Sogge - lecture 3.

(M,g) bdyless, $\dim = n$.

$$\sqrt{-\Delta_g}\, e_\lambda = \lambda e_\lambda . \qquad 0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots , \quad \{e_j\} \text{ o.n. ass. eigenfunc.}$$

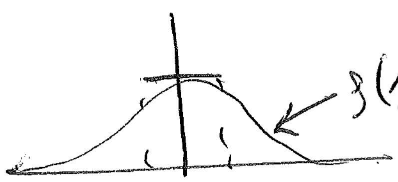
$$\chi_\lambda f = \sum_{\lambda_j \in [\lambda, \lambda+1]} E_j f , \qquad E_j f(x) = \langle f, e_j \rangle e_j(x).$$

$$\|\chi_\lambda f\|_{L^p} \lesssim \begin{cases} \lambda^{n(\frac12 - \frac1p) - \frac12} \|f\|_p , & \frac{2(n+1)}{n-1} \le p \le \infty . \\ \lambda^{\frac{n-1}{2}(\frac12 - \frac1p)} \|f\|_p , & 2 < p \le \frac{2(n+1)}{n-1} . \end{cases}$$

(In FIO '93 Book - cannot do better).

$\rho \in \mathcal{S}(\mathbb{R})$, $\rho(0) = 1$, $\operatorname{supp} \hat\rho \in (\delta/2, \delta)$.

$$P = \sqrt{-\Delta_g}, \qquad \rho(\lambda - P) f = \sum_{j=0}^{\infty} \rho(\lambda - \lambda_j) E_j f .$$



$\rho(\lambda - \lambda_j)$ ← smooth out indicator func.

χ_λ.

$$E_j f(x) = e_j(x) \int f(y) \overline{e_j(y)}\, dy .$$

$$\Rightarrow \rho(\lambda - P)(x,y) = \sum_{j=0}^{\infty} \rho(\lambda - \lambda_j) e_j(x) \overline{e_j(y)} .$$

$$\|\rho(\lambda-P)f\|_{L^p} \le C\lambda^{\sigma(p)}\|f\|_{L^2}.$$

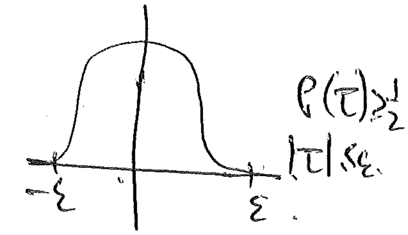
$$\Updownarrow$$

$$\|\chi_\lambda f\|_{L^p} \le C\lambda^{\sigma(p)}\|f\|_{L^p}.$$

$$\Longleftrightarrow \quad \|(\Delta+\lambda^2)f\|_2 \le C\lambda\|f\|_2,$$
$$\|f\|_p \le C\lambda^{\sigma(p)}\|f\|_2.$$

Pf ($\Rightarrow$):

$$\|\rho(\lambda-P)f\|_{L^2}^2 = \sum_{j=0}^{\infty} |\rho(\lambda-\lambda_j)|^2 \|E_j f\|_2^2$$



$$\ge \frac14 \sum_{\lambda_j\in[\lambda,\lambda+\varepsilon]} \|E_j f\|_2^2 = \frac14\|\chi_{[\lambda,\lambda+\varepsilon]}f\|_2^2. \qquad \square$$

$$\rho(\lambda-P) = \frac{1}{2\pi}\int \hat\rho(t)\,e^{i\lambda t}\,e^{-itP}\,dt.$$

$$\left(u := e^{-itP}f = \sum_j e^{-it\lambda_j}E_j f \Leftarrow (\partial_t + iP)u = 0,\ u|_{t=0}=f. \right).$$

$$\rho(\lambda-P)(x,y) = \frac{1}{2\pi}\int_{\delta/2}^{\delta} \hat\rho(t)\,e^{i\lambda t}\underbrace{\left(e^{-itP}\right)(x,y)}_{\text{parametrix}}\,dt.$$

<u>Hadamard Parametrix 1929</u>: For small $|t|$ held away from 0, parametrix for $(e^{-itP})(x,y)$ is given by

$$(e^{-itP})(x,y) = \int_0^\infty \theta^{\frac{n-1}{2}}\,a(t,x,y,\theta)\,e^{i\theta(d_g(x,y)-t)}\,d\theta + \underline{\text{smooth}}$$

