Last time 08/11/2017. (11)

Fibre bundles: $\cdot\ F \to E \xrightarrow{\pi} M$

$\cdot\ \{(\mathcal{U}_\alpha, \Phi_\alpha)\}_{\alpha\in A}$, $\Phi_\alpha : \mathcal{U}_\alpha \times F \xrightarrow{\cong} \pi^{-1}(\mathcal{U}_\alpha)$.

$\Phi_\alpha\big|_{\{x\}\times F} : \{x\}\times F \xrightarrow{\cong} E_x = \pi^{-1}(\{x\})$.

$\cdot$ **Th^m** Given transition functions $\{\psi_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \times F \to F\}_{\alpha,\beta\in A}$ corresponding to an open cover $\{\mathcal{U}_\alpha\}$ s.t. the conditions $\psi_{\alpha\alpha}(x,\cdot) = \mathrm{id}_F$. &

$\forall \alpha,\beta,\gamma \in A$, $x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$

$\psi_{\alpha\beta}(x, \psi_{\beta\gamma}(x,\cdot)) = \psi_{\alpha\gamma}(x,\cdot)$ | cocycle conditions.

$\exists!$ upto iso fibre bundle $F \to E \xrightarrow{\pi} M$ with an atlas $\{(\mathcal{U}_\alpha, \Phi_\alpha)\}_\alpha$ s.t.

$(\Phi_\alpha^{-1} \circ \Phi_\beta)(x,f) = (x, \psi_{\alpha\beta}(x,f))$.

$\forall x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$, $f \in F$.

Explicitly $E = \{[\alpha,x,f] : \alpha\in A, x\in \mathcal{U}_\alpha, f\in F\}$.

and $[\cdot]$ are equivalence classes.

$(\alpha,x,f) \sim (\beta,\tilde{x},\tilde{f}) \iff \tilde{x}=x,\ \tilde{f} = \psi_{\alpha\beta}(x,f)$.

G-bundle over M: $\cdot$ G Lie group,

$\cdot$ open cover $\{\mathcal{U}_\alpha\}_\alpha$ of ~~M~~ G.

$\cdot$ Smooth mappings $\{g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to G\}_{\alpha,\beta\in A}$.

s.t. $g_{\alpha\beta}(x)\cdot g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)$. $\forall \alpha,\beta,\gamma\in A$, $\forall x\in \mathcal{U}_\alpha\cap\mathcal{U}_\beta\cap\mathcal{U}_\gamma$.

Sps G acts on F from the left.

Define: $\{ \psi_{\alpha\beta} : (U_\alpha \cap U_\beta) \times F \to F \}_{\alpha\beta}$

s.t. $\psi_{\alpha\beta}(x, f) := g_{\alpha\beta}(x) \cdot f$

$\forall x \in U_\alpha \cap U_\beta, \quad f \in F$. $\Big\}$ (*)

Note: ① $\psi_{\alpha\alpha}(x,f) = g_{\alpha\alpha}(x)\cdot f = e\cdot f = f$.

② $$\psi_{\alpha\beta}(x, \psi_{\beta\gamma}(x,f)) = g_{\alpha\beta}(x)\cdot (g_{\beta\gamma}(x)\cdot f)$$
$$= (g_{\alpha\beta}(x)\cdot g_{\beta\gamma}(x))\cdot f$$
$$= g_{\alpha\gamma}(x)\cdot f$$
$$= \psi_{\alpha\gamma}(x,f).$$

Corollary: Sps ξ is a G-bundle and $G \circlearrowright F$, F a smooth mfld. Then we obtain a unique fibre bundle

$$F \to E(\xi, F) \xrightarrow{\pi} M.$$

with transn fncs given by (*).

Def. $E(\xi, F)$ is said to be associated to ξ and the action $G \circlearrowright F$.

If a fibre bundle E may be realized in this way, i.e., if its transn fncs take the form (*) we say that E has structure group G.

