25/10/2013.

Recap

Introduced Lie groups:

- $(G, \cdot)$ C^∞ mfld s.t.

$$\mu_1 : G \times G \to G, \quad (g,h) \longmapsto g\cdot h.$$
$$\mu_2 : G \to G \quad g \longmapsto g^{-1}.$$

are smooth.

- Ex. $GL(n), O(n), SO(n), \dots$
- Transformation group on M (C^∞ mfld) $=$ $(G, \cdot)$ together with a left action. $\cdot : G \times M \to M$. s.t.

$$\forall g,h \in G, \ x \in M, \quad \begin{cases} g\cdot(h\cdot x) = (g\cdot h)\cdot x. \\ e\cdot x = x. \end{cases}$$

- Important ~~for~~ maps.

Left / right translation: $\forall g \in G$
$$\left.\begin{aligned} \lambda_g &: G \to G. \\ h &\mapsto g\cdot h. \\ \rho_g &: G \to G. \\ h &\mapsto \cancel{g}\, h\cdot g. \end{aligned}\right\} \text{Smooth.}$$

For actions:

$L_g : M \to M,$ $\quad g \mapsto g\cdot y$ $\quad$ (g fixed)

$R_x : G \to M.$ $\quad h \mapsto h\cdot x$ $\quad$ (x fixed)

- Infinitesimal transformations of a trans. group.

$$\mathfrak{g}_M := \{X_v : M \to TM \text{ with } v \in T_eG\}.$$

$$X_v(x) := d_e R_x(v).$$

- If the action is effective: $\forall x$ $g \cdot x = x$ ~~$\forall x$~~ $\Rightarrow g = e$.

Compared with free: ~~g~~ $\exists x$ $g \cdot x = x \Rightarrow g = e$.

- Effective $\Rightarrow$ $\dim \mathfrak{g}_M = \dim G =: r$.

$$\left.\begin{array}{l} v \mapsto X_v \\ T_eG \to \mathfrak{g}_M \end{array}\right\} \text{ v.s. isomorphism}.$$

- $\mathfrak{g}_G = \{$ right inv. v.f. on $G\}$ (action is gp mult).

$= \{ X : G \to TG$ s.t. $\forall g, h \in G$ ~~$d_g(X(h)) = X(h\cdot)$~~

$$d\rho_g (X(h)) = X(h \cdot g) \}.$$

$\rightsquigarrow (\mathfrak{g}_G, [\cdot,\cdot])$ is a Lie Algebra.

$G = GL(n) \Rightarrow [X_A, X_B] = BA - AB$ (opp. sign).

- Ex. $\mathfrak{g}_G \cong \mathfrak{g}^L_G$ $\left(= \{\text{left inv. v.f. on } G\}\right.$

as Lie Alg. $= \{X : M \to TM$ s.t. $\forall g, h \in G,$

$d\lambda_g(X(h)) = X(gh)\}$.

explicitly, $\varphi : \mathfrak{g}_G \to \mathfrak{g}^L_G$ $X \mapsto (g \mapsto d\rho_{i_2}(X(g^{-1})))$.

$T_eG \to \mathfrak{g}^L_G$ $v \mapsto X^L_v$ $X^L_v(g) = d_e\lambda_g(v)$.

Pf ($\Rightarrow$) $\sigma(0) = \sigma(0+0) = \sigma(0)^2 \Rightarrow \sigma(0) = e$.

OTOH, (✱) $\Rightarrow$ $\frac{d}{dt}\Big|_t \sigma(t) = \frac{d}{ds}\Big|_{s=0} \sigma(t+s) = \frac{d}{ds}\Big|_{s=0} \underbrace{\sigma(t)\sigma(s)}_{\rho_{\sigma(t)}(\sigma(s))}$

$= d_e \rho_{\sigma(t)}(\dot\sigma(0)) = X^L_{\dot\sigma(0)}(\sigma(t))$.

$\sigma(t+s) = \sigma(t)\sigma(s) \Leftrightarrow$ (✱) $\Rightarrow$ $\dot\sigma(t) = X^L_{\dot\sigma(0)}(\sigma(t))$.

($\Leftarrow$). Sps σ is an integral curve of

$$\begin{cases} \dot\sigma(t) = X_v(\sigma(t)) \\ \sigma(0) = e. \end{cases} \qquad v \in T_eG \text{ fixed.}$$

ODE $\Rightarrow$ σ exists and unique in time int. $(-\delta_1, \delta_2)$.

