Suzuki - Convergence of ~~RCD~~ Brownian motions on RCD spaces. 31/08/2016.

- measured Gromov Convergence of m.m. spaces (Geometric)
- weak convergence of the laws of BM (Stochastic).

$RCD(K,\infty)/RCD^*(K,N)$

$$\begin{array}{ccc} X_n & \xrightarrow{mG} & X_\infty \\ \Big| Ch_n & & \Big| Ch_n \\ \mathbb{B}_n & \xrightarrow{\text{"weak"}} & \mathbb{B}_\infty \end{array}$$

$\mathbb{B}_n$ - Brownian motion.

$\boxed{\text{Q: } (A) \Leftrightarrow (B) \text{ ?}}$

Brownian motion on RCD spaces:

$(X, d, m) \in RCD(K,\infty)$ space.

- $Ch(f) = \frac{1}{2}\int_X |\nabla f|^2 dm \quad f \in W^{1,2}(X,d,m)$
- $H_t : L^2(X,m) \to L^2(X,m)$.

$\Rightarrow \exists\, (\Omega, \mathcal{M}, \mathbb{P}^x)$ a.e. $x \in X$. — probability space.

$\exists!\, B_{(\cdot)} : \Omega \to C([0,\infty); X) = \{w : [0,\infty) \to X \text{ cts}\}$.

$$\mathbb{E}^x[f(B_t)] = \int_\Omega f(B_t)\, d\mathbb{P}^x = (H_t f)(x) \quad \forall f \in L^2 \cap \mathcal{B}_b.$$

Brownian motion: $(\Omega, \mathcal{M}, \mathbb{P}^x, B_{(\cdot)})$.

mG ~~#~~ conv: $\mathcal{X}_n = (X_n, d_n, m_n)$, $m_n(\mathcal{X}_n) = 1$.

$\mathcal{X}_n \to \mathcal{X}_\infty$ if $\left\{ \begin{array}{l} \exists \text{ cmp. sep. met space } (X, d) \\ \exists \text{ isom embed } i_n : X_n \hookrightarrow X. \end{array} \right.$

s.t. $i_{n\#} m_n \xrightarrow{\text{weak}} i_{\infty\#} m_\infty$ in $\mathcal{P}(X)$

Note: weaker than mGH, no doubling condition here.

Weak conv of BM

$B^n_{(\cdot)} : (\Omega^n, \mathcal{M}^n, \mathbb{P}^x_n) \to C([0,\infty); X_n)$ measurable.

$(B^n_{(\cdot)})_\# \mathbb{P}^x_n \in \mathcal{P}(C([0,\infty); X_n))$ law of BM.

$(i_n B^n_{(\cdot)})_\# \mathbb{P}^x_n \in \mathcal{P}(C([0,\infty); X))$.

$\parallel$

$\mathbb{B}_n$.

$\mathbb{B}_n \longrightarrow \mathbb{B}_\infty$ if $\mathbb{B}_n(f) \to \mathbb{B}_\infty(f)$. $\forall f \in C_b(C([0,\infty); X))$.

Main Th$^{\underline{m}}$: $\mathcal{X}_n = (X_n, d_n, m_n)$ m.m. spaces. s.t.

$X_n \in RCD^*(K,N)$, $\mathrm{Diam}(X_n) \le D$, $m_n(X_n) = 1$. ~~[illegible]~~

then (I) $\mathcal{X}_n \xrightarrow{mG} \mathcal{X}_\infty \iff \exists$ cpct (X, d) $\exists$ isom embed.

$X_n \hookrightarrow X$, ~~$\exists x_n \in X_n$~~

$\boxed{N \in (0,\infty)}$: $\left(\begin{array}{c} \iff mGH \text{ by} \\ \text{Ass } 1 \end{array} \right)$.

Given $x_\infty \in X_\infty$ $\exists x_n \in X_n$.

s.t. $i_n(x_n) \to i_\infty(x_\infty)$ in (X, d)

s.t. $i_n(B^n_{(\cdot)})_\# \mathbb{P}^{x_n}_n \longrightarrow i_\infty(B^\infty_{(\cdot)})_\# \mathbb{P}^{x_\infty}_\infty$ in $\mathcal{P}(C([0,\infty); X))$.

(2). $N=\infty$, and <u>no</u> diameter assumption.

Thm $X_n \xrightarrow{mG} X$. iff $\begin{cases} \exists (X,d') \text{ complete sep.} \\ \exists X_n \hookrightarrow X \text{. isom. s.t.} \\ i_n(B^n_{(\cdot)})_\# \mathbb{P}_n^{m_n} \to i_\infty(B^\infty_{(\cdot)})_\# \mathbb{P}_\infty^{m_\infty} \end{cases}$

$$\mathbb{P}_n^{m_n}(\cdot) = \int_{X_n} \mathbb{P}_n^x(\cdot)\, dm_n(x).$$

Pf of (1): $(\Rightarrow)$ (a) Tightness of $\{ i_n(B^n_{(\cdot)})_\# \mathbb{P}_n^{x_n} \}_n$.
relative cptness.

(b) Convergence of finite-dim distributions.

Tightness: uniform heat kernel est. $p_t(x,y) \leq C_1 \exp\{-C_2 \frac{d^2(x,y)}{t}\}$

C_1, C_2 only dep on. K, N, D.

$\underbrace{\phantom{C_1 \exp\{-C_2 \frac{d^2(x,y)}{t}\}}}_{\text{not true! Hmm.}}$

need a more careful estimate.

fin. dim. distr. Mosco conv. $Ch_n \to Ch_\infty$