Rmk Usually identify bundle atlas

$$\{(U_\alpha, \Phi_\alpha)\}_{\alpha\in A}, \quad \{(V_\gamma, \tilde{\Phi}_\gamma)\}_{\gamma\in\Pi}$$

with $\Phi_{\alpha\beta}(x,f) = g_{\alpha\beta}(x)\cdot f$ and

$\tilde{\Phi}_{\gamma\delta}(x,f) = \tilde{g}_{\gamma\delta}(x)\cdot f$.

s.t. $\{(U_\alpha, \Phi_\alpha)\}_\alpha \sim \{(V_\gamma, \tilde{\Phi}_\gamma)\}_\gamma$.

$\Leftrightarrow$ $\exists \{\tau_{\alpha\gamma} : U_\alpha \cap V_\gamma \to G\}_{\alpha\in A, \gamma\in\Pi}$.

s.t. $\tilde{g}_{\gamma\delta} = \tau_{\alpha\gamma}^{-1}\, g_{\alpha\beta}\, \tau_{\beta\gamma}$ $\quad \forall \alpha,\beta\in A, \forall \gamma,\delta\in\Pi$.

on $U_\alpha \cap U_\beta \cap V_\gamma \cap V_\delta$.

Example 1: Recall G_{triv} was the $\{e\}$-bundle

with $\{U_\alpha\} = \{M\}$. and $g_{\alpha\alpha}(\cdot) = e$.

Let $\{e\} \curvearrowright F$ be the 'trivial action', $e\cdot f = f$ $\forall f\in F$.

$\rightsquigarrow$ $E(G_{triv}, F)$ with a single ~~chart~~ bundle.

trivialisation $\Phi : M\times F \to E(G_{triv}, F)$

$\Rightarrow$ $E(G_{triv}, F) = M\times F$, the trivial bundle over M

with fibre F. $\square$

Example 2: Recall ~~A~~ was the $GL(n,\mathbb{R})$-bundle s.t.

$\{U_\alpha\}_{\alpha\in A}$ are the domains of charts.

$$\{\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subset \mathbb{R}^n\}.$$

and $g_{\alpha\beta}(x) = D(\varphi_\alpha \circ \varphi_\beta^{-1})(\varphi_\beta(x))$.

Now, let $F = \mathbb{R}^n$, and $GL(n,\mathbb{R}) \curvearrowright \mathbb{R}^n$ viewed as column vectors from the left,

$\rightsquigarrow$ $E(\mathcal{A}, \mathbb{R}^n) \cong TM$.

(one way to do this: $T_xM \ni \sum_{i=1}^n v^i \partial_i|_x \mapsto [\alpha, x, \sum_{i=1}^n v^i e_i]$.)

Could also consider the dual representation $GL(n,\mathbb{R})$ on $(\mathbb{R}^n)^* \rightsquigarrow E(\mathcal{A}, (\mathbb{R}^n)^*) \cong T^*M$.

Similarly obtain tensor bundles by taking a representation of $GL(n,\mathbb{R})$.

For $E(\mathcal{A}, (\mathbb{R}^n)^*)$, $E_x(\mathcal{A}, (\mathbb{R}^n)^*) = \{[\alpha, x, \nu_\alpha]_x : \nu \in (\mathbb{R}^n)^*\}$.

$$[\alpha, x, \nu_\alpha] = [\beta, x, \nu_\beta] \iff \nu_\alpha = \rho^*(g_{\alpha\beta}(x))\nu_\beta.$$
$$\iff \forall v \in \mathbb{R}^n$$
$$(\nu_\alpha, v) = (\nu_\beta, \rho(g_{\alpha\beta}(x)^{-1})v)$$

obtain a well-defined bilinear pairing

$$E_x(\mathcal{A}, (\mathbb{R}^n)^*) \times E_x(\mathcal{A}, \mathbb{R}^n) \to \mathbb{R} \quad \text{s.t.}$$

$$([\alpha, x, \nu_\alpha]_x, [\alpha, x, v_\alpha]) = (\nu_\alpha, v_\alpha).$$

Note that for vectors, $v_\alpha = \rho(g_{\alpha\beta}(x))v_\beta$,

hence $(\nu_\alpha, v_\alpha) = (\nu_\beta, v_\beta)$.