($d_g(x,y)$: Reimannian distance.)

$$|\partial_{x,y}^\alpha \partial_\theta^j a(t,x,y,\theta)| \le C_{\alpha,j}(1+|\theta|)^{-j}$$

mod $\lambda^{-\infty}$:

$$\rho(\lambda-P)(x,y) = \frac{1}{2\pi}\int_{\delta/2}^{\delta}\int_0^\infty \theta^{\frac{n-1}{2}} a\, e^{it\lambda} e^{i\theta(d_g(x,y)-t)} \cdot \hat{\rho}(t)\, d\theta\, dt.$$

<u>Lemma</u> $\rho(\lambda-P)(x,y) = \lambda^{\frac{n-1}{2}} b_\lambda(x,y)\, e^{i\lambda d_g(x,y)} + O(\lambda^{-N})$.

where $b_\lambda(x,y) = 0$ if $d_g(x,y) \notin [\delta/2, \delta]$,

$|\partial^\alpha_{x,y} b_\lambda(x,y)| \leq C_\alpha$.

<u>Cor.</u> $\|\chi_\lambda f\|_\infty \leq C \lambda^{\frac{n-1}{2}} \|f\|_{L^2}$.

<u>Pf.</u> $\|\rho(\lambda-P)\|_{L^1\to L^\infty} = O(\lambda^{\frac{n-1}{2}})$.

<u>Pf of Lemma</u>: $d_g(x,y)\notin[\delta/2,\delta] \Rightarrow O(\lambda^{-\infty})$ i.b.p. in θ.

Sps not, $t \longmapsto \Phi(t,x,y,\theta) := \hat{\rho}(t)a(t,x,y,\theta) \in \mathcal{S}(\mathbb{R})$.

$$\text{Kernel} = \int_0^\infty \Phi(\lambda-\theta,x,y,\theta)\,\theta^{\frac{n-1}{2}} e^{i\theta d_g(x,y)}\, d\theta.$$

$$= e^{i\lambda d_g(x,y)} \int_{-\lambda}^\infty \Phi(-\theta,x,y,\ )\left(1+\frac{\theta}{\lambda}\right)^{\frac{n-1}{2}} e^{i\theta d_g}\, d\theta. \quad \square$$

L^∞ estimate ~~β~~ in corollary, L^2 trivial,

So only need to prove:

$$\|\chi_\lambda f\|_{\frac{2(n+1)}{n-1}} = \|\rho(\lambda-P)f\|_{\frac{2(n+1)}{n-1}} \leq C\lambda^{\frac{n-1}{2(n+1)}}\|f\|_{L^2}.$$

$$T_\lambda f = \lambda^{\frac{n-1}{2}} \int e^{i\lambda d_g(x,y)} \, b_\lambda(x,y) f(y)\, dy .$$

$\varphi(x,y) = d_g(x,y)$ has Hess rank $\frac{\partial^2 \varphi}{\partial x_i \partial y_j} = n-1$ (Gauss lem. on Exp map $\Rightarrow$ linear along geodesics.)

$y \mapsto \nabla_x \varphi(x_0, y) = \Sigma_{x_0}$ hypersurface.

$= \{ \xi : \sum g^{jk}(x,y) \xi_j \xi_k = \}$ $g = g_{ij}^{??}$ [illegible]

has $\neq 0$ principal curvatures.

Stein result: Hess rank hyp as above and $\neq 0$ principal curvature as above:

$$\| \int_{\mathbb{R}^n} e^{i\lambda\Phi} \, b f(y)\, dy \|_q \leq C \lambda^{-n/q} \|f\|_p .$$

$1 \leq p \leq 2$, $q = \frac{n+1}{n-1} p'$.

For us. $p = 2 \Rightarrow q = \frac{2(n+1)}{n-1}$. $\Rightarrow$ $\|T_\lambda\|_{L^2 \to L^{\frac{2(n+1)}{n-1}}} = O\left(\lambda^{\frac{n-1}{2}} \cdot \lambda^{\frac{-n(n-1)}{2(n+1)}}\right)$

Remark Same procedure works for arbitrary symbol ρ satisfying principal curvature condition. But, different parametrix.

Lecture 3: Slides.