22/11/2017

Recall $\quad G \longrightarrow P \xrightarrow{\pi} M \quad$ PFB

$$0 \to VP \longrightarrow TP \xrightarrow{d\pi} \pi^{-1}TM \to 0$$

Ie, $\quad VP \cong (\ker d\pi)$

Bundle morphism $\quad d\pi : TP/VP \to \pi^{-1}TM$.

~~Bienne~~ $\cancel{T_pP/V_p \cong T_{\pi}M}$ $\quad d_p\pi \quad T_pP/V_pP \xrightarrow{\cong} T_{\pi(p)}M$.

$\forall g \in G, \quad dR_g(V_pP) = V_{p\cdot g}P$, right action.

$TP/VP \circlearrowleft G$ and $dR_g(T_pP/V_pP) = T_{p\cdot g}P/V_{p\cdot g}P$.

$\to TP/VP$ is G-invariant.

Given local v.f. $X : U \to TM$, $U \subset M$ open, may lift to a local section of TP/VP

$\bar{X} : \pi^{-1}(U) \to TP/VP$.

$\bar{X}(p) = (d_p\pi)^{-1}(X(\pi(p)))$.

Can show, $d_pR_g(\bar{X}(p)) = \bar{X}(R_g(p)) = \bar{X}(p\cdot g)$

OTOH, given G-invariant section $\bar{X} : \pi^{-1}(U) \to TP/VP$, obtain v.f. $X : U \to TM$ as follows.

G-bundle $\begin{cases} \forall \alpha \in A, \text{ have local section} \\ \sigma_\alpha : U_\alpha \to \pi^{-1}(U_\alpha). \end{cases}$

Define for each α s.t. $\mathcal{U}_\alpha \cap \mathcal{U} = \emptyset$,

$$X_\alpha : \mathcal{U}_\alpha \cap \mathcal{U} \to TM.$$

$$X_\alpha(x) = d_{\sigma_\alpha(x)}\pi\left(\bar{X}(\sigma_\alpha(x))\right).$$

Turns out that when $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U} \neq \emptyset$, $X_\alpha = X_\beta$ on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}$. (since $\bar{X}$ is G-invariant)

Define $X(x) = X_\alpha(x)$. $\forall x \in \mathcal{U}_\alpha \cap \mathcal{U}$.

<u>Note:</u> TP/VP is ...

~~TP/~~

<u>Def^n</u>. A connection P is a G-invariant distribution on P complementary to VP, i.e. a subbundle $HP < TP$ ~~(fibrewise~~ s.t.

$$TP = HP \oplus VP \quad \text{(fibrewise)}$$

and $\forall g \in G$, $p \in P$

$$d_pR_g(H_pP) = H_{p\cdot g}P.$$

We call HP a <u>horizontal subbundle</u>.

Clearly, we have an isom. $HP \cong TP/VP \cong \pi^*(TM)$.

<u>Rmk</u>. $VP = (\ker d\pi) \longrightarrow$ Fixed. But lots of choices of HP.

<u>Rmk</u>. Fundamental issue in gauge theory, find a "good" HP.

Lemma Connections always exist.

Pf. It suffices to construct a G-invariant Riem. metric on TP. i.e. $a : P \to T^*P \otimes T^*P$ with inner prod. conditions. and s.t. $\forall v, w \in T_pP$

$$(a(p\cdot g), d_pR_g(v) \otimes d_pR_g(w)) = (a(p), v \otimes w).$$

Let $$HP = (VP)^\perp = \bigcup_{p \in P} \underbrace{\{ v \in T_pP : \forall w \in V_pP, (a, v\otimes w) = 0\}}_{H_pP}$$

Note: If $v \in H_pP$, then $\forall \tilde{w} \in V_{p\cdot g}P$. $\tilde{w} = d_pR_g(w)$.

$$(a(p\cdot g), d_pR_g(v) \otimes \tilde{w}) = (a(p\cdot g), d_pR_g(v) \otimes d_pR_g(w))$$
$$= (a(p), v \otimes w) = 0.$$
$$\uparrow v \in H_pP.$$

$$\rightsquigarrow d_pR_g(v) \in H_{p\cdot g}(P).$$

Construct such a Riem metric as follows:

- Fix Riem a on M.
- Fix an IP on $\mathfrak{g}$ and define:

$$\tilde{a} : G \to T^*G \otimes T^*G. \quad \forall g \in G, \; v, w \in T_gG.$$

$$(\tilde{a}(g), v \otimes w) = \langle d_g R_{g^{-1}}(v), d_g R_{g^{-1}}(w) \rangle.$$

$\longrightarrow$ right invariant metric. $(R_g^* \tilde{a} = \tilde{a} \quad \forall g \in G)$

- These give rise to a metric a_α on $U_\alpha \times G$, using $T_{(x,g)}(U_\alpha \times G) \cong T_x U_\alpha \oplus T_g G_g$.

define for $(x,g) \in U_\alpha \times G$,

$(v_1 \oplus w_1) \in T_{x,g} (U_\alpha \times G)$
$(v_2 \oplus w_2)$

$$(a_\alpha(x,g), (v_1 \oplus w_1) \otimes (v_2 \oplus w_2)) := (\underline{a}(x), v_1 \otimes v_2) + (\tilde{a}(g), w_1 \otimes w_2).$$

This metric is invariant w.r.t. $\underline{R}$ (R mult. on $U_\alpha \times G$ in second comp.).

