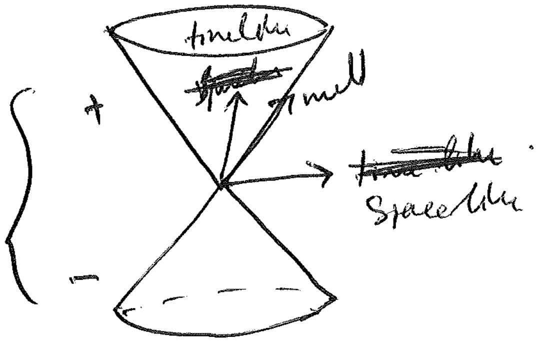
Some old and new issues about mass in math. relativity. 15/01/2016.

Spacetime: $\mathbb{R}^{3+1}$ w/ signature $(-,+,+,+)$ — γ.

In each tangent space $\exists$ light cone.



$\gamma(x,x)<0$ timelike.
$\gamma(x,x)=0$ null.
$\gamma(x,x)>0$ spacelike.

Metric γ solves: $G^{\gamma} := Ric^{\gamma} - \frac{1}{2} Sc^{\gamma} - \Lambda^{\gamma} = 8\pi T$.

T encodes all other interactions than gravitation.

Examples: ① $\mathbb{R}^{n,1}$ $\quad -dt^2 + \sum_{i=1}^{n} (dx_i)^2$.

② Anti de Sitter. $AdS^{n,1} \simeq \mathbb{R}^{n+1}$.

$$\gamma = -\cosh^2 r\, dt^2 + dr^2 + \sinh^2 r\, g_{S^{n-1}}.$$

~~All~~ Isolated systems: All interactions decay to zero as you go to ∞.

Problem: no consistent way to measure distances.

①

the "unphysical" decomposition: $N = M \times (a,b)$ locally.

Def^n AF: $\exists \varphi : M \setminus K \to \mathbb{R}^n \setminus B$ (chart at ∞),
and $\tau > 0$ s.t.

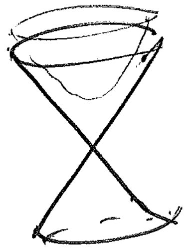
$$g_{ij} - \delta_{ij} \in O(|x|^{-\tau}), \quad \partial_k g_{ij} \in O(|x|^{-\tau-1}).$$
$$\partial_k \partial_l g_{ij} \in O(|x|^{-\tau-2}).$$

If $\exists$ 2nd ff, $\quad k_{ij} \in O(|x|^{-\tau-1})$, ; $\partial_a k_{ij} \in O(|x|^{-\tau-2})$.

Caveat: condition stated in a chart, but geometric by result in 1989. (Bartnik - K - N).

Asymptotically Hyperbolic: replace $\mathbb{R}^n$ by $\mathbb{H}^n$ and
δ_{ij} by $g^{\mathbb{H}^n}$, $|x|^{-k}$ by $e^{-\alpha r}$.

Other model: $\underbrace{k - g^{\mathbb{H}^n}}_{} \in O(e^{-\alpha r})$.
approaches $g^{\mathbb{H}^n}$, rather than 0 as in 2nd ff.



Ex. 1916: $N = \mathbb{R} \times \mathbb{R}^3 \setminus B(0,2m)$.

$$\gamma = -\left(1 - \frac{2m}{s}\right) dt^2 + \frac{ds^2}{1 - \frac{2m}{s}} + s^2 g_{S^2}$$

(Really not whole Schwarzschild metric. just exterior region).

- Static sol^n ! $-V(x)^2 dt^2 + g(x)$ timelike killing vector
- $\Lambda = 0$
- Each slice AF and totally geodesic $k = 0$.
- $r = m/2$ is the <u>horizon</u> of the black hole.

Schwarzschild - AdS : $N = \mathbb{R} \times \mathbb{R}^3 \setminus B(0, s_0)$.