$\Rightarrow E(\mathcal{A}, \mathbb{R}^n)^* \cong E(\mathcal{A}, (\mathbb{R}^n)^*)$.

Intuition: — for reduction. See Ex 1 also as a larger bundle,

$\{U_\alpha\}_{\alpha \in A}$ an open cover, with same action,

get charts $\Phi_\alpha : U_\alpha \times F \to F$

Seems like a non-trivial bundle, but we can find equivalent chart $\Phi : M \times F \to F$.

(or see it in maximal atlas).

which allows us to see it as a trivial bundle.

So, going to a subgroup as a structure group is like finding a sub atlas thus learning more information about bundle.

Ex 3. Recall $\mathcal{A}_0$ was the $O(n, \mathbb{R})$ bundle s.t. given a cover $\{U_\alpha\}_{\alpha \in A}$ of $(M, \langle \cdot, \cdot \rangle)$.

together with local o.n. frames

$$\{ (e_1^\alpha, \dots, e_n^\alpha) : U_\alpha \to (TM)^n \}.$$

* Note, assume $(M, \langle \cdot, \cdot \rangle)$ is Riem to obtain $O(n, \mathbb{R})$. bundle on M.

Set: $g_{\alpha\beta}(x) = (\langle e_i^\alpha(x), e_j^\beta(x) \rangle)_{i,j=1,\dots}^n$.

Then, using $O(n, \mathbb{R}) \curvearrowright \mathbb{R}^n$, obtain bundle $E(M_0, \mathbb{R}^n)$. Turns out $E(\mathcal{A}_0, \mathbb{R}^n) = TM$.

Eg. Consider $T_x M \to \sum_{i=1}^n v^i e_i^\alpha \mapsto [\alpha, x, \sum_{i=1}^n v^i e_i] \in E(\mathcal{H}_0, \mathbb{R}^n)$.

$\rightsquigarrow$ tangent bundle is also in $O(n, \mathbb{R})$ bundle.

Ex 4 (Vector bundles).

More generally, given a G-bundle $\mathcal{G}$, representation $\rho : G \to GL(V)$ with V a vector space, we obtain a fibre bundle

$$V \longrightarrow E(\mathcal{G}, V) \xrightarrow{\pi} M,$$

and a v.s. structure on the fibres.

$$k \cdot [\alpha, x, v] + [\alpha, x, w] =: [\alpha, x, k \cdot v + w] \quad \forall k \in \mathbb{F} (= \mathbb{R} \text{ or } \mathbb{C}), \; v, w \in V.$$

Note: · Can view vector bundles as $GL(N)$-bundles, consider $\tilde{g}_{\alpha\beta} := \rho \circ g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V) \cong GL(N, \mathbb{F})$

matrices. $N = \dim V$.

· Can always reduce the structure group to $O(N, \mathbb{F})$. Use p.o.u. to glue together a metric $\langle \cdot, \cdot \rangle$ and define cocycles analogously to Ex. 3.

· If $E(\mathcal{G}, V)$ is an $O(n)$-bundle, obtain a canonical metric: $x \in U_\alpha \subset M$.

$$\langle [\alpha, x, v_\alpha], [\alpha, x, w_\alpha] \rangle := \langle v_\alpha, w_\alpha \rangle$$

preserve inner prod.

!!!

well defined since $\langle v_\alpha, w_\alpha \rangle = \langle \tilde{g}_{\alpha\beta} v_\beta, \tilde{g}_{\alpha\beta} w_\beta \rangle =$ $\boxed{\tilde{g}_{\alpha\beta} \in O(n)}$

Say v.b with $O(n)$ as its struct. group

or v.B with metric is Riemannian.

- E_1, E_2 v.b. on M.