~~Sps $\{U_\alpha\}$ loc. finite.~~

Sps w.l.o.g. $\{U_\alpha\}$ locally finite and $\{\psi_\alpha\}$ p.o.u. subordinate to it. Then, desired metric, is

$$a(p) = \sum_\alpha \psi_\alpha(\pi(p)) \left((\underline{F}_\alpha^{-1})^* a_\alpha \right)(p).$$ ⊠

Remk. 22/4/2017.

Given horizontal subbundle, we obtain an associated projection $pr_V : TP \to VP$ that commutes with R_g. Explicitly,

$$u \oplus v \in T_pP, \quad pr(u \oplus v) = v.$$

and $$pr_V(dR_g(u \oplus v)) \in T_{p\cdot g}P$$
$$= pr_V(\underbrace{dR_g(u)}_{\in H_{p\cdot g}P} \oplus \underbrace{dR_g(v)}_{\in V_{p\cdot g}P})$$
$$= dR_g(v) = dR_g(pr_V(u \oplus v)).$$

Conversely, given a $pr_V : TP \to VP$ s.t. pr_V bundle morphism, $pr_V = id_{VP}$ & $pr_V \circ dR_g = dR_g \circ pr_V$.

$HP = \ker pr_V$ defines a connection.

Why pr_V rather than pr_H? Because $VP \cong P \times \mathfrak{g}$.

In particular, given pr_V, we may define a $\mathfrak{g}$-valued one form $\omega : P \to \mathfrak{g} \otimes T^*P$ s.t. $\forall p \in P, v \in T_pP$,

$$(\omega(p), v) := (d_eL_p)^{-1}(pr_V(v)).$$

L_p - "orbit map"
$L_p : G \to P$
$g \mapsto p \cdot g = R_g(p)$
$d_eL_p : \mathfrak{g} \to T_pP$
Showed $d_eL_p(\mathfrak{g}) = V_pP$.

Note: we have the following property of ω

- If $X_v = (p \mapsto d_eL_p(v))$ is an infinitesimal transform with $v \in \mathfrak{g}$, then
$$(\omega(p), X_v(p)) = v.$$
(pairing is in 2nd coord. $\omega(p) \in \mathfrak{g} \otimes T_p^*P$.)

(X_v is also called the fundamental VF on P associated with v).

$$((R_g^* \omega)(p), v) \overset{\text{def}^n}{=} (\omega(p\cdot g), d_pR_g(v)).$$

$$\overset{\text{def}^n \text{ of } \omega}{=} (d_e L_{p\cdot g})^{-1} \mathrm{pr}_V (d_pR_g(v)).$$

$$= (d_e L_{p\cdot g})^{-1} (d_pR_g) (\mathrm{pr}_V v).$$

$$= ((d_pR_g)^{-1} \circ d_e L_{p\cdot g})^{-1} (\mathrm{pr}_V v).$$

$$= (d_{p\cdot g} R_{g^{-1}} \circ d_e L_{p\cdot g})^{-1} (\mathrm{pr}_V v).$$

$$= d_e(R_{g^{-1}} \circ L_{p\cdot g})(\mathrm{pr}_V v).$$

$$= (d_e \cancel{L_p} \circ \underbrace{d_e C_g}_{Ad_g})^{-1} (\mathrm{pr}_V v).$$

$$= Ad_{g^{-1}} (d_e L_p)^{-1} (\mathrm{pr}_V v)$$

$$= Ad_{g^{-1}} (\omega(p), v).$$

I.e., $R_g^* \omega = Ad_{g^{-1}} \omega$. (explicitly $= (Ad_{g^{-1}} \otimes Id_{T^*P})\omega$).

As it turns out, given such an ω (with the infinitesimal transformation and ~~Ad~~ invariance) ... as a defn of a conn.

<u>Def</u>

Def^n. A connection on $G \to P \to M$ is a $\mathfrak{g}$-valued one-form $\omega : P \to \mathfrak{g} \otimes T^*P$ s.t.