~~Sch~~

$$\gamma = -\left(1 - \frac{2m}{s} + s^2\right) dt^2 + \frac{ds^2}{1 - \frac{2m}{s} + s^2} + s^2 g_{S^2}.$$

Solves Einstein $\Lambda < 0$, each slice AH, $k = 0$.

<u>Energy of isolated systems:</u> $\int_{t_0}^{t_1} \int_M (\mathrm{Scal}^\gamma - 2\Lambda)\, d\mathrm{vol}^\gamma$.

Lagrangian for Einstein eq^{n}s.

~~B~~ Perform formally a Legendre transform near AF or AH.

<u>Problem:</u> $N = M \times (a, b)$ unphysical. One identifies this with the flow of an arbitrary transverse vector field : $T = N\nu + V$.

Hamiltonian: $H(T, g, h) = H_M + H_\infty = \int_M (\dots) + \lim_{r\to\infty} \int_{S_r} (\dots)$

T is arbitrary, but term doesn't matter, all the ~~problem~~ ~~is in~~ geometry is in $\lim_{r\to\infty} \int_{S_r} (\dots)$ term :

Only bdry term is <u>dynamically relevant</u>.

Asymptotic conditions: @ ∞, v.f.d T. must preserve AF/AH.

$$T \to_\infty T_\infty \quad (\text{infinitesimal isom. of } AdS^{n+1} \text{ or } \mathbb{R}^{n+1}).$$

$H(T; g, k)$ linear in T.

<u>Def^{n}</u>. (M^n, g) AF. $n \geq 3$. Mass of g:

$$m(g) = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} (\partial_i g_{ij} - \partial_j g_{ii}) \nu_r^j \, d\mathrm{vol}_{S_r}$$

S_r crd sphere in chart at ∞.

(M^n, g, k) AF initial data:

$$p^i(g,k) = \frac{1}{(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} (k_{ij} - k_{mm} g_{ij}) \nu_r^j \, dS_r$$

Energy momentum: (M, g, k) $(p^0 = m, p^1, \dots, p^n)$

"<u>Boosts</u>": $c^i(g)$ centre of mass.

Infinitesimal spacelike rot: $\Omega^{ij}(g,k)$. ang. momentum.

$$\begin{tikzcd}
 & (M \setminus K, g, k) \arrow[dl, "\Phi_1"'] \arrow[dr, "\Phi_2"] & \\
(\mathbb{R}^n \setminus B, \delta) \arrow[rr, "\Phi"'] & & (\mathbb{R}^n \setminus B, \delta)
\end{tikzcd}$$

$g_i = (\Phi_i)^* g$.

Th^{m} (Chruściel, <u>Bartnik</u>). $\tau > \frac{n-2}{2}$. Scal^g integrable, $m(g)$ exists & independent of choice of chart.

~~Chart~~ Fact: change of chart at ∞ asymptotic to Eucl. Isom. $\Phi = A + o(1)$.

Chruściel 1985: $\tau > \frac{n-2}{2}$, Scal^g, $|k|^2$, ∇k integrable, $P(g,k)$ exists, ~~...~~ & ~~...~~ equivariant under change of Charts at ∞ $P(\Phi^* g, \Phi^* k) = A^{-1} P(g,k)$.

The AH case: expressed ~~as~~ in terms of Lie alg ~~SO~~ $(n,2)$.

B. Michel 2011 (student of Marc).

All masses come from single functional $\Phi_V(g,k)$.

Assumes a "common cancellation" of Bartnik's.

Open Question to finish: 2012 higher order mass, can it be obtained similar to B. Michel's analysis?

Marc Herzlich - lec 2. 16/01/2017.

- Remarks for last time dominant energy.
- Under natural conditions, gravitation should be non-neg.

DEC. (N, γ) satisfies (DEC) along a Riem slice M if $(G^\gamma - \Lambda\gamma)(e_0, \cdot)$ is timelike, or null or positively oriented.

