S. Fackler! 27/05/2016.

$V \hookrightarrow H$ two Hilbert spaces.

$a: V \times V \to \mathbb{C}$. sesquilinear form.

$\mathrm{Re}\, a(v,v) \geq \alpha \|v\|_V^2$

$|a(u,v)| \leq M \|u\|_V \|v\|_V \quad \forall u, v \in V.$

$\exists$ unbounded $B: \mathcal{D}(B) \to H$.

$$\begin{cases} \mathcal{D}(B) = \{ v \in H : a(u,v) = \langle f, v \rangle_H \quad \forall v \in H \}. \\ Bu = f. \end{cases}$$

Ex. $A: \Omega \to \mathbb{R}^{n \times n}$. $\sum_{j,i=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \alpha |\xi|^2$, & bdd dom.

$V = H_0^1(\Omega)$, $H = L^2$. $a(u,v) = \int_\Omega A \nabla u \cdot \overline{\nabla v}$ $L \sim -\mathrm{div}(A\nabla)$.

NOTE: coefficients, real, but rough.

L_p - L^p realisation.

Facts about L_p: $p \in (1, \infty)$.

(I) L_p has a bdd H^∞-calculus.

(II) L_p has maximal L^q-regularity,

v solution to $v_t + L_p v = f$, $v(0) = 0$ satisfies.

$\|v_t\|_{L^q((0,T), L^p)} + \|L_p v\|_{L^p(L^q)} \lesssim \|f\|_{L^q((0,T), L^p)}$.

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For this, one considers the singular integral

$$Kf = \int_0^t L_p T(t-s) f(s)\, ds, \quad \|L_p T(t)\| \lesssim t^{-1}$$

if $A = L_p$ generates analytic semigroup.

<u>How to show this?</u>

(I) Harmonic analysis: off-diag. estimates for e^{-tL_p} & domination by maximal functions.

(II). Op. theory: Fendler's dilation theorem & abstract transference principle of Coifmann-Weis.

$S(t): L^p(\mathbb{R}) \to L^p(\mathbb{R})$ shift op.

$(S(t)f)(x) = f(x-t)$.

$$\left\| \int_0^\infty b(t) e^{-tL_p} dt \right\| \le \left\| \int_0^\infty b(t) S(t)\, dt \right\|. \quad (*)$$

usual convolution theorem.

This holds in the following abstract setting: $e^{-tL_p} \sim T(t)$

(I) $\|e^{-tL_p}\| \le 1$, (II) $e^{-tL_p} \ge 0$ & $f \ge 0 \Rightarrow e^{-tL_p} f \ge 0$.

Check:

(I). Nittka's criterion. $B^p = \{v \in L^2 : \|v\|_{L^p} \le 1\}$

$P: L^2 \to B^p$ orth. proj provided $PV \subset V$ &

$a(v, |v|^{p-2} v) \ge 0 \quad \forall v \in V \quad |v|^{p-2} v \in V$.

(II) Special case of Ombuba's criterion.

Open problem: Does (*). holds if one removes
(I) positivity (II) contractivity.

Remark (II): Matsaev's conjecture $\Rightarrow$ (*) still holds.

$S: \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$. $Sx_n = x_{n-1}$. $T: L^p \to L^p$, $\|T\| \leq 1$

$\|P(T)\| \leq M \|P(S)\|$ $\forall P$ polynomial

(open since 60s).

L^p is difficult, tried $L^p(L^q)$ $p \neq q$.