First show (✱). Fix s, s, t; $s+t \in (-\delta_1, \delta_2)$.

$\sigma_1(t) = \sigma(t+s)$, $\sigma_2(t) = \sigma(t)\sigma(s)$.

Note: $\sigma_1(0) = \sigma(s) = \sigma_2(0)$. and

$$\begin{cases} \dot\sigma_1(t) = \dot\sigma(t+s) = X_v(\sigma(t+s)) = X_v(\sigma_1(t)) \\ \dot\sigma_2(t) = \frac{d}{dt}\Big|_t \rho_{\sigma(s)}(\sigma(t)) = d\rho_{\sigma(s)}(\dot\sigma(t)) = d\rho_{\sigma(s)}(X_v(\sigma(t))) \end{cases}$$

$= X_v(\sigma(t)\sigma(s)) = X_v(\sigma_2(t))$

Want to show: $\delta_1, \delta_2 = \infty$ (either via contradiction or multiplication).

3 Lie groups & Lie algebras

$\forall g \in G$, $R_g, L_g : G \to G$, are diffeo

$\Rightarrow d_e L_g, d_e R_g : T_eG \xrightarrow{\cong} T_gG$ (diffeo).

$G \times T_eG \xrightarrow{\varphi} TG \quad (g,v) \mapsto d_eL_g(v)$.

'preserves fibres', i.e., $\varphi(\{g\} \times T_eG) = T_gG$.

__Prop__. (TG is trivial $\Leftarrow$ G is parallelisable)

__Def^n__. The Lie algebra $\mathfrak{g}$ of G is the v.s. T_eG together with the Lie bracket defined by $[v,w] := [X_v^L, X_w^L](e) = -[X_v, X_w](e)$, $\forall v,w \in T_eG$.

One parameter subgroups

__Def^n__. Smooth curve $\sigma : \mathbb{R} \to G$ is a one-parameter subgroup of G if $\forall t,s \in \mathbb{R}$. $\sigma(s+t) = \sigma(s)\sigma(t)$

__Th^m__. σ is a one parameter subgroup iff it is the integral curve of an invariant v.f. passing through e.

Sps $\delta_2 < \infty$. Consider $\sigma_+ : \left(\frac{\delta_2}{2} - \delta_1, \frac{3\delta_2}{2}\right) \to G$.

st. $t \mapsto \sigma\left(t - \frac{\delta_2}{2}\right) \cdot \sigma\left(\frac{\delta_2}{2}\right)$.

We have for $t \in \left(\frac{\delta_2}{2} - \delta_1, \delta_2\right)$ $\quad \sigma_+(t) = \sigma\left(t - \frac{\delta_2}{2}\right) \cdot \sigma\left(\frac{\delta_2}{2}\right)$

$\overset{*}{=} \sigma(t)$.

Also, $\dot{\sigma}_+(t) = d\rho_{\sigma\left(\frac{\delta_2}{2}\right)}\left(X_v\left(\sigma\left(t - \frac{\delta_2}{2}\right)\right)\right) = X_v(\sigma_+(t))$

$\tilde{\sigma} : \left(-\delta_1, \frac{3\delta_2}{2}\right) \to G$ extends $t \mapsto \begin{cases} \sigma(t) & t < \delta_2 \\ \sigma_+(t) & \delta_2 < t < \frac{3\delta_2}{2} \end{cases}$

contradicting maximality.

Now sps $\delta_1 < \infty$. Define $\sigma_- : (-\infty, \delta_1) \to G$, $t \mapsto \sigma(-t)^{-1}$.

(Note, $e = \sigma(0) = \sigma(t) \cdot \sigma(-t) \Rightarrow \sigma(t) = \sigma(-t)^{-1}$).

for $t \in (-\delta_1, \delta_1)$ $\quad \sigma_-(t) = \sigma(t)$. and

$\dot{\sigma}_-(t) = -d\mu_2(\dot{\sigma}(-t)) = -d\mu_2(X_v(\sigma(-t))) = -d\mu_2(d\rho_{\sigma(-t)} \cdot (v))$

$= -d(\mu_2 \circ \rho_{\sigma(t)}) = -d\mu_2(d_{\sigma(-t)}(v)) = d_e\lambda_{\sigma(-t)^{-1}}(d_e\mu_2(v))$

$= d_e\lambda_{\sigma_-(t)}(v) = X_v^L(\sigma_-(t))$.