E_1 is a vec. <u>sub bundle</u> if it's a vector bundle & there is an inclusion $\varphi: E_1 \to E_2$ s.t. $\varphi|_{E_1|_x}: E_1|_x \hookrightarrow E_2|_x$ is linear.

Quotient bundles for sub-bundle $E_1 < E_2$.

$\rightsquigarrow E_2/E_1$ with fibers $(E_2/E_1)_x = E_2|_x / \varphi(E_1|_x)$.
(exercise).

- As before, can form $E_1 \otimes \dots \otimes E_n$, E^*, ΛE, ...

By uniqueness $T_pM \to \sum_{i=1}^{n} v^i e_i^{\alpha} \mapsto [x, n, \sum_{i=1}^{n} v^i e_i] \in F(M)_{o}, \mathbb{R}^n)$

$\rightarrow$ tangent bundle is also an $O(n, \mathbb{R})$ bundle.

Lecture 2 ... (L2) 08/11/2018.

Earlier, G-bundle $\mathcal{G} = \{ g_{\alpha\beta} : U_\alpha \cap U_\beta \to G \}_{\alpha\beta\in A}$ cocycles, $M = \bigcup_\alpha U_\alpha$.
Group action $\mathcal{G} \curvearrowright F$.

Thm. $F \to \underset{E''}{E(\mathcal{G}, F)} \xrightarrow{\pi} M$ .. E fibre bundle w/ struct. gp. $\mathcal{G}$.

· If $\mathcal{G}$ is a representation, $F = V$ a v. space,
(action is a rep).
$\rho : \mathcal{G} \to GL(V) \cong GL(\dim V, \mathbb{R})$.

then E is a v.b.

Ex. 5. Sps $F \to E(\mathcal{G}, F) \xrightarrow{\pi} M$ is a fibre bundle w/ st. group G, and let $f : N \to M$ be smooth.
Consider:

$$(f^{-1}g)_{\alpha\beta} : f^{-1}(U_\alpha \cap U_\beta) \to \mathcal{G}$$

by $(f^{-1}g)_{\alpha\beta}(x) := g_{\alpha\beta}(f(x))$.

Then define cocycles in N since $\bigcup_\alpha f^{-1}(U_\alpha) = N$.
This determines a resulting bundle

$$F \to E(f^{-1}\mathcal{G}, F) \to N$$

called the pullback of $E(\mathcal{G}, F)$ by f.

Ex. 6 Let $\mathcal{G}$ be a G-bundle and $\mathcal{G} \curvearrowright G$ by left multiplication. (λ_g, $g \in \mathcal{G}$). Obtain.

$$G \to P := E(\mathcal{G}, G) \xrightarrow{\pi} M.$$

Sps $p = [\alpha, x, g_\alpha] \in P$, note $\boxed{\lambda_g \rho_h = \rho_h \lambda_g}$
(left, right mult. commute)

Define $p \cdot g := [\alpha, x, g_\alpha \cdot g]$, $g \in G$.

well-defined: $p = [\alpha, x, g_\alpha] = [\beta, x, g_{\beta\alpha}(x) \cdot g_\alpha]$.

then
$$[\alpha, x, g_\alpha \cdot g] = [\beta, x, g_{\beta\alpha}(x) \cdot g_\alpha \cdot g] = [\beta, x, g_{\beta\alpha}(x) g_\alpha] \cdot g.$$

$\rightarrow$ Global right action that is smooth!

$$\cdot : P \times G \longrightarrow P.$$
$$(p, g) \longmapsto p \cdot g = R_g(p) = L_p(g).$$

Note: R_g preserves fibres; $\pi(R_g(p)) = \pi[\alpha, x, g_\alpha \cdot g] = x = \pi(p)$

and acts transitively, $\forall p_1, p_2 \in P_x$

$\exists\, g \in G$ s.t. $p_2 = p_1 \cdot g$.

