Representations of Lie Groups 01/11/2017.

Recall: A group action of a Lie group G on a mfld M is a smooth mapping $\cdot : G\times M \to M$ s.t. $\forall g,h \in G$, $x\in M$,

$$\begin{cases} g\cdot(h\cdot x) = (g\cdot h)\cdot x \\ e\cdot x = x \end{cases}$$

Write: $G \curvearrowright M$.

Ex. $G = GL(n,\mathbb{R})$, $M = \mathbb{R}^n$
matrix multiplication: $(A,x) \mapsto A\cdot x$

Def^n. A <u>representation</u> of G on a vector space V is a group action s.t. $\forall g\in G$, $V \ni x \mapsto g\cdot x \in V$ is <u>linear</u>.

Notation: $\rho : G \to GL(V)$. (Ie, see "$\rho: G \to \mathcal{L}(V)$").
$(GL(V) := Iso(V,V) \cong GL(n,\mathbb{R}))$.

Rmk. If we have representations $\{\rho_i : G \to GL(V_i)\}_{i=1}^N$, obtain natural induced representations on $V_1 \oplus \dots \oplus V_N$, $V_1 \otimes \dots \otimes V_N$ & V_i^* via:

$\rho_\oplus : G \to GL(V_1 \oplus \dots \oplus V_N)$. $g \mapsto \left((v_i)_{i=1}^N \mapsto (\rho_i(g)v_i)_{i=1}^N\right)$

$\rho_\otimes : G \to GL(V_1 \otimes \dots \otimes V_N)$ $g \mapsto \left((v_1\otimes \dots \otimes v_n) \mapsto \rho_1(g)v_1 \otimes \dots \otimes \rho_N(g)v_N\right)$

$\rho_* : G \to GL(V_i^*)$ $g \mapsto \rho(g^{-1})^{tr}$ ie $\forall \nu \in V_i^*$, $v\in V_i$.
$\langle {}^t\rho(g^{-1})\nu, v\rangle = \langle \nu, \rho(g^{-1})v\rangle$.

Ex. For each $g \in G$, consider the map

$$C_g : G \to G, \quad h \mapsto ghg^{-1}. \quad \text{(conjugation)}.$$

This defines a group action $G \circlearrowright G$, distinct from left multiplication. $G \times G \to G$, $(g,h) \mapsto C_g h$.

Note: C_g keeps e fixed. $\Rightarrow$ $Ad_g := d_e C_g : \mathfrak{g} \to \mathfrak{g}$.

Satisfies. $Ad_e = d_e C_e = id_{\mathfrak{g}}$.

$Ad_{gh} = Ad_g \circ Ad_h \quad \forall g, h \in G$.

(If $G = GL(n)$, $Ad_g(X) = gXg^{-1}$, $g \in GL(n)$, $X \in \mathfrak{gl}(n)$).

$\boxed{Ad_g : \mathfrak{g} \to \mathfrak{g} \text{ is "the adjoint" representation of } G \text{ on } \mathfrak{g}}$

Can differentiate $g \mapsto Ad_g Y$ at e; $\forall Y \in \mathfrak{g}$,

$$ad_X(Y) := \frac{d}{dt}\Big|_{t=0} Ad_{\exp(tX)}(Y) \in \mathfrak{g}.$$

(adjoint rep of $\mathfrak{g}$ on $\mathfrak{g}$ — representation of <u>algebra</u>:
$\forall X, Y \in \mathfrak{g} \quad ad_{[X,Y]} = [ad_X, ad_Y]$).

<u>Prop</u>. $\forall X, Y \in \mathfrak{g}$, $ad_X Y = [X, Y]$.

<u>Pf</u> $ad_X Y = \frac{d}{dt}\Big|_{t=0} Ad_{\exp(tX)} Y$. and notice $\left(C_g = \lambda_g \circ \rho_{g^{-1}} = \rho_{g^{-1}} \circ \lambda_g \right)$.

$$= \frac{d}{dt}\Big|_{t=0} d_{\exp(tX)} \rho_{\exp(-tX)} \underbrace{\left(d_e \lambda_{\exp(tX)} Y\right)}_{X_Y^L(\exp(tX))}. \quad \text{left, right mult commute.}$$

$$= \frac{d}{dt}\Big|_{t=0} d\rho_{\exp(-tX)} \left(X_Y^L(\exp(tX))\right).$$

$$= [X_X^L, X_Y^L](e) = [X, Y]$$

<u>Recall</u>: $N \trianglelefteq G$ (N is a normal subgroup of G).

