<u>Honda</u> – New Stab. results for sequences of m.m. spaces with uniform Ricci bds from below. 31/08/2016.

M = n-dim cpct Riem mfld with $\mathrm{Ric} \geq n-1$.

(or $\mathrm{RCD}^*(N-1, N)$ $N \in (1, \infty)$.)

<u>Thm.</u> (Cheng, Meyers, Lichnerowicz, et. al.) ← Rigidity Result.

(I) $\lambda_1(M) \geq n = \lambda_1(S^n)$.

(II) $\mathrm{diam}\, M \leq \pi = \mathrm{diam}\, S^n$.

$h(M) \geq h(S^n)$ $h(M) := \inf_{\substack{\Omega \subset M \text{ open} \\ \partial\Omega \subset C^\infty \\ H^n(\Omega) \leq \frac{1}{2} H^n(M)}} \frac{H^{n-1}(\partial\Omega)}{H^n(M)}$.

Cheeger's isoperimetric constant.

n-dim Haus. measure.

(III) one Equality holds iff. all eq. holds iff $M \overset{\text{isom.}}{\cong} S^n$.

<u>Thm.</u> (Bayle, Colding, Croke, et. al.). ← Almost rigidity.

(I) $\forall \varepsilon > 0$ $\exists$ $\delta = \delta(\varepsilon, n) > 0$.

$$|\lambda_1(M) - n| < \delta \Rightarrow |\mathrm{diam}\, M - \pi| < \varepsilon.$$

(Cheeger–Colding) $\Longrightarrow$ $d_{GH}(M, S^0 * X) < \varepsilon$. $S^0 * X$ spherical suspension.

(II) $|\mathrm{diam}\, M - \pi| < \delta \Rightarrow |h(M) - h(S^n)| < \varepsilon$.

(III) $|h(M) - h(S^n)| < \delta \Rightarrow |\lambda_1(M) - n| < \varepsilon$.

Thm. (Matei, Valtorta).

(I). $\underbrace{\lambda_{1,p}(M)}$ $\geq \lambda_{1,p}(S^n)$. $\forall p \in (1,\infty)$.

1st +ve eigenvalue of p-Laplacian.

(II). The equality holds iff $M \overset{\text{isom}}{\cong} S^n$.

(III). $\forall \varepsilon > 0$, $\forall p \in (1,\infty)$. $\exists\, \delta = \delta(\varepsilon, p, n) > 0$. s.t.

ⓐ $|\lambda_{1,p}(M) - \lambda_{1,p}(S^n)| < \delta \Rightarrow |\operatorname{diam} M - \pi| < \varepsilon$.

ⓑ $|\operatorname{diam} M - \pi| < \delta \Rightarrow |\lambda_{1,p}(M) - \lambda_{1,p}(S^n)| < \varepsilon$.

Current work with L. Ambrosio; motivation:

Give a unified understanding of all these thms.

Main Thm. (Ambrosio, H.).

$\forall \varepsilon > 0$, $\exists\, \delta = \delta(n, \varepsilon) > 0$. s.t. for $p \in [1,\infty]$.

important that δ is indep of p.

$\Rightarrow$. $|\lambda_{1,p}(M)^{\frac{1}{p}} - \lambda_{1,p}(S^n)^{\frac{1}{p}}| < \delta \Rightarrow |\lambda_{1,q}(M)^{\frac{1}{q}} - \lambda_{1,q}(S^n)^{\frac{1}{q}}| < \varepsilon$.

where:

$$\lambda_{1,p}(M)^{\frac{1}{p}} := \begin{cases} \text{ordinary sense} & p \in (1,\infty). \\ h(M) & p = 1. \\ \dfrac{2}{\operatorname{diam} M} & p = \infty. \end{cases}$$

Key points of Pf :

① Compactness of the space.
② Rigidity for singular sps.
③ Continuity

Step 1, ②: Ketterer, Cavalletti-Mondino ;
Establish all inequalities before for $RCD^*(N-1,N)$ spaces.
except for $\dim X = \pi \Rightarrow \lambda_{1,p}(X) = \lambda_{1,p}(S^n)$.

Step 2 : ③ prove the continuity

$$\mathcal{M}(N,k,d) \times [1,\infty] \longrightarrow (0,\infty] .$$
$$(X,p) \longmapsto \left(\lambda_{1,p}(X)\right)^{\frac{1}{p}}$$

$RCD^*(K,N)$ spaces with $\operatorname{diam} \leq d$.

Step3: Finish pf, by contradiction argument.

If not, then $\exists X_i \in RCD(n-1,n)$ spaces and $\exists p_i, q_i \in [1,\infty]$
$\exists \tau > 0$ s.t. $|\lambda_{1,p_i}(X_i)^{\frac{1}{p_i}} - \lambda_{1,p_i}(S^n)^{\frac{1}{p_i}}| \to 0$.
but $|\lambda_{1,q_i}(X_i)^{\frac{1}{q_i}} - \lambda_{1,q_i}(S^n)^{\frac{1}{q_i}}| > \tau > 0$.

By compactness of $\mathcal{M}(N,k,d)$. ~~w.l.o.g.~~ X_i w.l.o.g.,
$X_i \to X \in M(n-1,n-1,\pi)$.

By continuity $p_i \to p$, $q_i \to q$.
$\lambda_{1,p}(X)^{\frac{1}{p}} = \lambda_{1,p}(S^n)^{\frac{1}{p}}$ but
$\lambda_{1,q}(X)^{\frac{1}{q}} \neq \lambda_{1,q}(S^n)^{\frac{1}{q}}$. ↯