Riem translation on (M, g, k),

$$J = 2(\mathrm{div}^g k + d\,\mathrm{tr}_g k), \quad \mu = \mathrm{Scal}^g - |k|^2 + (\mathrm{tr}_g k)^2 - 2\Lambda.$$

$$(\mathrm{DEC}) \iff \mu \geq |J|_g.$$

$k=0$: $\mathrm{Scal}^g \geq 0$ in AF, $\mathrm{Scal}^g + n(n-1) \geq 0$ in AH.

Conjecture: (M, g, k) sat. DEC, then energy-momentum should be positive unless (M, g, k) can be isometrically embedded into model spacetime. (where energy should vanish).

Issues: - positive? what's that?
- does DEC $\Rightarrow$ non-neg.
- equality case ~~embed~~: non-neg + embed $\Rightarrow 0$..

AF mass: true is +ve!

E-M +ve: $(m, p^1, \dots, p^n)$ $m > \left(\sum_{i=1}^n (p^i)^2\right)$

Mass AH: same as $\mathbb{R}^{n+1}$.

Full energy-mom is in $so(n,2)$, ↓ Naive. to positive means .
positive w.r.t. to killing metric of the Lie algebra.

~~Naive idea~~ :
Isometry group of $\mathbb{H}^n$ is $O_+(n,1)$ and EM is an orbit of this gp in $so(n,2)$.

$$\begin{tikzcd}
 & M \setminus K \arrow[dl, "\Phi_2"'] \arrow[dr, "\Phi_1"] & \\
\mathbb{H}^n \setminus \overline{B} \arrow[rr] & & \mathbb{H}^n \setminus \overline{B}
\end{tikzcd}$$

$\Phi = \Phi_2 \circ \Phi_1^{-1} : A + o(1)$, $A \in O_+(n,1)$.

(M,g,h) (Schoen-Yau 1978, Witten 81, Bartnik 86) positive mass theorem.
AF , $Sc^g \geq 0$.
① $3 \leq n \leq 7$ or M spin $\Rightarrow m(g) \geq 0$.
② $m(g) = 0 \Leftrightarrow (M,g)$ is isometric to $(\mathbb{R}^n, \delta)$.

AF satisfies DEC : ① $\Rightarrow m \geq |p|$.
② $n=3$ or M spin, $m = |p| \Leftrightarrow (M,g,h)$ embeds to isometric to (~~Mink~~) Minkowski.

Open q: non-spin equality $4 \leq n < 7$.

AH: (M^n, g) AH spin $Sc_g \geq -n(n-1)$. mass is either timelike, fut oriented or zero and in this case $(M,g) \cong (\mathbb{H}^n, g_0)$.

- Satisfying DEC instead of $R_g \geq -n(n-1)$, then EM satisfies same. non-neg. and $= 0 \Rightarrow (M,g,k) \cong (\mathbb{H}^n, g_0, 0)$.

Open Q's: · mass in non-spin ($n=3$ by Andersson-Galloway 2008).
· Right notion of +ve for EM
· Equality case.

Ideas of Pf: . Assume mass neg.

① $\exists$ new mfd with modern AF/AH end with neg mass.

②. Explicit barrier estimates $\Rightarrow \exists$ complete stable minimal or CMC hypersurfaces.

③ . $n=3$ use Gauss-Bonnet to get contra. $n \geq 4$, repeat argument on hypersurface (decrease dimension).

Drawbacks: needs to have a well understood standard model ~ sensitive to dimension (reg. of hypersurfaces)

AF: standard model Schwarzschild.

AH: Schwarzschild-AdS, $(m, 0, 0 \ldots, 0)$. and m can be negative here.

Ideas of Pf (spin case). $\Sigma \longrightarrow M$ spin bundle,

$(*)$ $D^*D\sigma = \nabla^*\nabla\sigma + R\sigma$.
right choice of D R related to (DEC).] Discovery of Witten.