Also integral one of right inv. system since $\sigma_-(0) = \sigma(0) = e$.

$\tilde{\sigma} : (-\infty, \infty) \to G \quad t \mapsto \begin{cases} \sigma_-(t) & t < 0 \\ \sigma(t) & t \geq 0 \end{cases}$

extends σ to $\mathbb{R}$. $\boxtimes$

Ex. $G = GL(n)$ (or any of its subgroups!).

The one-parameter subgroups are sol'ns to

$$\begin{cases} \dot\sigma^{ij}(t) = \sum_{k=1}^{n} \sigma^{ik}(t)\, v^{kj}. \\ \sigma^{ij}(0) = \delta^{ij} \end{cases} \qquad V = \sum_{i,j=1}^{n} v^{ij}\, \partial_{ij}\big|_I \in T_e GL(n) \cong M_{n\times n}(\mathbb{R})$$

basis matrices.

We know $t \mapsto \exp(tV)$ solves this system where for $X \in M_{n\times n}(\mathbb{R})$, $\exp(X) := \sum_{i=0}^{\infty} \frac{X^i}{i!}$ ☐

Motivates the following.

Defⁿ The exp map $\exp : \mathfrak{g} \to G$ is defined s.t. $\exp(v) = \sigma(1)$ where σ solves

$$\begin{cases} \dot\sigma(t) = X_v(\sigma(t)) \\ \sigma(0) = e \end{cases}$$

with $v \in \mathfrak{g} = T_e G$.

Exp is smooth by ode theory (smooth dep. on parameters).

We may rename σ for exp, ie, $\sigma(t) = \exp(tv)$.

Pf. Look at $t \xrightarrow{\gamma} \sigma(st) \rightsquigarrow \gamma(0) = e$,

$$\dot\gamma(t) = s \cdot \dot\sigma(st) = s \cdot X_v(\sigma(st)) = X_{sv}(\gamma(t))$$

$\Rightarrow \gamma(1) = \exp(sv)$, but $\gamma(1) = \sigma(st)\big|_{t=1} = \sigma(s)$ ☐.

• $h \in H \Rightarrow h = g_1 \cdots g_l \quad , \; g_1, \dots, g_l \in V$

$h^{-1} = g_l^{-1} \cdots g_1^{-1} \in V^{-1} = (U \cap U^{-1}) = U \cap U^{-1} \subset H.$

② V is an open nbh of e, $\rightsquigarrow \lambda_h(V)$ is also open for any $h \in H$ and $\lambda_h(V) \subset H \Rightarrow H$ is open.

③ Recall: $G_e = \bigcup_{g \in G_e} \underbrace{g \cdot H}_{\lambda_g(H)}$, can write

can write $H = \underbrace{G_e \setminus \Big(\underbrace{\bigcup_{g \in G_e \setminus H} g \cdot H}_{\text{open}} \Big)}_{\text{closed}}$.

$\Rightarrow H$ is open & closed in G_e and $e \in H \Rightarrow H \neq \emptyset$.

$\Rightarrow H = G_e$. ∎

Prop: exp is a local diffeo. in a nbhd of 0.

Pf. $v \in \mathfrak{g}$, $\frac{d}{dt}\Big|_{t=0} \exp(tv) = X_v(e) = v$

$\Rightarrow$ $d_0 \exp : T_0\mathfrak{g} \to \mathfrak{g}$ isomorphism.

IFT gives claim $\square$

Thm Let G_e be the connected component of G containing e.

Sps $\mathcal{U}$ is an open nbhd of e in G_e.

Then G_e is a Lie subgroup of G, and

$$G_e = \bigcup_{i=1}^{\infty} \mathcal{U}^i \quad , \quad \mathcal{U}^i = \{g_1 \cdots g_i : g_1, \ldots, g_i \in \mathcal{U}\}.$$

Pf G_e open by defn. $\Rightarrow$ open submfld.

But need to show closed under μ_1, μ_2.

Note μ_2 cts $\Rightarrow$ $\mu_2(G_e)$ conn'd. and $e \in \mu_2(G_e)$

$\Rightarrow$ $\mu_2(G_e) \subset G_e$.

Same $G_e \times G_e$ connected $\mu_1(G_e \times G_e) \subset G_e$.

