Kaj Nyström - Parabolic eq^{n}s. w/ Complex coeff. 14/09/2016.

$$\mathcal{H}u = (\partial_t + \mathcal{L})u := \partial_t u - \mathrm{div}_x A(x,t)\nabla_x u = 0.$$

t-indep coeff: $A(\lambda, x, t) = A(x)$.

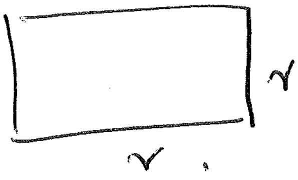
Off diag estimates. $\Theta_\lambda = \{\mathcal{E}_\lambda, \lambda \nabla_\parallel \mathcal{E}_\lambda\}$, $\Theta_\lambda^* = \{\lambda \mathcal{E}_\lambda \mathrm{div}_\parallel, \lambda^2 \nabla_\parallel \mathcal{E}_\lambda \mathrm{div}_\parallel\}$

$$d_p(E,F) = \inf\{\|(x-y, t-s)\| : (x,t)\in E, (y,s) \in F\}.$$

parabolic distance.

Reduce to ~~Carleson~~ Carleson measure est; use:

$$f^{\varepsilon}_{\Delta, w} = \left(1 + (\varepsilon \ell(\Delta)^2)\mathcal{H}_\parallel\right)^{-1}\left(\chi_\Delta(\varphi_\Delta \cdot \bar{w})\right).$$

Δ - parabolic cube, 

. $\ell(\Delta) = r$.

centred at (x_Δ, t_Δ)

and ~~$\beta_{\Delta}\ \varphi_\Delta$~~ χ_Δ cutoff φ_Δ smoothing.

This is very similar to test fnctn in Elliptic case,

so this is for time indep. coeff.

~~De Giorgi~~ Applications to BVP 2nd order approach.

De Giorgi - Moser - Nash. (P. Auscher).

$K_t(x,y)$ kernel of e^{-tL}

Introduce the single layer pot.

Th (K. N.) A_0, A_0^*, A_1, A_1^* sat.
De Giorgi - M - N. & A_0, A_0^* have
bdd invertible layer pot.
Then $\exists\, \varepsilon_0$ s.t. $\|A_0 - A_1\| < \varepsilon_0$.
Same for A_1, A_1^*.

Core estimates needed:

$$|||\cdot|||_+ = \left(\iiint_{\mathbb{R}^{n+2}_+} |\cdot|^2 \frac{dx\,dt\,d\lambda}{|\lambda|}\right)^{\frac{1}{2}}.$$

$$\begin{cases} (\mathrm{I}) & \sup_{\lambda \neq 0} \|\partial_\lambda S_\lambda^{A} f\|_2 + \sup_{\lambda \neq 0} \|\partial_\lambda S_\lambda^{A^*} f\|_2 \leq T \|f\|_2. \\ (\mathrm{II}) & \|\lambda \partial_\lambda^2 S_\lambda^{A} f\| + \|\lambda \partial_\lambda^2 S_\lambda^{A^*} f\|_2 \leq T \|f\|_2. \end{cases}$$

First-order p.o.v, these always hold.

• Get parabolic measure, which is doubling.

Local Tb th^m reduces to $\exists \{b_Q\}$ functions satisfying B condition, exactly like Elliptic, using single & double layer potentials cone estimates.

First order approach

$$A(\lambda, x, t) = A(x, t).$$

Connect all reinforced weak solutions via F.O. system

$$\partial_\lambda F + M F = 0.$$

$$\boxed{F = D_A u.}$$

Goal: prove Q estimates.

$$\int_0^\infty \| \lambda PM (1 + \lambda^2 (PM)^2)^{-1} h \|_2^2 \frac{d\lambda}{\lambda} \sim \|h\|_2^2.$$

in Moritz's thesis.

Lemma (By Šneiberg)

$\exists \delta_0 > 0$ s.t. if p, q $\left|\frac{1}{p} - \frac{1}{2}\right| < \delta_0$, $\left|\frac{1}{q} - \frac{1}{2}\right| < \delta_0 \Rightarrow (1 + i\lambda PM)^{-1}$ hold on

$L^p(\mathbb{R}, L^q(\mathbb{R}^n, \mathbb{C}^{n+2}))$ with uniform bounds w.r.t. λ.

Pf. by hidden coercivity.

$\partial_t + L_A$ invertible on $H_{2,2} \longrightarrow H^*_{2',2'}$.

bdd on $H_{p,q} \longrightarrow H^*_{p',q'}$.

Off-diag estimates:

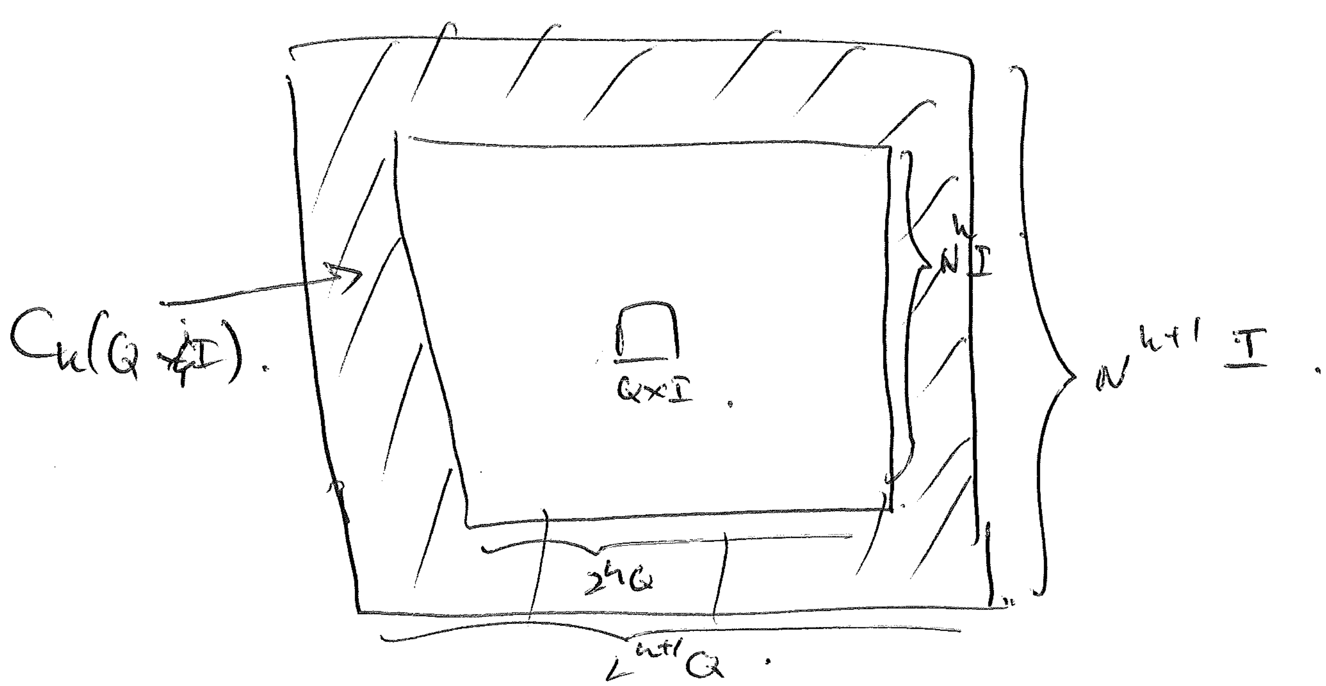
Prop 1: $\exists\, \varepsilon_0 > 0$, $N_0 > 1$ s.t. $\left|\frac{1}{q} - \frac{1}{2}\right| < \varepsilon_0$,

$\exists\, \varepsilon = \varepsilon(n, q, \varepsilon_0)$ with: $\forall N \geq N_0$,

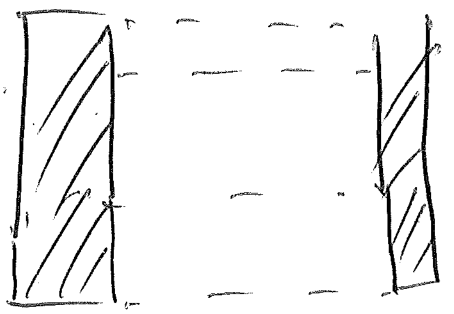
$$\iint_{Q \times 4^j I} |(I + i\lambda PM)^{-1} h|^q \, dy\, ds \leq C N^{-q\varepsilon k} \iint_{C_k(Q \times 4^j I)} |h|^q \, dy\, ds.$$

$Q = B(x, r) \subset \mathbb{R}^n$, $I = (t - r^2, t + r^2)$, $\lambda \sim r$, $j \in \mathbb{N}$,

with $h \in L^2 \cap L^q(\mathbb{R}^{n+1}; \mathbb{C}^{n+2})$. Spt $h \subset C_k(Q \times 4^j I)$

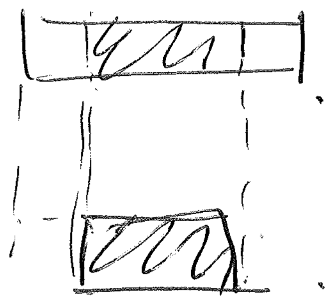


Break into two parts.



Stretch in space

this is fine.
exp off diag.



Stretch in time.

unbounded.

Take cut off.

$$\eta = \begin{cases} \boxed{Q \times I} \\ - \end{cases}$$

$$(N^k \ell(J))\,\|\partial_t \eta\|_\infty \lesssim 1.$$

Take $p > 2$, from Šneĭberg

$$\frac{1}{|Q|} \fint_J \int_{\mathbb{R}^n} |(1 + i\lambda PM)^{-1} u_2|^2 \, dy \, ds$$

$$\lesssim \frac{1}{|Q|} \left(\fint_J \left(\int_{\mathbb{R}^n} \underline{\qquad\qquad} \right)^{p/\cdot} \right)$$

Write in terms of commutators.

$$\eta(1+i\lambda PM)^{-1}h_2 = (1+i\lambda PM)^{-1} i\lambda [\eta, P] M (1+i\lambda PM)^{-1} h_2.$$

$$[\eta, P] = \begin{bmatrix} 0 & 0 & -[\eta, D_t^{\frac{1}{2}}] \\ 0 & 0 & 0 \\ -[\eta, H_t D_t^{\frac{1}{2}}] & 0 & 0 \end{bmatrix}.$$

$$\underbrace{\| [\eta, D_t^{\frac{1}{2}}] \|_{L^2(\mathbb{R}) \to L^p(\mathbb{R})} \lesssim N^4 \ell(J)^{2-2/p} \| \quad \|_{L^2 \to L^2}}_{\text{check.}} \; ?$$

Key thing: stretch more in time than space, + Sneiberg.

Rmk
test functions for Tb requires more care to handle non-local term.

In Q est, $((\Theta_\lambda - \gamma_\lambda S_\lambda) P_\lambda P v$ term,

get fractional estimate in time.

$$\int_0^\infty \| \lambda^{2\alpha} D_t^\alpha P_\lambda P v \| \frac{d\lambda}{\lambda}.$$

Choose α depending on ε coming from ε in off-diag estimates.