(since $p_1 = [\alpha, x, g_1]$, $p_2 = [\alpha, x, g_2]$
$p_1 \cdot (g_1^{-1} g_2) = [\alpha, x, g_1^{-1} g_2] = p_2$.

Unique since it acts freely:
If for some $p \in P$, $R_g(p) = p$
$\Longleftrightarrow [\alpha, x, g_\alpha g] = [\alpha, x, g_\alpha]$, then $g = e$.

Also, have right action $(U_\alpha \times G) \times G \longrightarrow U_\alpha \times G$.
$((x, g), h) \longmapsto (x, gh) = R_h(x, g) = (x, g) \cdot h$.

& by def^n.

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[r, "R_g"] \arrow[d, "\Phi_\alpha^{-1}"'] & \pi^{-1}(U_\alpha) \arrow[d, "\Phi_\alpha^{-1}"] \\
(U_\alpha \times G) \arrow[r, "R_g"] & (U_\alpha \times G)
\end{tikzcd}$$

Commutes:
$$\Phi_\alpha((x, g) \cdot h) = \Phi_\alpha(x, g) \cdot h.$$

#

__Def^n__ A __principal bundle__ is a fibre bundle $G \to P \xrightarrow{\pi} M$ together with a right action $P \times G \to P$ and bundle atlas $\{(U_\alpha, \Phi_\alpha)\}_\alpha$ w.r.t. which the ~~[illegible]~~ $\Phi_\alpha(x,g)\cdot h = \Phi_\alpha(x, g\cdot h)$.

(Ex 6. gives principal bundles).

~~Conversely~~ __Conversely__, given a principal bundle we may recover $\{g_{\alpha\beta}\}$ (i.e. the underlying G-bundle).

Since
$$
\begin{aligned}
&(\Phi_\alpha^{-1} \circ \Phi_\beta)(x,g) \\
&= \Phi_\alpha^{-1}\big(\Phi_\beta(x,e)\cdot g\big) \\
&= \Phi_\alpha^{-1}\big(\Phi_\beta(x,e)\cdot g\big) \\
&= R_g\, \Phi_\alpha^{-1}\big(\Phi_\beta(x,e)\big). \\
&= R_g\,\big(x, \Phi_{\alpha\beta}(x,e)\big). \\
&= \big(x, \psi_{\alpha\beta}(x,e)\cdot g\big).
\end{aligned}
$$

So take $g_{\alpha\beta}(x) = \cancel{\Phi_{\alpha\beta}(x,e)}\ \psi_{\alpha\beta}(x,e)$.

__Rmk__ This is not a simple "lifting" of G over each x, because this doesn't preserve group structure. But $G \circlearrowleft G$ via __conjugation__ does. (See later in Gauge transform).

__Def^n__ {__Principal bundle morphism__} is a bundle morphism $\varphi : P_1 \to P_2$ s.t. $\forall p \in P_1,\ g \in G$, $\varphi(p\cdot g) = \varphi(p)\cdot g$.

If φ is a __diffeo__, it is said to be a __gauge transformation__. (Here $P_1 = P_2$).

Ex. 7 Recall: Given charts $\{(\mathcal{U}_\alpha, \varphi_\alpha)\}$ of smooth mfld M^n, define $\mathcal{A}$ to be the $GL(n,\mathbb{R})$ - bundle with $g_{\alpha\beta}(x) = D(\varphi_\alpha \circ \varphi_\beta^{-1})(\varphi_\beta(x))$

(Remark $\mathcal{A}$ is not a "bundle", but it is a $G = GL(n,\mathbb{R})$ bundle by def^n of nomenclature)

Consider the principal bundle obtained from $\mathcal{A}$

$$GL(n,\mathbb{R}) \longrightarrow E(\mathcal{A}, GL(n,\mathbb{R})) \xrightarrow{\pi} M.$$

<u>Define</u> for $P_x = [\alpha, x, g_\alpha] \in E_x(\mathcal{A}, GL(n,\mathbb{R}))$ the mapping

$$T(P_x, \cdot) = (\mathbb{R}^n \ni v \longmapsto [\alpha, x, g_\alpha \cdot v]_\sim \in E_x(\mathcal{A}, \mathbb{R}^n) \cong T_x M)$$

<u>well-defined</u> since

$$[\alpha, x, g_\alpha \cdot v] = [\beta, x, (g_{\beta\alpha}(x) \cdot g_\alpha) \cdot v] = [\beta, x, g_\beta \cdot v].$$

Also, $T(P_x, \cdot) : \mathbb{R}^n \to T_x M$ is linear and invertible.