Integrate $(*)$ with $\sigma \in N(D)$ and asymptotic to model section σ_0 $\Rightarrow$ $0 = \int |\nabla\sigma|^2 + \langle R\sigma, \sigma\rangle . + \int_{\text{infinity}} \ldots$

Mass / EM induces a quadratic (Hermitian) form on constant (model) sections at ∞ and this is non-neg.

Idea of H: Natural idea: use other bundles to circumvent the spin assumption.

Th^m. (H. 2016): Whatever the bundle and operator, the value of the quadratic form is always a (computable) multiple of mass.

I.e, c depends only on representation, not metric.

Big Q: Forget spin $n \geq 4$.

Quasi-local mass.

- Mass in bdd region. If c, can't do local, but quasi-local.

Setting: Σ^2 spacelike surface in (N, g), Ω^3 spacelike domain s.t. $\partial\Omega = \Sigma$.

Expect q.l.m:

1. $\exists$ for any C^∞ Ω.
2. depend only on extrinsic geom of Σ.
3. non-neg under DEC.
4. Zero iff Ω embeds isom. into $\mathbb{R}^{n+1}$, AdS^{n+1}.
5. monotone, also ... ?
6. —— ?

Attempt 1: Hawking mass $m_H(\Sigma) = \frac{|\Sigma|^{\frac{1}{2}}}{64\pi^{3/2}}\left(16\pi - \int_\Sigma H_\Sigma^2\right)$.

$Sc_g \geq 0$, ~~for~~ Σ genus 0, not inside B/H $\Rightarrow$ $m_H(\Sigma) \geq 0$.

monotone along IMCF and $\lim_{r\to\infty} m_H(\Sigma_r) = m(g)$.

Attempt 2: Brown-York mass.

$\Sigma^2 = \partial\Omega^3$ $K_\Sigma > 0$ $\exists$ min isom embed $\Sigma \hookrightarrow \mathbb{R}^3$.

$$m_{BY}(\Sigma) = \int_\Sigma (H_0 - H)\, d\mathrm{vol}_{g_\Sigma}.$$

Attempt 3: AF case $k=0$, $n=3$.

$\Sigma^2 = \partial\Omega^3$. and g on Ω with $Sc_g \geq 0$.

$$\mathcal{E}(\Sigma,\Omega,g) = \{(M^3,\bar{g}) : Sc_{\bar g} \geq 0,\ (\Omega, g) \hookrightarrow (M^3,\bar{g}),\ M\setminus\Omega \text{ horizon free}\}.$$

Def: Bartnik EQ $m_B(\Omega) = \inf\{m(\bar g) : (M,\bar g) \in \mathcal{E}(\Sigma,\Omega,g)\}$.

$K_\Sigma > 0$ $\exists$ unique ext. $(M,\bar g)$ with min mass static in int. $M\setminus\Omega$ and $H_{\Sigma,\Omega} = H_{\Sigma, M\setminus\Omega}$.

More Huisken

20/01/2017

Recall:

1. AF mass$(g) \in \mathbb{R}$, EM $\in \mathbb{R}^{n,1}$, center of mass $\mathbb{R}^n$.
2. AH mass $\in \mathbb{R}^{n,1}$, EM $\in SO(n,2)$.
3. Positive mass Th^m's. (M^3, g) AF, $Sc_g \geq 0 \Rightarrow m(g) > 0$ unless g flat.
4. Quasi-local masses:
   (Hawking) $m_H(\Sigma) = \frac{|\Sigma|^{\frac{1}{2}}}{64\pi^{3/2}} \left(16\pi - \int_\Sigma H^2\right)$.
   (Brown–York) $m_{BY}(\Sigma)$.
   (Bartnik) $m_B(\Sigma)$.

