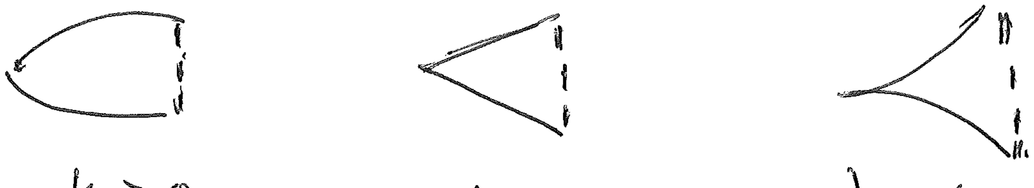
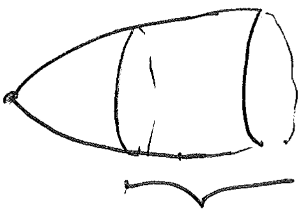
Kell — Sectional curvature-type conditions on metric spaces. 31/08/2016.



$k > 0$ $\qquad$ $k = 0$ $\qquad$ $k < 0$

¦ = "Riemann" ie "triangle comparisons".

Ric ≥ 0.:



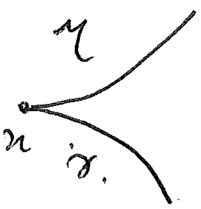
measure contraction property.

Finsler model: "sectional curv" $\not\Rightarrow$ Ric ≥ 0.

Motivation: figure out so $\Rightarrow$ holds.

Busemann: 1st Finsler approach to "sectional" curvature. "Busemann convexity".

$\eta, \gamma \in Geo_{[0,1]}(M,d)$.



$d(\eta_t, \gamma_t) \leq t\, d(\eta_1, \gamma_1)$.

2nd approach "convexity" of good balls.
CAT(0) inequality.

$$d^2(x,m)^2 \leq \frac{1}{2} d(x,y)^2 + \frac{1}{2} d(x,z)^2 - \frac{1}{4} d(y,z).$$

This is very "Hilbertian" → smells Riem.
"uniform convexity".

~~Getting~~
Getting inspiration from B-space people.

$$d(x,m)^p \le \tfrac{1}{2} d(x,y)^p + \tfrac{1}{2} d(x,z)^p - \tfrac{C}{4} d(y,z)^p .$$

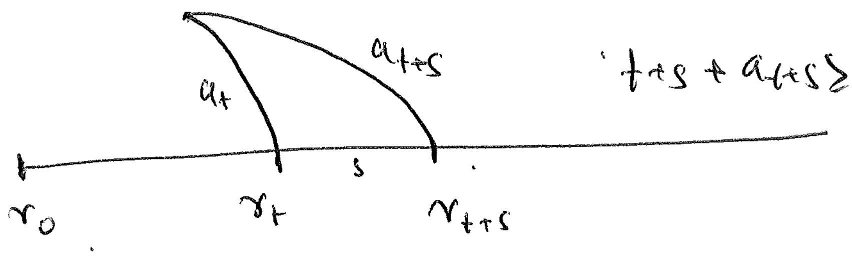
$$\text{"} \quad \ge \quad \underline{\text{"} \qquad \text{"} \qquad \text{"}}$$

^ p-uniform. "smoothness"

Ex. for:
- all cpt. Finslers.
- all Riem sec ≥ 0.
- all Berwald mflds. sec ≥ 0.

<u>Busemann functions</u>: $\gamma : [0,\infty) \to M$, $d(\gamma_t, \gamma_s) = |t-s|$.

$$b_\gamma(x) = \lim_{t\to\infty} t - d(x,\gamma_t) \le d(x,\gamma_0).$$



$t+s + a_{t+s} \ge t+s - (a_t + s) \ge t - a_t$.

<u>Prop.</u> b_γ is 1-Lipschitz.

<u>Prop.</u> (M,d) is p-uniform smooth and γ a ray then b_γ is geodesically convex.

<u>Corollary</u>: If (M,d) p-uniformly smooth, then $\exists$ convex fct $b : M \to [0,\infty]$ with compact and convex sublevels.

Pf. $b(x) = \sup_{\gamma_{x_0}} b_{\gamma_{x_0}}(x)$ $\quad \gamma_{x_0}$: Starting at x_0.

+ sup of convex is convex.

Busemann concavity $\eta, \gamma \in \mathrm{Geo}_{[0,1]}(M,d)$.

$$d(\gamma_t, \eta_t) \geq t\, d(\gamma_1, \eta_1).$$

Ex: · all Alex(0).

· all L^p products of of Alex(0).

· at least all Banach spaces of $\kappa \geq 0$.

Hausdorff mass: $\mathcal{H}^k_\delta(A) = \inf\left\{ \sum c_k (\mathrm{diam}\, U_i)^k : A \subset \bigcup_{i \in I} U_i,\ \mathrm{diam}\, U_i \leq \delta \right\}$

$$\mathcal{H}^k(A) = \sup_{\delta > 0} \mathcal{H}^k_\delta(A).$$

Idea: prove something like MCP (meas. contract property).

$\varphi_{t,x} : M \to M, \quad x \mapsto \hat{\gamma}_{xy}(t)$,

$A_{t,x} = \{ \gamma_{xy}(t) ; y \in A, \gamma_{xy} \in \mathrm{Geo}(x,y) \}$.

$A_{t,x} \subset \bigcup_i U_i \quad N_i = \varphi_{t,x}^{-1}(U_i)$.

Assume $U_i \subset A_{t,x}$ (by intersection, still get cover).

· $\mathrm{diam}\, V_i \leq 2 \mathrm{diam}\, U_i$ (via mid point).

$$\Rightarrow \quad \mathcal{H}^n_\delta(A \cap V_i) \leq 2^n \mathcal{H}^n_{2\delta}(A_{t/2,x})$$ midpoint of γ.

$$\Rightarrow \mathcal{H}^n(A) \leq 2^n \mathcal{H}^n(A_{1/2,x}).$$

<u>Corollary</u>. If (M,d) Busemann concave, $0 < \mathcal{H}^n(B_1(x)) < \infty$.

then (M,d) satisfies (I) MCP$(0,n)$, (II). doubling (III) Poincare inequality (IV) Doubling-Poinc. ⊕ unique tangents $\Rightarrow$ $\mathcal{H}^n$-a.e. tangent spaces are normed spaces.

<u>bi-Lipschitz splitting Th^m</u>: (idea).

Technical lemma: If (M,d) is geodesic & proper, and $\exists\ (\varphi_t)_{t\in\mathbb{R}}$ Bouches. $d(x, \varphi_t(x)) = |t|$. and $b: M \to \mathbb{R}$ 1-Lipschitz. $b(\varphi_t(x)) = t + b(x)$.

Then, (M,d) splits off line in a bilipschitz way.