Takatsu - CD Cond. from. Info geometry. 02/09/2016

(M,g) complete Riem mfld.

(M,g) sat. $CD(K,N) \iff Ric \geq K$ & $\dim M \leq N$.

$\iff$ convexity of the N-Renyi entropy in $(P_2^{ac}(M, vol_g), W_2)$.

$$E_N(\mu) = -\int_M \left(\frac{d\mu}{dvol_g}\right)^{1-\frac{1}{N}} dvol_g.$$

Q. How to find the N ~~entry~~ Renyi-Entropy.

A. Generalise the Boltzmann entropy via info geometry.

Boltzmann: $E_\infty(\mu) = \int_M \frac{d\mu}{dvol_g} \ln\left(\frac{d\mu}{dvol_g}\right) dvol_g.$

Heat eq./exp-distribution.

exp: sol to $\dot{x}(t) = x(t)$ ~~&~~ & $x(0) = 1$.

& inverse $\longleftrightarrow$ $\ln(t) = \int_1^t \frac{1}{s}\, ds$.

$u(r) = \int_1^r \ln(t)\, dt$ ~~convex~~ $= r\ln r - r$. convex.

(X,ω) measure space $E_\omega(\mu) = \int_X u\left(\frac{d\mu}{d\omega}\right) d\omega$. $\mu \in P(X,\omega)$

$$= \int_X \frac{d\mu}{d\omega} \ln\left(\frac{d\mu}{d\omega}\right) d\omega - 1.$$

W_2-gradient flow of Entropy

$$\partial_t \rho = \Delta\rho = \mathrm{div}(\rho \nabla \ln\rho).$$

on $\mathbb{R}^n$: $c_1 t^{-n/2} \exp(-\lambda t^{-1}|x|^2)$.

Information Geometry: $\varphi : (0,\infty) \to (0,+\infty)$. cont., non-dec.

Back to Boltzmann. $\exp_\varphi$ sol^n. $\dot{x}(t) = \varphi(x(t))$ & $x(0)=1$.

Inverse: $\ln_\varphi(t) = \int_1^t \frac{1}{\varphi(s)}\, ds$.

$U_\varphi(r) = \int_0^r \ln_\varphi(t)\, dt$ <u>if it exists.</u>

$$\theta_\varphi := \sup_{s>0} \left\{ \frac{s}{\varphi(s)} \limsup_{\varepsilon \downarrow 0} \frac{\varphi(s+\varepsilon)-\varphi(s)}{\varepsilon} \right\}.$$

$$E_{w,\varphi}(\mu) = \int_X U_\varphi\left(\frac{d\mu}{dw}\right) dw.$$

W_2-grad flow of $\mathrm{Ent}_{\mathrm{vol}_g,\varphi}$:

$$\partial_t \rho = \mathrm{div}(\rho \nabla \ln_\varphi(\rho)).$$

<u>Ex.</u> $\varphi_q(s) = s^q \quad (0<q<2,\ q\neq 1)$. $\theta_{\varphi_q} = q$.

$$\ln_q(t) = \frac{t^{1-q}-1}{1-q},$$

$$\exp_q(t) = \left(1+(1-q)t\right)_+^{\frac{1}{1-q}} \longrightarrow \exp t.$$

$$U_q(r) = \frac{r^{2-q}}{(2-q)(1-q)} - \frac{r}{1-q}, \quad N_q = \frac{1}{q-1}.$$

$$E_{q,w}(\mu) = -\frac{1}{N_w}\left(1-\frac{1}{N_q}\right)^{-1} E_{N_q,w}(\mu) + \frac{1}{N_q}.$$

Info geo. $\alpha = \dfrac{1}{n(1-q)+2} \xrightarrow{q\to 1} \dfrac{1}{2}$.

Rem. (1) $0 < q' < 1 < q < 2$, $CD(0, N_{q}) \Rightarrow CD(0,\infty)$.

$N_q > 0$, $N_1 = \infty$..

$CD(0,\infty) \Rightarrow CD(0, N_{q'})$ $\boxed{N_{q'} < 0.}$

Neg dimension has meaning here.

(2). $\omega = e^{-\Phi} \mathrm{vol}_g$, $\mathrm{Ent}_\omega(\mu) = \mathrm{Ent}_{\mathrm{vol}_g}(\mu) + \int_M \Phi \, d\mu$.

But $\mathrm{Ent}_{\omega, N} \neq \mathrm{Ent}_{\mathrm{vol}_g, N}(\mu) +$ " "

$=: \mathrm{Ent}^{\Phi}_{\omega, \mathrm{vol}_g}(\mu)$.

"Thm" (Ohta-T). (modulo technical assumptions).

(M,g): Riem mfld, $\omega = e^{-f} \mathrm{vol}_g$, $f \in C^\infty(M)$, $\Phi \in C(M)$, $\varphi : (0,\infty) \to (0,\infty)$ cts non-dec.

$$\mathrm{Ent}^{\Phi}_{\omega, \varphi} := \int_M u_\varphi\left(\frac{d\mu}{d\omega}\right) d\omega + \int_M \Phi \, d\mu.$$

$K \in \mathbb{R}$, $N \in \mathbb{R} \setminus (-1, n)$, TFAE:

(1). $\mathrm{Hess}_{W_2} \mathrm{Ent}^{\Phi}_{\omega, q} \geq K$, $\frac{1}{q-1} = N$.

(2). $(M, \omega) \in CD(0, N)$ & $\mathrm{Hess}_g \Phi \geq K$.

(3). $\mathrm{Hess}_{W_2} \mathrm{Ent}^{\Phi}_{\omega,\varphi} \geq K$. $\forall \varphi$ with $\theta_\varphi \leq q$.

$K > 0 \Rightarrow \exists! \exp_\varphi(\lambda - \Phi)\,\omega$: unique minimizer of $\mathrm{Ent}^{\Phi}_{\omega,\varphi}$.