Clearly $\bigcup_{i=1}^{\infty} \mathcal{U}^i \subset G_e$ since $\nearrow$.

"$\supset$". Let $V = \mathcal{U} \cap \mu_2(\mathcal{U})$.

Consider $H = \bigcup_{i=1}^{\infty} V^i \subset \bigcup_{i=1}^{\infty} \mathcal{U}^i \subset G_e$.

① H is a subgroup of G_e

⑦

Last lecture 25/10/2017.

- Introduced smooth map $\exp\ \mathfrak{g} \to G$
  $\left([v,w] := [X_v^L, X_w^L](e) \right)$

  s.t. $t \mapsto \exp(tv)$ satisfies

$$\left|\begin{array}{ll} \dot\sigma(t) = X_v(\sigma(t)), & \sigma(0) = e \\ \dot\sigma(t) = X_v^L(\sigma(t)), & \sigma(0) = e \end{array}\right\} \Leftrightarrow \sigma(t+s) = \sigma(t)\sigma(s).$$

- $\exists\ \mathcal{U} \subset \mathfrak{g}$, $V \subset G$ open, $0 \in \mathcal{U}$, $e \in V$ s.t.
  $\exp|_{\mathcal{U}} : \mathcal{U} \to V$ diffeo.

- G connected $\Rightarrow$ $G = \bigcup_{i=1}^{\infty} V^i$, where $V^i = \{g_1 \cdots g_i ;\ g_i \in V\}$.

<u>Now</u>: Suppose $\varphi : H \to G$ is a Lie group homom,

is $\varphi \circ \exp_H = \exp_G \circ d_{e_H}\varphi$?

<u>Lem</u>. φ as above, $\exp_G : \mathfrak{g} \to G$, $\exp_H : \mathfrak{h} \to H$. Then,

- $\varphi \circ \exp_H = \exp_G(d_{e_H}\varphi)$.
- If $\varphi : H \to G$ is an embedding, then
  $\exp_G(d\varphi(\mathfrak{h})) \subset \varphi(H)$.

  Conversely, if $\exp_G(tv) \subset \varphi(H)$. $\forall t \in (a,b)$.
  then $v \in d_e\varphi(\mathfrak{h})$.

Pf. Claim 1: Consider $\sigma_1, \sigma_2 : \mathbb{R} \to G$ s.t.

$$\sigma_1(t) = \varphi(\exp_H(tv)), \quad \sigma_2(t) = \exp_G(d_{e_H}\varphi(tv)).$$

where $v \in \mathfrak{h}$ fixed.

Have $\sigma_1(0) = e_G = \sigma_2(0)$.

$$\sigma_1(t+s) = \varphi(\exp_H((t+s)v)) = \varphi(\exp_H(tv)\exp_H(sv))$$

$$= \varphi(\exp_H(tv)) \cdot \varphi(\exp_H(sv)).$$

$$= \sigma_1(t)\sigma_1(s).$$

Similarly, $\sigma_2(t+s) = \sigma_2(t)\sigma_2(s)$.

Finally,
$$\begin{cases} \dot\sigma_1(0) = d\varphi\left(\frac{d}{dt}\Big|_{t=0} \exp_H(tv)\right) = d\varphi(v). \\ \dot\sigma_2(0) = \frac{d}{dt}\Big|_{t=0} \exp_G(t\, d_{e_H}\varphi(v)) = d\varphi(v) \end{cases}$$

$\Rightarrow \sigma_1(t) = \sigma_2(t) \quad \forall t.$

Claim 2: first part. claim 1 $\Rightarrow$ ~~$v\in\mathfrak{h}$~~ $\Rightarrow$ ~~$\exp_G(d\varphi(v))$~~.

$v \in \mathfrak{h} \Rightarrow \exp_G(d\varphi(v)) = \varphi\exp_H(v) \in \varphi(H)$.

Second part: Sps $t \overset{\gamma}{\mapsto} \exp_G(tv) \in \varphi(H)$. $\forall t \in (a,b)$.

$\Rightarrow \gamma(t) = \varphi(\sigma(t))$ for some smooth $\sigma : (a,b) \to H$.

Both γ & σ satisfy semigroup prop.

Can be extended to maps $\mathbb{R} \to G$ and $\mathbb{R} \to H$.

with $\gamma(t) = \varphi(\sigma(t)) = \varphi(\exp_H(tv))$

$\frac{d}{dt}\Big|_{t=0} \longrightarrow \dot\gamma(0) = d\varphi(v)$.