$\Leftrightarrow$ $N < G$ and $\forall g \in G$ $gNg^{-1} = N \Leftrightarrow C_g(N) = N$.

and "$\underline{\mathfrak{n} \trianglelefteq \mathfrak{g}}$" $\mathfrak{n}$ is an ideal in $\mathfrak{g}$ if $\mathfrak{n}$ is a Lie-Subalgebra of $\mathfrak{g}$ & $\forall X \in \mathfrak{g}$, $Y \in \mathfrak{n}$ $[X, Y] \in \mathfrak{n}$.

<u>Thm</u> Sps G is a connected Lie group, $N < G$ connected Lie subgroup. Then $N \trianglelefteq G \Leftrightarrow \mathfrak{n} \trianglelefteq \mathfrak{g}$.

(connected is important since G connected $\Rightarrow G = \bigcup_{i=1}^{\infty} \mathcal{U}^i$, where $\mathcal{U} \subset G$ with $e \in \mathcal{U}$, $\mathcal{U}^i = \{g_1 \cdots g_i : g_i \in \mathcal{U}\}$.)

<u>Pf</u>. Let $g_t = \exp(tX)$, $n_s = \exp(sY)$.

Compute $C_{g_t}(n) = C_{g_t}(\exp(sY)) \overset{C_{g_t} \text{ homo}}{=} \exp(d_e C_{g_t}(sY))$.

$$= \exp(s \operatorname{Ad}_{g_t} Y). \qquad \operatorname{Ad}: G \to GL(\mathfrak{g}).$$

$$\overset{\operatorname{Ad} \text{ homo}}{=} \exp_G(s \exp_{GL(\mathfrak{g})}(\operatorname{ad}_{tX}) \cdot Y).$$

$$= \exp_G\left(s \sum_{i=0}^{\infty} \frac{(\operatorname{ad}_{tX})^i}{i!} \cdot Y\right).$$

$$= \exp_G\left(s \cdot \sum_{i=0}^{\infty} \frac{t^i}{i!} (\operatorname{ad}_X)^i Y\right).$$

Sps $N \trianglelefteq G \Rightarrow C_{g_t}(n) \in N$.

$\Rightarrow \exp_G\left(s \sum_{i=1}^{\infty} \frac{t^i}{i!} (\operatorname{ad}_X)^i Y\right) \in N$. $\forall s \in \mathbb{R}$.

$\Rightarrow Z(t) := \sum_{i=0}^{\infty} \frac{t^i}{i!} (\operatorname{ad}_X)^i Y \in \mathfrak{g}(N)$.

$$\Rightarrow \left.\frac{d}{dt}\right|_{t=0} z(t) \in \mathfrak{g}(N)$$

$$(\mathrm{ad}_X)Y = [X,Y]$$

$\Rightarrow$ $X \in \mathfrak{g}$, $Y \in \mathfrak{g}(N)$ arbitrary

$\Rightarrow$ $\mathfrak{g}(N) \subset \mathfrak{g}$.

(Note no need for connectedness here).

Sps $\mathfrak{g}(N) \subset \mathfrak{g}$

$$\Rightarrow z(t) = s \cdot \sum_{i=1}^{\infty} \frac{t^i}{i!} (\mathrm{ad}_X)^i Y \in \mathfrak{g}(N).$$

because $\mathrm{ad}_X Y \in \mathfrak{g}(N)$. (def^{n}).

$\Rightarrow (\mathrm{ad}_X)^i Y \in \mathfrak{g}(N) \quad \forall i$

$$\Rightarrow \quad c_{g_t}(n_s) = \exp(z(t)) \in N.$$

Restrict attn to $s = t = 1$, and $X \in \mathcal{U}$ of 0 in $\mathfrak{g}$, $Y \in V$ of 0 in $\mathfrak{g}(N)$.

s.t. $\exp|_{\mathcal{U}}$, $\exp|_V$ are injective.

and $W_1 = \exp(\mathcal{U}) \subset G$, $W_2 = \exp(V) \subset N$ are open nbh. of e.

$\Rightarrow c_g(n_{\mathbb{Z}}) \in N \quad \forall g \in W_1, n \in W_2$. but then.

$$c_{g_1 \cdots g_i}(n_1 \cdots n_j) = c_{g_1}(c_{g_2}(\cdots c_{g_i}(n_1 \cdots n_j)))$$

Now for $g \in W_1$, $n_1, \ldots, n_i \in N$.