$\rightsquigarrow$ $E(\mathcal{A}, GL(n,\mathbb{R}))$ is the <u>frame-bundle</u> of M.

Sps. $\sigma : \mathcal{U} \to P$ is a section. Then $\{T(\sigma(x), e_i)\}_{i=1}^n$ is a basis of $T_x M$ <u>and</u> $\mathcal{U} \ni x \longmapsto T(\sigma(x), e_i) \in T_x M$ is smooth.

If $\mathcal{U} = M$, then $\{T(\sigma(x), e_i)\}_{i=1}^n$ is a basis of $T_x M$ $\forall x \in M$ and obtain a trivialisation:

$$\begin{cases} M \times \mathbb{R}^n \xrightarrow{\cong} TM \\ (x, v) \longmapsto (x, T(\sigma(x), v)) \end{cases}$$

Th$^{\underline{m}}$. There is a one-one correspondence b/w trivialisations $U \times G \to \pi^{-1}(U)$ and local sections $U \to P$ of $G \to P \to M$ where $U \subset M$ open.

Pf Let $\Phi : U \times G \to \pi^{-1}(U)$ be a triv. Then, $\sigma := \Phi(\cdot, e)$ is a section $U \to P$.

Conversely, given $\sigma : U \to P$, define $\Phi : U \times G \to \pi^{-1}(U)$ s.t. $\Phi(x,g) = \sigma(x) \cdot g$ $\square$

From now on, use correspondence $\Phi_\alpha \longleftrightarrow \sigma_\alpha$.

Def$^{\underline{n}}$ For fixed $v \in \mathfrak{g} = T_eG$, define the v.f.

$$\underline{X}_v : P \to TP \,, \quad p \mapsto d_e L_p(v).$$

Recall vertical subbundle $VP < TP$ with fibres

$$V_pP = \ker(d_p\pi).$$

Note: $d_p\pi(\underline{X}_v(p)) = d_p\pi(d_eL_p(v)) = d_e(\pi \circ L_p)(v) = 0$

$\Rightarrow \quad \underline{X}_v(p) \in V_pP \qquad \left(\pi(L_p(g)) = \pi(p \cdot g) = \pi(p)\right)$

Moreover,

Prop The mapping $P \times \mathfrak{g} \to VP$, $(p,v) \mapsto \underline{X}_v(p)$ is a bundle isom.

Pf. Note: $d_eL_p : \mathfrak{g} \to V_pP$ as before. Also,

$$(\Phi_\alpha^{-1} \circ L_{\Phi_\alpha(x, g_\alpha)})(g) = \Phi_\alpha^{-1}(\Phi_\alpha(x, g_\alpha) \cdot g) = (x, g_\alpha g)$$

$$\Rightarrow \quad (\mathrm{pr}_2 \circ \Phi_\alpha^{-1}) \circ L_{\Phi_\alpha(x, g_\alpha)}(g) = g_\alpha g = L_{g_\alpha}.$$

$\Leftrightarrow d(\lambda_{g_\alpha^{-1}} \circ pr_2 \circ \Phi_\alpha^{-1}) \circ d_e L_{\Phi_\alpha(x, g_2)} = id_{\mathfrak{g}}$.

$\Rightarrow d_e L_{\Phi_\alpha(x,g)}$ is injective (left inv $\Rightarrow$ inj).

and $\dim G = \dim \ker (d_{\Phi_\alpha(x,g)}) = \dim \mathfrak{g}$. $\square$