- $\forall p \in P, v \in \mathfrak{g}$, $(\omega(p), d_eL_p(v)) = v$.
- $\forall g \in G$, $R_g^* \omega = Ad_{g^{-1}} \omega$. (Ad-inv.).

Go one step further and "pull ω down to M":

Let $\{\sigma_\alpha : U_\alpha \to P\}_{\alpha \in A}$ be local sections associated to the bundle atlas of P.

<u>Recall</u> $\sigma_\alpha(x) = [\alpha, x, e]$

$$= [\beta, x, g_{\beta\alpha}(x)\cdot e] = [\beta, x, e\cdot g_{\beta\alpha}(x)] =$$

$$= [\beta, x, e]\cdot g_{\beta\alpha}(x) = \sigma_\beta(x)\cdot g_{\beta\alpha}(x).$$

$\forall x \in U_\alpha \cap U_\beta$.

Define the collection of $\mathfrak{g}$-valued $\{\omega_\alpha : U_\alpha \to \mathfrak{g}\otimes T^*M\}_{\alpha\in A}$ s.t. $\omega_\alpha = \sigma_\alpha^*\omega$, where ω is a given connection on P.

<u>Sps</u>. $x \in U_\alpha \cap U_\beta \neq \emptyset$, we now relate $\omega_\alpha(x)$ to $\omega_\beta(x)$.

Fix $v \in T_xM$;

$$(\omega_\alpha(x), v) \overset{def}{=} (\omega(\sigma_\alpha(x)), \underbrace{d_x\sigma_\alpha(v)}_{\text{calculate} \dots}).$$

$$\omega_\alpha(x) = \mathrm{Ad}_{g_{\beta\alpha}(x)^{-1}}\,\omega_\beta(x) + d\lambda_{g_{\beta\alpha}(x)^{-1}} \circ d_x g_{\beta\alpha}.$$

Conversely given such a collection $\{\omega_\alpha\}$, we can ~~construct~~ reconstruct ω as follows: for $p \in \pi^{-1}(U_\alpha)$, define $\omega(p)$ s.t.

$$(\omega(p), d_{\pi(p)}\sigma_\alpha(w) + d_e L_p(v)) = ((\omega_\alpha(\pi(p)), w) + v.$$

for $w \in T_{\pi(p)}M$, $v \in \mathfrak{g}$.

'Easy' to see that this ~~defin~~ yields a well-defined smooth connection $\omega : P \to \mathfrak{g}\otimes T_G^*P$.

We have decomposition.

$$T_{(x,g)}(M\times G) = d_x i_g(T_xM) \oplus d_g i_x(T_gG).$$

~~$\mathcal{H} \to V_{x,P}$~~

Set $\mathcal{H}_{(x,g)}P = d_x i_g(T_x M)$.

$$\rightsquigarrow \mathcal{H}P = \bigcup_{(x,g)\in M\times G} \mathcal{H}_{(x,g)}P \subset T(M\times G).$$

Also, $d_{(x,g)}R_h(d_x i_g(v)) = d_x(R_h \circ i_g)(v) = d_x i_{g\cdot h}(v) \in \mathcal{H}_{(x,gh)}P$

$\rightsquigarrow \mathcal{H}P$ is G-inv. $\rightsquigarrow \mathcal{H}P$ is a connection.

Compute ω: $\quad (\omega(x,g), v) = d_g \lambda_{g^{-1}}(d_{(x,g)}pr_{\mathcal{V}}(v))$.

because $\quad pr_{\mathcal{V}} = dpr_2$.

On M: $\quad \sigma_\alpha : M \to M\times G$

$\qquad x \mapsto (x,e) = i_e(x)$. global sect.

So, $\quad (\omega_\alpha(x), w) = (\omega(x,e), d_x\sigma_\alpha(w))$

$= d_e\lambda_{e^{-1}}(d_{(x,e)}pr_2(d_x i_e(w)))$

$= d(\underbrace{\lambda_e \circ pr_2 \circ i_e}_{\text{"const.}})(w)\cdot x$.

$\stackrel{!}{=} 0$.

$\Rightarrow \omega_\alpha(x) = 0$. $\square$

<u>Example</u>. Let $GL(n,\mathbb{R}) \to LM = E(M, GL(n,\mathbb{R}))$ frame bundle.

and M Riem. Recall Christoff. $\{\overset{\alpha}{\Gamma}{}^k_{ij} : i,j,k \in \{1,\dots,n\}\}$

lansing for L.C. conn. One has

$$\Big\{\omega_\alpha = \sum_{i=1}^n \Big(\sum_{j,k=1}^n \overset{\alpha}{\Gamma}{}^k_{ij}\, e_{kj}\Big) \otimes dx^i_\alpha\Big\}_{\alpha\in A}$$

defines a connection on LM.