Isoperimetric Profile

(M^n, g) Riem.

$$I_g(V) = \inf \{ \text{Area}(\partial\Omega) : \Omega \text{ set of finite perimeter } |\Omega| = V \}.$$

If Ω realises iso. profile for volume, $\partial\Omega$ is C^∞ and CMC for $n \leq 7$. Ω called <u>Isoperimetric domain</u>.

<u>Examples</u>:
- round balls in $\mathbb{R}^n$, $n = 3$ $I_e(V) = \sqrt[3]{36\pi}\, V^{2/3}$.
- $r = $ const. surface for large r in Schwarzschild (Bray 1998) (M^3 = time fixed spacelike.)
- General mflds, isoperimetric domains may not exist.

CMC foliations.

Thm: (M^3, g) AF with $Sc_g \geq 0$, $m > 0$.

① Complement of a compact set in M is uniquely foliated by stable CMC top. spheres.

② Surfaces become nearly round at ∞, their approx centres converge to a point in $\mathbb{R}^3$ $\rightarrow$ geometric ~~centre~~ centre of mass.

Huang 2010: Geometric centre of mass $\equiv$ Centre of mass in Lec 1.

Thm ① $\exists$ iso. domain for any $V > 0$
② for large V, $\partial\Omega$ belongs to CMC foliation.

(Chodosh-Eichmair-Shi-Yu 2016).

Sketch of Pf (existence) $\rightarrow$ fix $V > 0$.

1. minimising sequence splits into converging part Ω_c, and diverging part which escapes to ∞.
goes outside any compact set.

2. On AF infd. div part converges to round ball B_∞ in $(\mathbb{R}^n, \delta)$ and disjoint union $\Omega \sqcup B_\infty$ is an isoperimetric profile for V.

3. Existence comes from showing $B_\infty = \emptyset$.
Comes from following Thm.

Shi 2016: (M^3, g) AF $Sc_g \geq 0$; g not flat outside cpct set, then $\forall V > 0$, $\exists \Omega_V$ s.t.

$$|\Omega_V| = V \text{ and } \text{Area}(\partial \Omega_V) < I_e(V).$$

$m > 0$, (M, g) not flat $\Rightarrow \exists \Omega_\infty$ s.t. $|\Omega_\infty| = |B_\infty|$ with $\text{Area}(\partial\Omega_\infty) < \text{Area}(\partial B_\infty)$

Existence & uniqueness of this from IMCF.

$$\partial_t x_t = H_t(x_t)^{-1} \nu_t(x_t) \qquad x_t \in \Sigma_t.$$

Thm (Huisken & Ilmanen) 1998 Strong thm about weak IMCF.

(M^3, g) AF, $Sc_g \geq 0$. x for any x in M, $\exists$ weak IMCF starting at x. exists for $t \in (0, T)$ with:

- $\Sigma_t = \partial \Omega_t$.
- $x \to \infty$, $T \to \infty$.
- $\text{Area}(\Sigma_t) = e^t$.
- Σ_t become round spheres for $t \gg 1$.
- x non-flat $m_H(\Sigma_t) > 0$, which increases and converges to $m(g)$.

Pf. Shi: Let $V(t) = |\Omega_t|$, for a.e. $t > 0$.

$$V'(t) = \int_{\Sigma_t} H^{-1} \geq \text{Area}(\Sigma_t)^{3/2} \int_\Sigma (H^2)^{-\frac{1}{2}} \cdot (\text{Hölder})$$

$$\geq \text{Area}(\Sigma_t)^{3/2} \left(16\pi - 64\pi^{3/2} m_H(\Sigma_t) \text{Area}(\Sigma_t)^{-\frac{1}{2}}\right)$$

Set $t(V)$. value of t for which $|\Omega_t| = V$, and $\Omega_V = \Omega_{t(V)}$.

$$B(V) = \mathrm{Area}(\Sigma_{t(V)}) \le \sqrt[3]{36\pi}\cdot\left(\int_0^V \left(1-\sqrt{16}\cdot B(u)^{-\frac12} m_H(\Sigma_{t(u)})\right)^{-\frac12} du\right)^{\frac23}.$$ $\square$

Thm $\Rightarrow$ isoperimetric profile (M^3,g) AF with $Sc_g \ge 0$.
is no larger than Euclidean isom. prof.
& equality for some volume $\Rightarrow$ flatness everywhere.