$\blacksquare$

$\underline{\text{Th}^{\underline{m}}}$. Sps $\mathfrak{h}$ is a Lie subalgebra of the Lie algebra $\mathfrak{g}$ of a Lie group G. Then, $\exists!$ connected Lie subgroup H of G s.t. $\mathfrak{h}$ is the Lie Algebra of H.

$\underline{\text{Pf}}$. Define the $\underline{r}$-dimensional distribution

$$D_g := d_e \lambda_g(\mathfrak{h}) \quad \forall g \in G.$$

($\dim \mathfrak{h} = r$).

This is a distribution because for a basis $\{e_i\}_{i=1}^{r}$ of $\mathfrak{h}$.

$\{X^L_{e_i}(g)\}_{i=1}^{r}$ span D_g, $g \in G$.

Also, $[X^L_{e_i}, X^L_{e_j}](g) = d_e\lambda_g([X^L_{e_i}, X^L_{e_j}](e)) = d_e\lambda_g([e_i, e_j]) \in D_g$.

$\rightsquigarrow$ $\exists!$ maximal connected submfld H of G 'integrating' D and passing thru e. (union over all submflds given by Frobenius).

Need to show: $g_1, g_2 \in H$, $g_1^{-1}g_2 \in H$.

Consider $g_1^{-1}H \subset G$, submfld passing thru e.

Uniqueness $g_1^{-1}H = H$.

Also, writing $i : H \to G$, $h \mapsto g^{-1}h$, we have

$$di(T_hH) = d\lambda_{g^{-1}}(T_hH) = d\lambda_{g^{-1}}(D_h) = D_{g^{-1}h} = D_{i(h)}.$$

By uniqueness $g_1^{-1}H = H \Rightarrow g_1^{-1}g_2 \in H$. $\square$

Thm Let H be an (algebraic) subgroup of a Lie group G with H closed. Then H is a Lie subgroup.

Rmk H is just a subgroup, but not necessarily a smooth mfld.

Pf. Fix an inner product on $\mathfrak{g}$. If H were a Lie subgroup, then $\mathfrak{h} = \{\dot\sigma(0) : \sigma : \mathbb{R} \xrightarrow{C^\infty} G$ with $\sigma(t) \in H \ \forall t \in \mathbb{R}$, $\sigma(0) = e\}$.

We would also expect $\mathfrak{h} \overset{?}{=} \{v \in \mathfrak{g} : \exp_G(tv) \in H \ \forall t\}$.

Define $\tilde{\mathfrak{h}} :=$ $\{v \in \mathfrak{g} : \exp_G(tv) \in H \ \forall t\}$.

First of all: $\mathfrak{h}$ is a v.s.; if $\dot\sigma_1(0), \dot\sigma_2(0) \in \mathfrak{h}$, then

$$\frac{d}{dt}\Big|_{t=0} (\sigma_1(t) \cdot \sigma_2(\alpha t)) \in h$$
$$\parallel$$
$$\dot\sigma_1(0) + \alpha \dot\sigma_2(t).$$

Secondly, $\tilde{\mathfrak{h}} \subset \mathfrak{h}$. Show the following: If $t_n \downarrow 0$ in $\mathbb{R}$, $X_n \to X$ in $\mathfrak{g}$ and $\exp(t_n X_n) \in H \ \forall n$, then $\exp(tX) \in H \ \forall t$.

Fix a sequence $\{m_n\}_{n=1}^\infty \subset \mathbb{Z}$ s.t. $m_n t_n \to t$ as $n \to \infty$.

Then, $\exp_G(tX) = \lim_{n\to\infty} \exp(m_n t_n X_n) = \lim_{n\to\infty} \underbrace{\exp(t_n X_n)}_{\in H}{}^{m_n} \in H$.

↑ closedness.

Define the curve: $v : (-\delta, \delta) \to \mathfrak{g}$ s.t. $v(t) = \exp_G^{-1}(\sigma(t))$ where $\dot\sigma(0) \in \mathfrak{h}$ and $\delta > 0$ small.

but $t_n = |X_n| \to 0$, $Y_n \to Y$. $\exp(t_n Y_n) = \exp(X_n) \in H$.

$\Rightarrow \forall t \in \mathbb{R}$ $\exp(tY) \in H$.

