Maas. 01/08/2016.

- $\mathcal{H}$ Hilbert space.
- a: unbounded op on $\mathcal{H}$ s.t. $aa^* - a^*a = id$.

Ex. $\mathcal{H} = L^2(\mathbb{R}, \gamma)$ γ: std Gauss, $a = \partial_x$, $a^* = x - \partial_x$.

Define. $\mathcal{L}_+$; $\mathcal{L}_-$ acting on $\mathcal{B}(\mathcal{H})$ by.

$$\mathcal{L}_+(\rho) = \frac{1}{2}\{[a^*\rho, a] + [a^*, \rho a]\}, \quad \mathcal{L}_-(\rho) = \frac{1}{2}\{[a\rho, a^*] + [a, \rho a^*]\}.$$

Set $\mathcal{L} = \mu^2 \mathcal{L}_- + \lambda^2 \mathcal{L}_+$ $(0 < \lambda < \mu)$.

$\Rightarrow$ $P_t = e^{t\mathcal{L}}$ defines a semigroup of completely positive trace-preserving maps.

↗ maps +ve s.a. ops to +ve s.a. maps.

Fact: $\exists$ stat. state σ $(P_t\sigma = \sigma)$

$$\sigma = \frac{1}{Z}\sum_n e^{-\beta n} E_n, \quad e^{-\beta} = \frac{\lambda^2}{\mu^2}.$$

where E_n is the orthogonal proj. onto space $\{h_n\}$.
space of Hermite poly's.

(P_t) is called Quantum O-U semigroup.

Define. $\mathrm{Ent}(\rho|\sigma) = \mathrm{Tr}[\rho(\log\rho - \log\sigma)]$.
for $\rho \in \mathcal{P} = \{\rho \in \mathcal{B}(\mathcal{H}) : \rho \geq 0, \mathrm{Tr}\,\rho = 1\}$.

<u>Conjecture</u> (see Huber-König-V —? & 2016.)

$$\mathrm{Ent}(\mathcal{P}_t\rho\,|\,\sigma) \le e^{-(\mu^2+\lambda^2)t}\,\mathrm{Ent}(\rho\,|\,\sigma) \quad \forall \rho\in\mathfrak{P}.$$

<u>Thm</u>. [Carlen-M.] <u>True</u>.

<u>Thm</u>. (Carlen-M). Eqn. $\partial_t\rho = \mathcal{L}\rho$ is the gradient flow of $\mathrm{Ent}(\cdot\,|\,\sigma)$ w.r.t. to metric on $\mathfrak{P}$,

$$\partial_t\rho = \mathcal{L}\rho = -\mathcal{K}(\rho)\,D\mathrm{Ent}(\rho\,|\,\sigma).$$

<u>Metric</u>: $\mathcal{K}(\rho)A := \partial^+(\hat{\rho}_+ \# \partial A) + \partial(\hat{\rho}_- \# \partial^+(A))$.

$\partial A = [a, A]$, $\partial^+ A = [a^*, A]$.

$$\hat{\rho}_\pm \# A := \int_0^1 (e^{\mp\beta/2}\rho)^{1-s}\, A\, (e^{\pm\beta/2}\rho)^s\, ds.$$

Caused on the log term in Entropy.

Thm $\mathcal{K}(\rho)$ - "inverse Riem metric on space of ops."

$$W^2(\rho_0,\rho_1) = \inf\left\{\int_0^1 \langle \mathcal{K}(\rho)A_t, A_t\rangle\,dt : \partial_t\rho_t + \mathcal{K}(\rho_t)A_t = 0\right\}$$

<u>Key identity</u>: $\underbrace{[a,\rho]}_{\partial\rho} = \int_0^1 \rho^{1-s}\,[a, \log(\rho)]\,\rho^s\,ds$.

Note: Commutators are derivations:

$$\partial(AB) = \partial A \cdot B + A \cdot \partial B.$$

$$\Rightarrow \quad \partial(A^n) = \sum_{k=0}^{n-1} A^{n-k-1} \cdot \partial A \cdot A^k,$$

$$A^n = \rho \Rightarrow \partial \rho = \frac{1}{n} \sum_{k=0}^{n-1} \rho^{1-\frac{k+1}{n}} \frac{\partial(\rho^{\frac{1}{n}} - I)}{\frac{1}{n}} \rho^{k/n}.$$

$$\xrightarrow{n \to \infty} = \int_0^1 \rho^{1-s} \partial(\log \rho) \rho^s \, ds.$$

- Simple Pf of geodesic convexity of Ent w.r.t. W_2. (Dolbeault-Nazaret-Savaré). $\boxed{\text{on } \mathbb{R}^n}$

$\rightsquigarrow$ Hard to know geodesics, but know heat flow.

Suffices to show EVI:

$$\frac{1}{2t}\left[W_2^2(\nu, P_t \rho) - W_2^2(\nu, \rho)\right] \leq \text{Ent}(\nu) - \text{Ent}(P_t \rho).$$

Pf. Let ρ_s be geodesic b/w. ν and ρ.

$$\partial_s \rho_s + \nabla \cdot v_s = 0.$$

Set $\rho_s^t = P_{st} \rho_s$, then,

$$\partial_s \rho_s^t = t \Delta P_{st} \rho_s + P_{st} \partial_s \rho_s:$$

$$= \nabla \cdot (t \nabla \rho_s^t) - \nabla \cdot P_{st} v_s.$$

$$= \nabla \cdot \left(-P_{st} v_s + t \nabla \rho_s^t\right).$$

Then: $$W_2^2(\nu, P_t \rho) \leq \int_0^1 \int \frac{|P_{st} v_s - t \nabla \rho_s^t|^2}{\rho_s^t} \, dx \, ds.$$

$$= \int_0^1 \int \frac{|P_{st} V_s|^2}{P_{st} P_s} \, dx\, ds - 2t \iint \frac{P_{st}V_s - t\nabla P_s^t \cdot \nabla P_s^t}{P_s^t} \, dx\, ds$$

$$\underbrace{-t^2 \iint \frac{|\nabla P_s^t|^2}{P_s^t} \, dx\, ds}_{\text{forget since } -\text{ve}}$$

Jensen $\frac{x^2}{y}$ convex

$$\leq \underbrace{\int_0^1 \int \frac{|V_s|^2}{P_s} \, dx\, ds}_{W_2^2(\nu,\rho) \text{ by geodesic prop.}} - 2t\left(\mathrm{Ent}(P_t\rho) - \mathrm{Ent}(\nu)\right).$$

$\boxed{}$.

Mimic $\mathbb{R}^n$ pf in non-commutative setting:

Two "$\mathbb{R}^n$" steps in Pf: ① $\nabla P_t = P_t \nabla$ incl.

② $\iint \frac{|P_{st} V_s|^2}{P_{st} P_s} \, dx\, ds \leq \iint \frac{|V_s|^2}{P_s} \, dx\, ds$.

② $\rightarrow$ Trace convexity inequality (HR-Petz, Lieb)

$(R, S, A) \mapsto \mathrm{Tr}\left[\int_0^\infty \frac{1}{x+R} A^* \frac{1}{x+S} A \, dx\right]$, $R, S \geq 0$

is jointly convex.

Basically "$\frac{a^2}{r}$".

① Use exponential term?