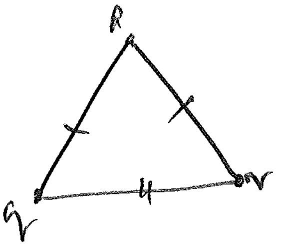
Galaz-Garcia : 3 dim Alex spaces with pos & non-neg curvature. 29/08/2016.

**Def.** Path connected met space, *length*: $d(x,y) = \inf_{r} L(r)$.

Model spaces: $S^2_k$ : 2-dim, simply connected Riem mflds. with sec curvature $= k$.

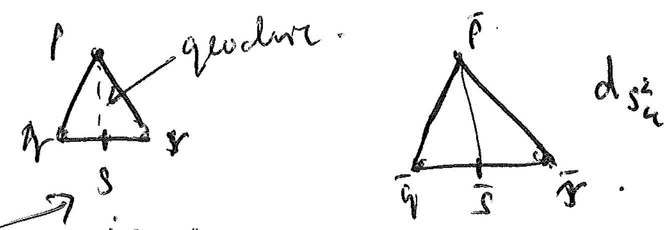
$$= \begin{cases} \mathbb{R}^2 & \text{Euclidean space } k=0. \\ S^2(k) & k>0 \text{ round 2-sphere}. \\ \mathbb{H}(k) & k<0 \text{ hyp}. \end{cases}$$

Triangle in $(X,d)$ consist of three distinct points $p,q,r$. and geodesics $[pq]$, $[pr]$, $[qr]$. $\Delta_{pqr}$.




Comparison triangle consist of 3 points. $\bar{p}, \bar{q}, \bar{r} \in S^2_k$.
s.t. lengths $[\bar{p}\bar{q}]$, $[\bar{p}\bar{r}]$ and $[\bar{q}\bar{r}]$ are the same as ~~...~~ $\Delta$ without bar

(Property $T_k$):




$d_{S^2_k}$.

$\forall s \longrightarrow$

$d(p,s) \geq d_{S^2_k}(\bar{p},\bar{s})$

**Def.** A complete, locally cpct length space with finite Haus-dim. is an Alex space with curvature bdd below by $k$ if $\forall p \in X$ $\exists$ nbhd $U_p$ where prop $T_k$ is satisfied $\forall$ triples $p,q,r$.

Examples: • $(M,g)$ with $\sec \geq k$.

• Quotients. $X \in \mathrm{Alex}^n(k)$, $\Gamma \curvearrowright X$ isom., closed orbits, $X/\Gamma \in \mathrm{Alex}(k)$.

Some important properties:

• Geodesics do not bifurcate.


• Globalisation (Top Th) holds globally.

• Define angles b/w geodesics,

Tangent directions = geodesics/angle zero $(\Sigma_p', \measuredangle)$.

Completion of $(\Sigma_p', \measuredangle)$ is $(\Sigma_p, \measuredangle)$. is the directions of $X$ at $p$.

Need completion:


$D^2 \subset \mathbb{R}^2$.

Th$^m$ (Burago, Gromov, Perelman)

$X \in \mathrm{Alex}^n(k) \Rightarrow \Sigma_p X \in \mathrm{Alex}^{n-1}(1)$.

Th$^m$ (Conical nbhd Th$^m$)

$\forall p \in X$ has nbhd that is ~~flat~~ pointed homeomorphic to the cone $C(\Sigma_p)$.

Important goal: classify Alex spaces with curv $\geq 1$.

Fact: 1- and 2-dim. ~~≈~~ homeomorphic to top. mflds.

dim 1: $X^1 \cong S^1$.
dim 2: $X^2 \simeq$ surface, curv $\geq 1 \Rightarrow |\pi_1(X)| < \infty$.
$\Rightarrow X \cong S^2$ or $\mathbb{RP}^2$.

Assume $\partial X = \emptyset$ and compact. $\partial X$ can be defined for these spaces.

<u>Question</u>: What are the closed 3-D Alex spaces with curv $\geq 1$.

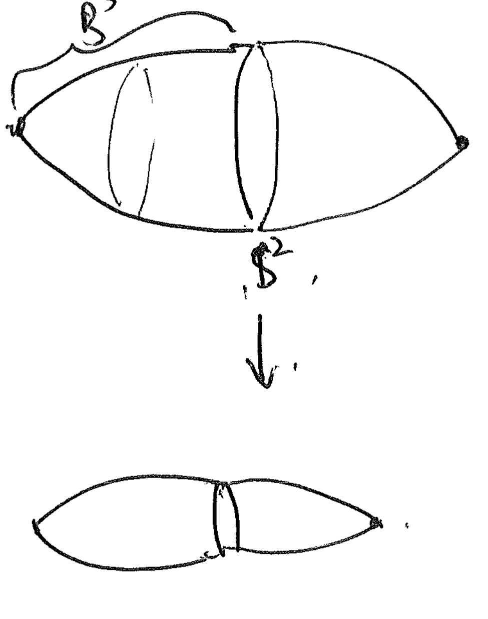
<u>Top</u>: $X^3$ closed and is <u>not</u> a 3-mfld.

$\Rightarrow$ • $\exists\, p \in X^3$ with $\Sigma_p = \mathbb{RP}^2$.

• By compactness, only finitely many $p$'s, isolated pts.

• $X^3 = X_0 \cup \bigcup_{i=1}^{k} C(\mathbb{RP}^2)$.
$\uparrow$ non-orientable mfld with bdry. $k$ copies of $\mathbb{RP}^2$.

• $\exists\, Y^3$ orientable and an orientation reversing involution $i: Y \to Y$ s.t. $X = Y/i$.



$\simeq S^3$.

$\cong$ susp$(\mathbb{RP}^2)$.

<u>Th$^m$</u> A w/ Grujano 2014, (H-S) Susp$(\mathbb{RP}^2)$ only non-mfld Alex space with curv $\geq 1$.

Th$^m$ B Classification of 3D-closed Alex. with curv $\geq 0$.

<u>Th$^m$</u> Th$^m$ A true with $CD^*(2,3)$ and B with $CD^*(0,3)$.