{0\})$



$M \overset{\text{homo.}}{\simeq} \mathbb{R}^3 \setminus \{0\} \simeq S^2$.

$\Rightarrow H^2(M) \cong H^2(S^2) \cong \mathbb{R}$.

Fact: $\forall k \in \mathbb{Z}$, $\exists$ line bundle $E \to S^2$ s.t.

$$\int_{S^2} c_1(E) = k.$$

Extend E to $\mathbb{R}^3 \setminus \{0\}$ radially, and then to M constant in time $\rightsquigarrow \mathcal{E} \to M$.

$$\frac{1}{2\pi i}\int_{S^2} \Omega = \int_{S^2} c_1(\mathcal{E}) = \int_{S^2} c_1(E) = k.$$

$\frac{1}{2\pi}\int_{S^2}$ Field ~~strength~~!

4

we have $\int_{S^2} c_1(\mathcal{E}) = \frac{1}{2\pi i}\int_{S^2} \Omega$.

because it's a line bundle. Since $c_1(\mathcal{E}) = [tr\,\Omega]$

and

$$\frac{1}{2\pi i}\int_{S^2}\Omega = \frac{1}{2\pi}\int_{S^2} F = \frac{1}{2\pi}\int_{S^2} B \leftarrow \text{Field strength}.$$

~~Total~~ this is indep of connection

"topological charge" (Dirac's magnetic monopole).

2) Additive & multiplicative chern classes.

$$g = g_0 + g_1 x^1 + g_2 x^2 + \dots \in \mathbb{C}[[x]]. \quad (\text{formal}).$$

$$\Lambda_g : H^{2j}(M) \to H^{2j}(M), \quad (-1)^{j+1} j\, g_j \cdot id$$

$c \in H^{even}(M)$, $c = 1 + c_1 + c_2 + \dots$.

Formally $\log(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} \mp \dots$,

$\log(c)$ well defined

$$\underset{\uparrow\, H^0}{r \cdot g_0} + \Lambda_g(\log(c)).$$

$(r+\tilde r) + \Lambda_g \log(c\tilde c) = r g_0 + \Lambda_g \log(c)$

$= r \qquad + \tilde r g_0 + \Lambda_g(\log(\tilde c))$.

If $c = c(E)$ Chern class of E and r its rank,

then $r g_0 + \Lambda_g(\log(c(E)))$. <u>additive</u> class for g.

$E_1 \oplus E_2 \Rightarrow \mathrm{add}(E_1 \oplus E_2) = \mathrm{add}(E_1) + \mathrm{add}(E_2)$.

Special case: $E = L_1 \oplus \cdots \oplus L_r$, L_j - line bundles

$\mathrm{add}(E) = \mathrm{add}(L_1) + \cdots + \mathrm{add}(L_r)$

$$\begin{aligned}\mathrm{add}(L) &= g_0 + \Lambda(\log(1+x)).\\ &= g_0 + \Lambda_g\Big(\sum_{j\geq 1} (-1)^{j+1}\frac{x^j}{j}\Big)\\ &= g_0 + \sum_{j\geq 1} g_j x^j = g(x).\end{aligned}$$

$\Rightarrow \mathrm{add}(E) = g(x_1) + \cdots + g(x_r)$, $x_j = c_1(L_j)$.

<u>Remark</u>. In the literature, splitting case as above is only considered, and for more general bundles, need a "Splitting principle".

<u>Ex</u>. $g(x) = e^x \rightsquigarrow ch(E)$ <u>Chern character</u>.

<u>Sps</u> $c = (1+x_1)\cdots(1+x_r) = 1 + (x_1 + \cdots + x_r) + \sigma_2(x_1, \ldots, x_r)$.

$ch = e^{x_1} + \cdots + e^{x_r} = r + \underbrace{(x_1 + \cdots + x_r)}_{c_1} + \underbrace{\frac{x_1^2 + \cdots + x_r^2}{2}}_{\frac{c_1^2}{2} - c_2}$.

$f(x) = 1 + f_1 x + f_2 x^2 + \dots \in \mathbb{C}[[x]]$.

$f_c(E) = \exp(\Lambda_{\log f} \log(c(E)))$.

<u>multiplicative class</u> for f.

Special case, $E = L_1 \oplus \dots \oplus L_r$.

$f_c(E_1 \oplus E_2) = f_c(E_1) \cdot f_c(E_2)$.

$\Rightarrow \quad f_c(E) = f_c(L_1) \dots f_c(L_r)$.

$f_c(L) = \exp(\Lambda_{\log f} \log(1+x)) = \exp(\log(f(x))) = f(x)$

$\Rightarrow \quad f_c(E) = f(x_1) \dots f(x_r)$, where $x_i = c_1(L_i)$.

<u>Ex 1</u> $\quad f(x) = 1+x$.

~~$f_c(L)$~~ $c(E) = (1+x_1) \dots (1+x_r)$,

$f_c(E) = c(E)$.

<u>Ex 2</u> $\quad f(x) = \dfrac{x}{1-e^{-x}} = 1 + \dfrac{x}{2} + \dfrac{x^2}{12} + \dots$

$\rightsquigarrow \quad f_c(E) = Td(E)$ Todd class.

<u>Thm. (Riemann - Roch - Hirzebruch)</u>: M $\mathbb{C}$ mfld, let $E \to M$ be holomorphic bundle. Then

$$\sum_q (-1)^q h^{0,q}(M,E) = \int_M Td(TM) \cdot ch(E).$$

$\uparrow$ Hodge numbers.

Pontryagin classes. $E \to M$ $\mathbb{R}$-v.b.

complexifying : $E \otimes \mathbb{C} \longrightarrow M$ $\mathbb{C}$-v.b.

$E \cong E^* \Rightarrow (E \otimes \mathbb{C})^* \cong E \otimes \mathbb{C}$, ..

$$c_k(E \otimes \mathbb{C}) = c_k((E \otimes \mathbb{C})^*) = (-1)^k c_k(E \otimes \mathbb{C}).$$

$\Rightarrow$ if k is odd, $c_k(E \otimes \mathbb{C}) = 0$.

Def^n $p_j(E) := (-1)^j c_{2j}(E \otimes \mathbb{C}) \in H^{4j}(M)$.

Th^m (I) $p(E_1 \oplus E_2) = p(E_1)\, p(E_2)$

(II) Not important. ($p_k(E^*) = p_k(E)$ since $E \cong E^*$)

(III) $E_1 \cong E_2 \Rightarrow p(E_1) = p(E_2)$.

$\Lambda_g = (-1)^{j+1} \cdot j \cdot y_j$ on H^{4j} . only diff

Ex. $\hat{a}(x) = \dfrac{\sqrt{x}/2}{\sinh(\sqrt{x}/2)} = 1 - \dfrac{x}{24} + \dfrac{7x^2}{5760} + \cdots$

$\rightsquigarrow$ mult. class for $\hat{a}$ and p_i.

$\hat{A}(M) \in H^{4\bullet}(M)$.

Ex. M spin, $D : S_L^{tw} \to S_R$. $S_L = \$ \otimes E$.

ind(D) = dim ker(D) − dim. coker(D).

AS. Th^m: ind$(D) = \int_M \hat{A}(M)\, ch(E)$.