$\Rightarrow Y \in \tilde{\mathfrak{h}} = \mathfrak{h}$ but $Y \in \mathfrak{k} \Rightarrow$ $\mathfrak{h} \cap \mathfrak{k} = \{0\}$ and $Y \neq 0$ contradiction!

Since F is a diffeo in nbh of $(0,0)$.

$\exists$ $\underset{\overset{\Uparrow}{0}}{\mathcal{U}} \subset \mathfrak{h}$, $\underset{\overset{\Uparrow}{0}}{V} \subset \mathfrak{k}$, $\underset{\overset{\Uparrow}{e}}{W} \subset G$. s.t. $F : \mathcal{U} \times V \to W$.

is a diffeo. Also, we assume $\exp(V) \cap H = \{e\}$.

Thus, if $F(v,w) = h \in H \Rightarrow \exp(v)\exp(w) = h$.

$\exp(w) = \underbrace{\exp(v)^{-1}}_{\in H} \underset{\overset{\Uparrow}{H}}{h}$. $\Rightarrow \exp(w) \in H$ for $w \in V$.

$\Rightarrow w = 0$.

and for $v \in \mathcal{U} \subset \mathfrak{h}$, $F(v,0) = \exp(v) \in H$.

$\Rightarrow$ $F(\mathcal{U} \times \{0\}) = W \cap H$. This gives us a local parametrisation of G in nbh of $\{e\}$.

We obtain adapted param at other $h \in H \subset G$

by considering $F_h := l_h \circ F$. ∎

Note that $\dot{v}(0) = \frac{d}{dt}\Big|_{t=0} \exp_G^{-1}(\sigma(t)) = \dot{\sigma}(0)$.

But $\dot{v}(0) = \lim_{n\to\infty} n \cdot v(\frac{1}{n})$. Let $t_n = \frac{1}{n}$, $X_n = n \cdot v(\frac{1}{n})$.

Then $\exp(t_n X_n) = \exp(v(\frac{1}{n})) = \sigma(\frac{1}{n}) \in H$.

$\Rightarrow$ By Lemma, since $X_n \xrightarrow{n\to\infty} X := \dot{\sigma}(0)$ in $\mathfrak{g}$.

we have that $\exp(t\dot{\sigma}(0)) \in H$ $\forall t \in \mathbb{R}$.

$\Rightarrow \dot{\sigma}(0) \in \tilde{\mathfrak{h}} \Rightarrow \mathfrak{h} = \tilde{\mathfrak{h}}$.

<u>Thirdly</u>, let $\mathfrak{k}$ be a vec. subspace of $\mathfrak{g}$ s.t.

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}.$$

and consider $F: \mathfrak{h} \times \mathfrak{k} \to G$. $(v, w) \mapsto \exp_G(v) \cdot \exp_G(w)$.

Note if σ_1, σ_2 are two curves in $\mathfrak{h}, \mathfrak{k}$ with $\sigma_1(0) = \sigma_2(0) = 0$

then $\frac{d}{dt}\Big|_{t=0} F(\sigma_1(t), \sigma_2(t)) = \dot{\sigma}_1(0) + \dot{\sigma}_2(0) \in \mathfrak{g}$.

$\Rightarrow$ F is a diffeo in nbh of $(0,0)$.

Want: $W \subset G$ open, $\mathcal{U} \subset \mathfrak{h}$ open s.t. $F(\mathcal{U} \times \{0\}) = H \cap W$.

Claim: $\exists V \subset \mathfrak{k}$ open, $0 \in V$ s.t. $\exp(V) \cap H = \{e\}$.

Pf. $\exists$ sequence $\{X_n\} \subset \mathfrak{k}$ s.t. $X_n \neq 0$, $X_n \xrightarrow{n\to\infty} 0$ and $\exp(X_n) \in H$.

consider $Y_n := \frac{X_n}{|X_n|}$, w.l.o.g. $Y_n \xrightarrow{n\to\infty} Y$ with $|Y| = 1$.

Ex. $O(n) = \{A \in GL(n) \cdot A^T A = I\}$ is closed.

$\rightsquigarrow$ Lie subgroup.

Ex. $\varphi: G \to H$. homom of Lie groups, then $\ker \varphi$ is a Lie subgroup of G and $T_e(\ker \varphi) = \ker d_{e_G} \varphi$.