$c_g(n_1 \ldots n_i) = c_g(n_1) \cdots c_g(n_i) \in (W_2)^i \subset N$.

$\Rightarrow c_g(n) \in N \quad \forall n \in N, g \in W_1$. (via connected of N)

Now fix $n \in N$, $g_1, \dots, g_i \in W_1$

$\Rightarrow C_{g_1 \dots g_i}(n) = C_{g_1}(n) \cdots C_{g_i}(n)$

$\Rightarrow C_{g_i}(n) \in N \Rightarrow C_{g_{i-1}} C_{g_i} \in N \Rightarrow C_{g_1 \dots g_i}(n) \in N.$

$\Rightarrow C_g(n) \in N \quad \forall g \in G$ can write $g = g_1 \dots g_i$

and via connectedness of G.

Corollary: A connected Lie group is abelian iff its Lie algebra is abelian.

<u>Th^{m}</u>. $H < G$ is closed $\Rightarrow G/H$ is a manifold.

& if $H \trianglelefteq G$, then G/H is a Lie group.

01/11/2017.

Fibre bundles (Fibré, Faserbündle).

Def^{n} A <u>fibre bundle</u> is a triple of smooth manifolds (F, E, M) together with a surjection $\pi : E \to M$, an open cover $\{U_\alpha\}_{\alpha \in A}$ of M and diffeomorphisms $\Psi_\alpha : U_\alpha \times F \to \pi^{-1}(U_\alpha)$ s.t. the follow diagram commutes:

$$\begin{tikzcd} U_\alpha \times F \arrow[r, "\Psi_\alpha"] \arrow[dr, "pr_1"'] & \pi^{-1}(U_\alpha) \arrow[d, "\pi"] \\ & U_\alpha \end{tikzcd} \qquad (5.1)$$

<u>Terminology</u>: F - standard fibre, M - base mfld, π - projection.

$\{(U_\alpha, \Psi_\alpha)\}_{\alpha \in A}$ - bundle atlas, $\{\Psi_\alpha\}$ - local trivialisations.

<u>Notation</u>: Write $F \to E \xrightarrow{\pi} M$ for such a bundle or simply $F \to E \to M$, $E \to M$ or E.

<u>Rmk</u>. Eq. (5.1) $\Rightarrow$

$$\begin{tikzcd} T(U_\alpha \times F) \arrow[r, "d\Psi_\alpha"] \arrow[dr, "dpr_1"'] & T\pi^{-1}(U_\alpha) \arrow[d] \\ & TU_\alpha \end{tikzcd}$$

i.e. $\Leftrightarrow$ $d\pi \circ d\Psi_\alpha = dPr_1$.

$\Leftrightarrow$ $d\pi = \underbrace{dPr_1}_{\substack{\text{epimorph} \\ \text{surjective}}} \circ \underbrace{d\Psi_\alpha^{-1}}_{\text{isomorphism}}$ $\Rightarrow$ π is a submersion. $\forall x \in M$, $E_x := \pi^{-1}(\{x\})$ is an embedded submfld of E

Also, $\Psi_\alpha|_{\{x\}\times F} : \{x\}\times F \xrightarrow{\cong} E_x$.

and hence for $p \in E_x$, $x \in M$,

$$T_pE_x \xrightarrow{\text{inclusion}} T_pE \xrightarrow{d_p\pi} T_xM \to 0$$

is exact.

I.e., $\begin{cases} \ker d_p\pi = T_pE_x \\ \operatorname{im} d_p\pi = T_xM \end{cases}$

(This motivates writing ~~R~~ $F \to E \to M$).

<u>And</u> since $d_p\pi$ is surjective, $\dim \ker d_p\pi$ = constant.

$\rightsquigarrow$ A distribution on E:

$$VE := \bigsqcup_{p\in E} \ker d_p\pi \subset TE.$$

(Can find $X_1, \dots, X_d : U \subset E \to TE$ with ~~$(VE)_p$~~ $(VE)_p = \operatorname{span}\{X_1(p), \dots, X_d(p)\}$ $\forall p \in U$ containing any point of your choosing)

Call VE the <u>vertical subbundle</u> of TE, which has E_x as integral manifolds.

<u>Def^n</u>. Let $F_1 \to E_1 \xrightarrow{\pi_1} M$ & $F_2 \to E_2 \xrightarrow{\pi_2} M$ be fibre bundles. A smooth mapping $\varphi : E_1 \to E_2$ is a bundle morphism if the diagram commutes:

$$\begin{array}{ccc} E_1 & \xrightarrow{\varphi} & E_2 \\ {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle\pi_2} \\ M & \xrightarrow[\mathrm{id}_M]{} & M \end{array} \qquad \pi_2 \circ \varphi = \pi_1.$$

φ is an isomorphism if it is a diffeomorphism.

