Sylvie Monniaux 26/05/2016.

First-order approach to L^p estimates for the Stokes operator on Lipschitz domains. (w/ Alan McIntosh).

(VWL) $\Omega \subset \mathbb{R}^3$. $\Omega = \bigcup_{i=1}^N \underbrace{\rho_i(B)}_{\Omega_j}$, B unit ball,

ρ_i bilipschitz.

$\chi_j : \Omega_j \to [0,1]$, $\operatorname{spt} \chi_j \subset \Omega_j$, $\sum \chi_j = 1$.

Motivation: Domains with cuts.

d: exterior derivative ; $\underline{\delta} = d^*$, $D_H = d + \underline{\delta}$.

no bdy conditions here, (i.e., d closure of C_c^∞). bar means bdy conditions.

$\Lambda = \bigoplus_j \Lambda^j$, $d: 0 \to L^p(\Omega, \Lambda^0) \xrightarrow{\nabla} L^p(\Omega, \Lambda^1) \xrightarrow{\text{curl}}$

$\to L^p(\Omega, \Lambda^2) \xrightarrow{\text{div}} L^p(\Omega, \Lambda^3) \to 0$

$\underline{\delta}$: goes other way.

$0 \to L^p(\Omega, \Lambda^3) \xrightarrow{-\nabla} L^p(\Omega, \Lambda^2) \xrightarrow{\text{curl}} L^p(\Omega, \Lambda^1) \xrightarrow{\text{div}}$
$L^p(\Omega, \Lambda^0) \to 0$

What is the bdy condition?

$\underline{\delta} u_3 = -\nabla u_3$, $u_3 = 0$ on $\partial\Omega$.

$\underline{\delta} u_2 = \text{curl}\, u_2$, $\nu_n u_2 = 0$ on $\partial\Omega$.

$\underline{\delta} u_1 = -\text{div}\, u_1$, $\nu \cdot u_1 = 0$ on $\partial\Omega$.

$D_H^2 = d\underline{\delta} + \underline{\delta}d = -\Delta_H$ Hodge Laplacian.

D_H^2	Λ^0	Λ^1	Λ^2	Λ^3
	$-\Delta_{\text{Neumann}}$	$\underline{\text{curl}}\,\text{curl} - \nabla \underline{\text{div}} = -\Delta$; $\nu \cdot n = 0$, $\nu \times \text{curl}\, n = 0$ on $\partial\Omega$	$\text{curl}\,\underline{\text{curl}} - \underline{\nabla}\,\text{div} = -\Delta$; $\nu \times n = 0$, $\text{div}\, n = 0$ on $\partial\Omega$	$-\Delta_{\text{Dirichlet}}$

operator to consider: on $N(\underline{\text{div}})$.

M. Mitrea & S.M. 2009: $-\Delta_H$ is sectorial,

$$\|(\lambda I - \Delta_H)^{-1}\| \lesssim \frac{1}{|\lambda|}.$$

But this is for Ω strongly Lipschitz, $p \in (3-\varepsilon, 3+\varepsilon)$

S. Hofmann, M. Mitrea, S.M. 2011.

$d(-\Delta_H)^{-\frac{1}{2}}$, $\underline{\delta}(-\Delta_H)^{-\frac{1}{2}}$ hold in $L^p(\Omega,\Lambda)$

for $p \in (3-\varepsilon, 3+\varepsilon)$.

Hodge decomposition. $L^2(\Omega,\Lambda) = \mathcal{R}^2(d) \oplus \mathcal{R}^2(\underline{\delta}) \oplus \Big(\underbrace{\mathcal{N}^2(d) \cap \mathcal{N}^2(\underline{\delta})}_{\text{finite dim.}}\Big)$

D_H is bisectorial in $L^2(\Omega,\Lambda)$.

+ off diagonal decay: $\|\chi_E (I - zD_H)^{-1}\chi_F\|_{\mathcal{L}(L^2(\Omega,\Lambda))} \leq C e^{-\alpha \frac{d(E,F)}{|z|}}$.

$E, F \subset \Omega$, bnd.

Potential operators $\quad 1 < p < \infty$

$\frac{1}{p^s} = \frac{1}{p} - \frac{1}{3}$. $\quad \uparrow$ dim

Poincare-type: $\exists R, k : L^p(\Omega, \Lambda) \to L^{p^s}(\Omega, \Lambda)$.

s.t. $dR + Rd = I - k$, $\quad k$ compact in $L^p(\Omega, \Lambda)$

and if $u \in \mathcal{R}^p(d)$, $\quad u = dRu$.

Bogovskii-type: $\quad \exists\ T, L : L^p(\Omega, \Lambda) \to L^{p^s}(\Omega, \Lambda)$.

s.t. $\underline{\delta} T + T \underline{\delta} = I - L$, $\quad L$ compact in $L^p(\Omega, \Lambda)$

and if $u \in \mathcal{R}^p(\underline{\delta})$, then $u = \underline{\delta} T u$.

So $\mathcal{R}^p(d)$ and $\mathcal{R}^p(\underline{\delta})$ form an interpolation scale

~~Sneiberg~~ $\Rightarrow$

$\Rightarrow$ So by Sneiberg $\Rightarrow$. $\exists\ p_H < 2 < p^H$ s.t.

$\forall p \in (p_H, p^H)$: $\quad L^p(\Omega, \Lambda) = \mathcal{R}^p(d) \oplus \mathcal{R}^p(\underline{\delta}) \oplus N^p(D_H)$

Thm $\quad \forall p \in \max(\{1, (p_H)_s\}, p^H) \qquad \frac{1}{p_s} = \frac{1}{p} + \frac{1}{3} \leftarrow$ dim.

$\forall u \in \mathcal{R}^p(d) \oplus \mathcal{R}^p(\underline{\delta}) \oplus N^p(D_H)$. $\forall f \in H^\infty(\Sigma_\theta)$

$\| f(D_H) u \|_p \lesssim \|u\|_p$.

Corollary $\quad \delta = -\Delta_H \big|_{\mathcal{R}^p(\underline{\mathrm{div}})}$ has a H^∞ f.c. for p as before.

Pf of Resolvent estimate:

$2 \longrightarrow 2_s = 6/5$. , $u \in \mathcal{R}^{6/5}(\underline{S})$. $u = \underline{S}Tu$.

Want: $\|(I - zD_H)^{-1} u\|_{6/5} \lesssim \|u\|$.

$Q_k = Q(x_k, |z|)$. $\bigcup_{\neq} Q_k = \mathbb{R}^3$, $\eta_k = 1$ on Q_k,

spt $\eta_k \subset Q(x_k, 2|z|)$. $\sum_k \eta_k^2 = 1$.

$$u = \sum_k \underbrace{\left(\underline{S}(\eta_k T(\eta_k u)) + \frac{1}{|z|} v_k \right)}_{\eta_k^2 u}, \quad v_k = \nabla\eta_k \lrcorner T(\eta_k u) \pm \eta_k L(\eta_k u) \pm \eta_k T(\nabla\eta_k \lrcorner u)$$

But then calculate, not that $T: L^p \to L^{p_s}$ or $L^2 \not\to L^{2s}$.

$\|w_k\|_2$, $\|v_k\|_2 \lesssim |z| \, \| \tfrac{1}{2} \eta_k u \|_{6/5}$.