Even something better: 2016 (Jauregui–Lee & CESY (Chodosh...)).

(M^3,g) AF, $Sc_g \ge 0$.

$$m(g) = \limsup_{V>0} \frac{2}{I_g(V)}\left(V - \frac{1}{6\sqrt{\pi}} I_g(V)^{3/2}\right). \quad \left(\overset{\text{RHS}}{=:} m_{iso}(g)\right)$$

If limsup = lim, then mass given by Euclidean iso + correction term due to Huisken 2006.

Sketch of Pf.

(1) Large coordinate sphere $\Rightarrow m_{iso}(g) \ge m(g)$.

(2) $V>0$, Ω iso dom $|\Omega| = V$, shrink Ω with constant speed.

$$H_{\partial\Omega} = \lim_{\varepsilon\to0} \frac{|\partial\Omega_\varepsilon| - |\partial\Omega|}{|\Omega_\varepsilon| - |\Omega|} \le \frac{I_g(V_\varepsilon) - I_g(V)}{V_\varepsilon - V}.$$

so that

$$\underbrace{(I_g)'_-}_{\text{half deriv.}} = H_{\partial\Omega} = \left(\frac{16\pi}{I_g(V)} - (16\pi)^{1/2}\frac{m_H(\partial\Omega)}{I_g(V)^{3/2}}\right)^{\frac12}$$

$$\ge \left(\frac{16\pi}{I_g(V)} - (16\pi)^{1/2}\frac{m(g)}{I_g(V)^{3/2}}\right)^{\frac12} \left(=: F(I_g(V))\right)$$

Equality Schwarzschild case! Do expansion geometry against this!

$m_{iso}(g_m) = m = m(g)$.

$\uparrow$ Schwarzschild met.

Elementary ODE: $\exists V_0 > 0$, $A > 0$ s.t.

$$I_g(V) \geq I_{g_m}(V) - A \quad \forall V \geq V_0.$$

Iso perimetric mass: $\limsup \left(\frac{2V}{I_g(V)} - \frac{I_g(V)^{\frac{1}{2}}}{6\sqrt{\pi}} \right)$,

function $f_V(x) := \frac{2V}{x} - \frac{x^{1/2}}{6\sqrt{\pi}}$ monotone in x.

So $f_V(I_g(V)) \leq f_V(I_{g_m}(V) - A)$.

passing to limit $V \to \infty$ yields estimate. $\square$

<u>Corollary</u> (p

mass as a function on space of AF metrics with $Sc \geq 0$ is lower semicts for $\underline{\underline{C_0}}$ topology: $g_i \to g$.

$$m(g) \leq \liminf m(g_i).$$

- No need to control derivatives,
- counterexamples to continuity.
- $Sc < 0$ counterexamples.

<u>AH results</u>: $Sc_g \geq -6$ timelike pos. mass vector.

- complement of cpct set foliated by CMC.
- centre of mass results. - approx. mass centre to point in $\mathbb{H}^3$.
- mass $= (m,0,0,0) \Rightarrow$ geometric centre of mass.

1978 Ashtekar & Hansen suggested mass.

$$\lim_{r\to\infty} \int_{S_r} G^g(X, \nu_r)\, d\mathrm{vol}_g.$$

X is a conformal Killing of the Euclidean metric.

Ie, $X = r\,\partial r$ yeilds mass.

<u>Thm</u>. Expection is true and extends to the AH case.

Opens this study to conformal geometry of Euclidean space.