<u>Def^n</u>. A local section $\sigma : \mathcal{U} \to E$ is a smooth function s.t. $\mathcal{U} \subset M$ open and the diagram commutes:

$$\begin{tikzcd} \mathcal{U} \arrow[r, "\sigma"] \arrow[dr, "\text{inc}"'] & E \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

If $\mathcal{U} = M$, then σ is a <u>global section</u>.

Look at the bundle atlas: $\{(\mathcal{U}_\alpha, \Phi_\alpha)\}_{\alpha \in A}$ more closely,

Consider $\alpha, \beta \in A$ & $\Phi_\alpha^{-1} \circ \Phi_\beta : (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times F \to (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times F$.

by (5.1):
$$\begin{aligned} Pr_1 (\Phi_\alpha^{-1} \circ \Phi_\beta)(x, f) &= (Pr_1 \circ \Phi_\alpha^{-1})(\Phi_\beta(x, f)) \\ &= \pi(\Phi_\beta(x, f)) \\ &= Pr(x, f) \\ &= x. \end{aligned}$$

$\to$ $\exists$ smooth mapping $\Psi_{\alpha,\beta} : (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times F \to F$.

s.t. $\forall (x, f) \in (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times F$,
$$(\Phi_\alpha^{-1} \circ \Phi_\beta)(x, f) = (x, \Psi_{\alpha\beta}(x, f)).$$

<u>Obtain</u>
$$\begin{aligned} (x, \Psi_{\alpha\gamma}(x, f)) &= (\Phi_\alpha^{-1} \circ \Phi_\gamma)(x, f) \\ &= (\Phi_\alpha^{-1} \circ \Phi_\beta)(\Phi_\beta^{-1} \circ \Phi_\gamma)(x, f) \\ &= (x, \Psi_{\alpha\beta}(x, \Psi_{\beta\gamma}(x, f))). \end{aligned}$$

$\Rightarrow \forall \alpha,\beta,\gamma,\ x \in U_\alpha \cap U_\beta \cap U_\gamma$ and $f \in F$,

(5.2) $\quad \psi_{\alpha\beta}(x, \psi_{\beta\gamma}(x,f)) = \psi_{\alpha\gamma}(x,f)$.

"Cocycle condition"!

Also, $\psi_{\alpha\alpha} = \mathrm{id}_{U_\alpha}$.

Can view $\psi_{\alpha\beta}$ as a mapping $U_\alpha \cap U_\beta \to \mathrm{Diff}(F)$.

Call $\{\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to F\}$. <u>transition functions</u>.

<u>Point</u>. Given transition functions, we can <u>construct</u> a fibre bundle.

<u>Thm</u>. Let M & F be smooth mflds, $\{U_\alpha\}_{\alpha\in A}$ an open cover of M and $\{\psi_{\alpha\beta} : U_\alpha \cap U_\beta \times F \to F\}$. smooth s.t. $\psi_{\alpha\alpha} = \mathrm{id}\ U_\alpha$ and (5.2) holds. Then, $\exists!$ (up to isomorphism) fibre bundle $F \to E \xrightarrow{\pi} M$. with $\{\psi_{\alpha\beta}\}$ as transition functions.

<u>Pf</u>. Set $E = \left(\bigcup_{\alpha\in A} \{\alpha\} \times U_\alpha \times F\right)/\sim$.

where $(\alpha, x, f) \sim (\beta, \tilde{x}, \tilde{f}) \iff x = \tilde{x},\ \tilde{f} = \psi_{\beta\alpha}(x,f)$.

(5.2) $\Rightarrow$ symmetry & transitivity. $\psi_{\alpha\alpha} = \mathrm{id}\,U_\alpha \Rightarrow$ reflexivity.

So $\sim$ is equivalence relation.

Define $\pi : E \to M$. s.t. $\pi([\alpha,x,f]) = x$.

well defined: $[\alpha,x,f] = [\beta,\tilde{x},\tilde{f}] \Rightarrow \tilde{x} = x$.
$\Rightarrow \pi([\beta,\tilde{x},\tilde{f}]) = x$.

Defin for each $\alpha \in A$:

$$\Phi_\alpha : U_\alpha \times F \to \pi^{-1}(U_\alpha).$$

s.t. $(x,f) \mapsto [\alpha, x, f]$.

