Hassannezhad - Eigenvalue problems in Sub-Riem. geometry. 02/09/2016.

$(M, H, g)$ $M$ - $C^\infty$ mfld, $H \subseteq TM$ subbundle. $g \in C^\infty$ metric.

- Hörmander condition: $H_x \subset H_x^2 \subset \dots \subset H_x^r = T_xM$. $\exists r \in \mathbb{N}$.
- Regular $n_i(x) = \dim H_x^i = \text{const}(i)$. $H_x^i = H_x^{i-1} + [H, H_x^{i-1}]$

Step of $H = \min\{r : H^r = TM\}$.

Caratheodory-Carnot dist:

$$d_{cc}(x,y) = \inf\left\{ \text{Length}(\gamma) : \begin{array}{l} \gamma:[0,1] \to M. \\ \gamma(0) = x,\ \gamma(1) = y. \\ \gamma'(t) \in H_{\gamma(t)}. \end{array} \right.$$

curves exist by Chow's Thm for $M$ connected.

Hausdorff dim $(M, d_{cc})$ = $h = \sum_{j=1}^{r} j(\dim H^j - \dim H^{j-1})$.

$\exists$ intrinsic $C^\infty$ measure $P_g$ in a regular Sub-Riem. mfld. called Popp's measure.

$$T_xM = H_x + [H, H_x] \underset{\text{non-can.}}{\cong} H_x \oplus \underbrace{\frac{H_x + [H,H]_x}{H_x}}_{= E}.$$

$$H \otimes H \longmapsto \frac{H + [H,H]_x}{H_x} \qquad x \otimes y \longmapsto [x,y].$$

$$\Lambda^n T^*M \cong \Lambda^n E^*.$$

Ex. • Heisenberg $\mathbb{H}^3$.

• Carnot group: simply connected Lie group s.t.

$$\mathfrak{g} = \oplus_{i=1}^{r} \mathfrak{g}_i .$$

• Contact structures (kernel of contact form), ~~Subs~~ CR manifolds etc.

Sub-Laplacian: $\nabla^H$ on $H$ given by.

$$g(\nabla^H u, X) = du(X).$$

$$\Delta_{sH} = -\mathrm{div}_g(\nabla^H u) \longleftarrow \text{since we have Popp's measure.}$$

$(M, H, g)$ cpct sub-Riem., $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots \nearrow \infty$.

Motivation: $(M, g)$ cpct Riem. of dim $n$.

• Buser's upper bd. 79: $Ric(M,g) \geq -(n-1)a^2$,

$$\lambda_k(\Delta_g) \leq \frac{(n-1)^2}{4} a^2 + c(n) \left(\frac{k}{Vol_g(M)}\right)^{2/n}.$$

• Korevaar 93: $\lambda_k(\Delta_g) \leq C(n, [g]) \cdot \left(\frac{k}{vol_g(M)}\right)^{2/n}$.

• H ~ 2001:

$$\lambda_k(\Delta_g) \leq C_1(n) \left(\frac{V[g]}{vol_g(M)}\right)^{2/n} + C_2(n) \underline{\qquad}.$$

conformal class.

$$V[g] = \inf \{ vol_{g_0}(M) : g_0 \in [g] ; \ Ric \geq -(n-1) \}.$$

Keevash for Contact metric mfd:

$(\theta, \xi, g, \varphi)$ Contact-metric structure.

$\theta$ - contact form $(d\theta)^n \wedge \theta \neq 0$.

$\theta(\xi) = 1, \quad d\theta(\xi, \cdot) = 0.$

$g$ - Riemannian metric, $\varphi$ 1-1 tensor field satisfying some conditions.

$$\frac{\int_M |\nabla_g^H f|^2 \, d\mathfrak{P}_g}{\int_M f^2 \, d\mathfrak{P}_g} \overset{\text{Hölder}}{\le} \frac{\left(\int_M |\nabla^H f|^n_{g_0} \, d\mathfrak{P}_{g_0}\right)^{2/n} \left(\int_M \cancel{\mathcal{D}}_{g \to f} \, d\mathfrak{P}_g\right)^{1-2/n}}{\int_M f^2 \, d\mathfrak{P}_g}$$

$g_0 \in [g]$

This quantity is conformally invariant, so have freedom to choose $g_0$.

$(M, d^{a}_{g_0}, \mathfrak{P}_g) \leftarrow$ m.m. space to consider.