Give E the smallest top for $\{\Phi_\alpha\}_{\alpha \in A}$ cts.
("quotient top").

Note: The $\{\Phi_\alpha\}_{\alpha\in A}$ are bijective and we have

$$(\Phi_\alpha^{-1} \circ \Phi_\beta)(x,f) = \Phi_\alpha^{-1}([\beta, x, f]) = \Phi_\alpha^{-1}([\alpha, x, \psi_{\alpha\beta}(x,f)])$$

$$= (x, \psi_{\alpha\beta}(x,f)).$$

$\to \Phi_\alpha^{-1} \circ \Phi_\beta : (U_\alpha \times U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$.
Smooth.

Taking composition ~~of~~ with class of $M \times F$,

$\exists$ smooth structure on E w.r.t. which $\{\Phi_\alpha\}_{\alpha\in A}$ are smooth.

Uniqueness: Let $\tilde{E}$ be another such bundle with atlas $\{(U_\alpha, \tilde{\Phi}_\alpha)\}_{\alpha\in A}$. Define $\varphi : E \to \tilde{E}$.
s.t. $[\alpha, x, f] \mapsto \tilde{\Phi}_\alpha(x,f)$. $\to$ well defined! $\boxed{}$.

$$\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to \underbrace{\mathrm{Diff}(F)}_{\text{infinite dim gp}}.$$

wish to consider a smaller, finite-dim class of symmetries.

Def: A G-bundle ξ over M ~~consists~~ consists of.

- a smooth mfld M.
- a Lie group G, and

- an open cover $\{U_\alpha\}_{\alpha \in A}$ of M together with smooth mappings $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to G\}$ called co-cycles s.t. the cocycle condition

$$g_{\alpha\beta}(x)\, g_{\beta\gamma}(x) = g_{\alpha\gamma}(x) \quad \text{holds.}$$

$$\forall x \in U_\alpha \cap U_\beta \cap U_\gamma .$$

Lie group + group action ⇒ transformation group.

G-bundle + transformation group $G \circlearrowright F$.

⇒ fibre bundle with std fibre F.

<u>Examples of G-bundles</u>

- $G = \{e\}$, $U_\alpha = M$, $g_{\alpha\alpha} = e$ (M C^∞ mfld.)

  'trivial G-bundle'.

- Let M be smooth mfld with charts

$$\{(U_\alpha, \varphi_\alpha = (\overset{\alpha}{x}{}^1, \dots, \overset{\alpha}{x}{}^n))\}_{\alpha \in A} \quad \text{and}$$

define the matrix $g_{\alpha\beta}(x) = (g_{\alpha\beta}(x)_{ij})_{i,j=1}^{n}$ for,

$x \in U_\alpha \cap U_\beta$ s.t.

$$g_{\alpha\beta}(x)_{ij} = \partial_j(\overset{\alpha}{x}{}^i \circ \varphi_\beta^{-1})(\varphi_\beta(x)).$$

$$= (i,j)^{\text{th}} \text{ entry of Jacobian matrix of } \varphi_\alpha \circ \varphi_\beta^{-1} \text{ at } \varphi_\beta(x).$$

⇝ $g_{\alpha\beta}(x) \in GL(n, \mathbb{R})$. ⇝ obtain $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$.

These are cocycles by Chain rule.

$GL(n, \mathbb{R})$ bundle denoted by $\cancel{\text{TM}}\,\Gamma$.

• Let (M,g) be a Riem mfld, sps $\{U_\alpha\}_{\alpha\in A}$ cover M and over each x an o.n. frame $\{(\overset{\alpha}{e}_1,\dots,\overset{\alpha}{e}_n): U_\alpha \to (TM)^n\}_{\alpha\in A}$ w.r.t. $g = \langle\cdot,\cdot\rangle$.

Define for $\alpha,\beta\in A$, $(g_{\alpha\beta}(x)_{ij})$ s.t.

$$\overset{\alpha}{e}_i(x) = \sum_{j=1}^{n} g_{\alpha\beta}(x)_{ij}\,\overset{\beta}{e}_j(x) \iff g_{\alpha\beta}(x)_{ij} = \langle \overset{\alpha}{e}_i(x), \overset{\beta}{e}_j(x)\rangle.$$

(Ex) Show that this defines cocycle $\{g_{\alpha\beta}: U_\alpha\cap U_\beta \to O(n,\mathbb{R})\}_{\alpha,\beta\in A}$

$\rightsquigarrow$ an $O(n,\mathbb{R})$ bundle on M.